**Effective cofinalities and admissibility in E-recursion**

E. R. Griffor and D. Normann (Oslo)

Abstract. In this paper we study the interplay of $\Sigma_2$-admissibility and $E$-recursion theory. If $\alpha \in \Omega$ and $E(\alpha)$ is its $E$-closure, we show that the $\Sigma_2$-admissibility of $E(\alpha)$ implies that its greatest cardinal has $RE \times CORE$ cofinality $\omega$. Let $\gamma$ denote $E(\alpha)$-cofinality of its greatest cardinal. A dynamic proof of selection on any $\delta < \gamma$ is given, which can therefore be relativized to recursion in an arbitrary relation on $E(\alpha)$. Among the applications of this selection result are: the consistency of the extended plus-one hypothesis with $\text{CH}$, $CO-RE$ cofinality of $\gamma$ is $\omega$, and an effective covering property for $CO-RE$ subsets of $\gamma$. Further, we show that for $\alpha, \beta \in \Omega$ with $\alpha < \beta$, if $\text{cf}(\beta) < \omega$ by a function $f$ recursive in $\alpha, \beta$ and some $\delta < \alpha$, then $\text{cf}(\beta) < \omega$ via some $f$ recursive in $\alpha, \beta$. Finally, let $\Gamma$ be monotone inductive over $\gamma < \gamma$. We prove that if $\phi(x, \cdot)$ is $\Delta_4$ and always has a solution in $\Gamma^\omega$, then the function giving the least level of such is $E$-recursive. Van de Wiele's characterization of the $E$-recursive functions follows as a corollary.

§ 0. Introduction. $E$-recursion was introduced by D. Normann [7] as a natural generalization of normal Kleene recursion in objects of finite type. Unless otherwise stated the $E$-closed sets we shall consider shall be of the form $E(\alpha)$ for some $\alpha \in \Omega$.

In § 1 we introduce the $RE \times CORE$ cofinality and show that $\Sigma_2$-admissibility of $E(\alpha)$ implies that its greatest cardinal has $RE \times CORE$ cofinality $\omega$. In addition we show that $RE$-cofinality $\omega$ does not imply admissibility.

Section 2 is devoted to a dynamic proof of selection (i.e. $\gamma = \text{cf}(E(\alpha))$) and we have uniform selection over $CO$ subsets of any $\delta < \gamma$ on $E(\alpha)$, which can therefore be relativized. This selection theorem thus has among its corollaries the consistency of the extended plus one hypothesis at the type three level with $\text{CH}$.

Applications of the proof of selection given in § 2 are presented in § 3. We show that if $\gamma$ is the cofinality of $\alpha$ in $E(\alpha)$, then the $CO-RE$ cofinality of $\gamma$ is $\gamma$. The proof of this gives rise to an effective covering property, namely, any $CO-RE$ subset of $\gamma$ can be covered by a REC set of the same order type. The final application makes clear the connection between selection and singularities. We show that for $\alpha < \beta$ such that $\text{cf}(\beta) < \omega$ by a function $f$ recursive in $\alpha, \beta$ and some $\delta < \alpha$, then $\text{cf}(\beta) < \omega$ by some $f$ recursive in $\alpha, \beta$.

The last section (§ 4) treats the interplay between monotone inductive definitions and $E$-recursive set functions using methods from Girard's $\beta$-logic [1], without introducing $\beta$-logic or its proof theory. If a $\Delta_4 \phi(x, \cdot)$ always has a solution in $\Gamma^\omega_4$.

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(the least fixed point of monotone inductive \( I \) over \( x \)), then the function giving that solution is \( E \)-recursive in \( x \). As a corollary we have an elementary proof of a theorem of Van der Wiel [14]:

If \( F : \mathcal{V} \to \mathcal{V} \) is uniformly \( \Sigma_\tau \)-definable and total over all admissible sets, then \( F \) is \( E \)-recursive.

Outside of \( \S 4 \), \( RE \), \( co-RE \) etc. are the boldface notions.

\section{Effective cofinalities.} Much attention has been given to various notions of definable cofinality, particularly in connection with priority arguments in \( E \)-recursion. We shall not attempt to give a complete picture and so the interested reader is directed to Griffor [2], Sacks [10] or Shanam [13]. The first question we address here was asked by Sacks, namely, is there a cofinality condition on \( x \) which characterizes when \( E(x) \) is \( \Sigma_\tau \)-admissible? The question was motivated by a result of Kriouos that: if \( E(x) \upharpoonright cf(\langle \rangle) = \omega \), then \( E(x) \) is \( \Sigma_\tau \)-admissible. Thus an attractive conjecture was that: \( E(x) \) is \( \Sigma_\tau \)-admissible if and only if \( E(x) \upharpoonright cf(\langle \rangle) = \omega \). However, Shanam noticed that if \( \gamma \) is the least ordinal where \( E(\langle \gamma \rangle \upharpoonright cf(\langle \gamma \rangle) > \omega \), then \( E(\langle \gamma \rangle) \) is \( \Sigma_\tau \)-admissible.

If \( E(x) \) is \( \Sigma_\tau \)-admissible Sacks [10] showed that there is a divergent computation without a Moschovakis witness in \( E(x) \). This witness induces an \( \omega \)-sequence through \( \mathbb{N} \) and we will first analyse the level of definability of one such sequence.

Definition. Consider \( E(x) \), \( x \in OR \), and without loss of generality assume that \( x \) is the greatest cardinal in \( E(\alpha) \). Define the \( RE \) join co-RE cofinality of \( x \) as:

\[
RE \land co-RE \upharpoonright cf(x) = \omega \land \xi < x \text{ such that there exists an } R \subseteq \alpha \text{ of order type } \tau \text{ unbounded in } x \text{ and } R \text{ is } RE \land co-RE, \text{ i.e. } R \text{ is the intersection of an } RE \text{ and a co-RE set.}
\]

Theorem 1.0 Suppose \( E(x) \) is \( \Sigma_\tau \)-admissible, then

\[
RE \land co-RE \upharpoonright cf(x) = \omega.
\]

Proof. As above we assume that \( x \) is the greatest cardinal in \( E(\alpha) \) (which is \( L_\alpha \) for some \( \alpha > x \)). If \( x \in \omega, \alpha \in E(\alpha) \), then associated with the computation tuple \( \langle e, a \rangle \) is the tree of subcomputations \( T_{\langle e, a \rangle} \) (which is recursive in \( \langle e, \langle \alpha \rangle \downarrow \uparrow \), but is in general only \( RE \) in \( \langle e, \langle \alpha \rangle \downarrow \)). Assume that \( E(\alpha) \) is \( \Sigma_\tau \)-admissible.

By Sacks [10] there exists an \( e \in \omega \) and \( a \in E(\alpha) \) such that \( T_{\langle e, a \rangle} \) is not well-founded, but

\[
L_\alpha \upharpoonright T_{\langle e, a \rangle} \text{ is well-founded.}
\]

Claim 1. The leftmost path in \( T_{\langle e, a \rangle} \) is in \( RE \land co-RE \).

Proof. We say that \( \sigma \) is on the leftmost path if

(i) \( \sigma \in T_{\langle e, a \rangle} \) (RE),

(ii) \( \sigma \uparrow \) (co-RE),

(iii) \( \tau < \sigma \) in the lexicographical ordering and \( n \) is minimal such that \( \tau(n) < \sigma(n) \), then \( \tau(n+1) \downarrow \) (RE).

This proves Claim 1.

Now assume that we have an effective coding of all finite sequences from \( x \) by \( x \) such that

\[
\langle \sigma^n \rangle \Rightarrow \langle \sigma \rangle, \text{ where } \sigma \neq \langle \rangle.
\]

Let \( \langle \beta_1, \ldots, \beta_\delta \rangle \in A \) if \( \beta_i \) is the index for the \( i \)th sequence of the leftmost path through \( T_{\langle e \rangle \langle a \rangle} \). Then \( A \) is the intersection of an \( RE \) set \( A_1 \) and a co-RE set \( A_2 \).

Claim 2. \( A \) is unbounded in \( x \).

Proof. If \( A \) is bounded by \( \lambda \), then use standard properties of the \( \Sigma_\tau \)-projection on admissible ordinals to show that \( A_1 \cap \lambda \in E(\alpha), A_2 \cap \lambda \in E(\alpha) \) and so \( \lambda \in E(\alpha) \), which is impossible.

This completes the proof of the theorem.

Definition. With \( E(x) \) as above let

(i) \( REC \upharpoonright cf(x) = \tau \in \omega \) such that there exists \( REC \downarrow \in \alpha \) of order type \( \tau \) unbounded in \( x \);

(ii) \( RE \upharpoonright cf(x) = \tau \in \omega \) such that there exists \( RE \downarrow \in \alpha \) of order type \( \tau \) unbounded in \( x \).

As one might expect the recursive cofinality is no stronger, on ordinals less than \( \omega \), than the cofinality in the sense of \( E(x) \).

Proposition 1.1. If \( \gamma < \omega \), then

\[
REC \upharpoonright cf(x) = cf^{E}(\gamma).
\]

Proof. \( \equiv: \) Let \( f : cf^{E}(\gamma) \to \gamma, f \in L_\gamma \) witness \( cf^{E}(\gamma) \) and without loss of generality we may assume that \( f \) is strictly increasing. Let \( R = im(f) \), then \( R \) witnesses

\[
REC \upharpoonright cf^{E}(\gamma) \subseteq cf^{E}(\gamma).
\]

\( \geq: \) Let \( \gamma \in \gamma \) witness the \( REC \upharpoonright cf(x) = \tau \), then \( \tau \in L_\gamma \) by the bounding principle and the function \( f : \tau \to \gamma \) given by \( \tau \leq \sigma \).

\[
f(\sigma) = \sigma \text{th element of } R
\]

is in \( L_\alpha \) and witnesses \( cf^{E}(\gamma) \subseteq REC \upharpoonright cf(x) \).

Corollary 1.2. If \( REC \upharpoonright cf(x) = \omega \), then \( E(x) \) is \( \Sigma_\tau \)-admissible.


\[
E(x) \upharpoonright cf(\langle \rangle) = \omega \Rightarrow E(x) \text{ is } \Sigma_\tau \text{-admissible}.
\]

We shall see now that \( REC \upharpoonright cf(x) = \omega \) is not enough to guarantee admissibility.

Theorem 1.3. \( RE \upharpoonright cf(x) = \omega \) non \( \Rightarrow E(x) \) is \( \Sigma_\tau \text{-admissible} \).

Proof. Begin with \( E(\mathbb{N}) \) (which is not \( \Sigma_\tau \)-admissible) and define the following \( \Sigma_\tau \)-sequence:

\[
\Sigma(0) = \Sigma; \quad \Sigma(n+1) = \Sigma^{2^n}.
\]

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Let \( \tau \) be a recursive function defined by:

\[
\tau(0) = 1; \quad \tau(\lambda) = \sup\{\tau(\beta) : \beta < \lambda\}
\]

if \( \lambda \) is a limit ordinal;

\[
\tau(\beta + 1) = \min(\mu_{\omega} \triangleright \beta + 1)
\]

Claim. \( \tau(\alpha) \geq \min(f) \).

Proof. Otherwise for each \( \beta < \alpha \) let \( h_\beta = \varphi_{\omega+1} \), then if \( \beta_1 < \beta_2 \), there is a \( \gamma < \delta \) such that

\[
h_{\beta_2}(\gamma) < h_{\beta_1}(\gamma).
\]

Let \( \beta_0 = h_\beta(\gamma) \). Then, if for some \( \gamma, \beta_0, \beta < \alpha \) is unbounded, we have \( ||f(\gamma)|| < \tau(\alpha) \), so this cannot be the case. Let \( \beta_0 = \sup\{\beta_0 : \beta < \alpha\} \). Since \( \delta < \gamma = \text{cf}^a(\alpha) \),

we have that

\[
\sigma = \sup(\beta_0, \gamma < \delta) < \alpha.
\]

But for each \( \beta < \alpha \) there is one minimal \( \gamma \) such that \( \beta_0 > \beta_1 \). This gives a one-to-one map of \( \delta \times \sigma \) into \( \delta \times \alpha \), which is impossible and gives the claim.

Since \( \tau(\alpha) \) is recursive, we have computed \( \min(f) \) from \( f \) giving the theorem.

**Corollary 2.** We have selection over \( \gamma = \text{cf}^a(\alpha) \) if and only if we have selection over \( \alpha \).

**Proof.** Selection over \( \alpha \) clearly implies selection over \( \gamma \). The other direction follows from the theorem and the dynamic proof of selection due to Sacks–Slaman (Theorem 2.8 in Slaman [13]) which inspired this proof.

Now assume that \( E(\alpha) \) is not \( \omega_1 \)-admissible and, hence, we do not have selection over \( \alpha \). The above corollary tells us we do not have selection over \( \gamma \), however the theorem tells us:

**Corollary 2.1.** Let \( \delta < \gamma, C \subseteq \delta \) be RE, then \( C \in E(\alpha) \).

**Proof.** Since we have selection over \( \delta \), it follows that

\[
\text{sup}\{\alpha : \gamma < \delta \} < \alpha
\]

and \( C \) can be defined this level in \( E(\alpha) \).

**Corollary 2.2.** (Further reflection). Let \( \delta, C \) be as above, then

(a) \( \delta^C < \delta \);  
(b) if \( B \in E(\alpha) \) is RE and \( B(C) \) holds, then there exists a \( \delta \)-recursive \( \beta \) such that \( B(C_\beta) \) holds.

**Proof.** Immediate.

**Corollary 2.3.** Suppose \( 2^\kappa = \kappa, \kappa \) is a regular cardinal and there is a well-ordering of \( 2^\kappa \) of height \( \kappa \) recursive in \( \text{E} \) and a real. Then the extended plus one hypothesis is true at the type 3 level.
This last corollary was pointed out to us by T. Slaman. The extended plus-one hypothesis (for reals) states: if $F$ is a normal type $n+2$ object and $n \geq 1$, then there exists a normal type 3 object $G$ such that
\[
\begin{align*}
&\text{se}(G) = \frac{1}{2} \text{se}(F),
\end{align*}
\]
where $\text{se}(F)$ is the collection of sets of reals recursive in $F$ and some real.

For background and further results on the extended plus-one hypothesis see Sacks [9] or Slaman [13].

§ 3. Applications: co-RE cofinality, effective covering and uniform computation of cofinality. We turn first to an application of the above selection result which will yield a covering property for many co-RE sets "preserving cofinality" and characterize what will call co-RE cofinality. Let $\alpha$ be an ordinal and consider again $E(\alpha) = L_\alpha$ for some $\alpha > \omega$. Without loss of generality we assume $\alpha$ is the greatest cardinal in $L_\alpha$ and let $\gamma = \text{cf}(\alpha)$.

**Definition.** Let $\beta \leq \gamma$ and define the co-RE cofinality of $\beta$ by:
\[
\text{co-RE-cf}(\beta) = \text{least } \delta \text{ such that there is a co-RE subset } A \text{ of } \beta \text{ of order type } \delta \text{ and unbounded in } \beta.
\]

**Lemma 3.0.** co-RE-cf($\alpha$) = co-RE-cf($\gamma$).

**Proof.** Let $f: \gamma \to \alpha$ be increasing and witness that $\text{cf}(\gamma) = \gamma$. If $\gamma$ is co-RE and of order type $\delta$, then $A_f = \{ f(\eta) \mid \eta \in A \}$ is the same order type through $\alpha$. If $A$ is unbounded in $\gamma$, then $A_f$ is unbounded in $f(\gamma) = \gamma$.

Let $\alpha \leq \omega$ be co-RE, unbounded and of order type $\delta$. Let $\gamma \leq \omega$, if there exists $\gamma_0 \in \gamma(\gamma), f(\gamma+1) \wedge A$. The RE sets are closed under the quantifiers $\forall \nu \in \nu$, so the RE sets are closed under $\forall \nu \in \nu$. Thus $\alpha(\gamma)$ is co-RE and clearly unbounded in $\gamma$. In addition $\text{ot}(\alpha(\gamma)) < \text{ot}(\gamma)$.

We shall show that $\text{co-RE-cf}(\gamma) = \gamma$. By the above selection theorem, $\beta \leq \gamma$ implies that the RE predicates are uniformly closed under $\beta \leq \gamma$ and, in addition, that
\[
L_\alpha \cap \text{WF}(\beta) \subseteq L_\alpha,
\]
where $\text{WF}(\beta)$ denotes the set of well-founded relations as $\beta \times \beta$ (the latter cannot in general be relativized).

**Theorem 3.1.** co-RE-cf($\gamma$) = $\gamma$.

**Proof.** Let $\gamma \leq \omega$ be co-RE, cofinal in $\gamma$ of order type $\beta$. Let $A$ be the $\alpha$th approximation to $A$ from the outside, i.e.
\[
A_f = \{ f(\gamma) \mid \gamma \in A \}.
\]

We will show that there is a recursive $\delta$ such that $\text{ot}(A) = \text{ot}(A_f)$.

Let $\gamma < \gamma_0$ then $\text{ot}(A \cap \gamma) < \beta$ and by further reflection applied to $\text{ot}(A)$, there is a $\delta$ recursive in $\gamma$ such that
\[
\text{ot}(A \cap \gamma) < \beta.
\]

Using this we construct a recursive increasing function $g: \gamma \to \alpha$ such that
\[
\forall \gamma < \gamma_0 \text{ ot}(\gamma \cap \alpha) < \beta.
\]

Let $\delta = \sup \{ g(\gamma) \mid \gamma < \gamma_0 \}$, then $\delta$ is recursive so let $C = A_\delta$. Thus $C$ is recursive and $A \subseteq C$. If $\text{ot}(C) > \beta$, then there exists a $\gamma < \gamma_0$ such that $\text{ot}(C \cap \gamma) = \beta$. But $C \cap \gamma \subseteq A_\delta \cap \gamma$ since $\gamma(\gamma) < \delta$. Since $\text{ot}(A_\delta \cap \gamma) < \beta$, we have a contradiction.

**Corollary 3.2.** (Covering Property). Any co-RE subset $A \subseteq \gamma$ can be covered by a REC set of the same order type.

The corollary is proven in the proof of the theorem and we used the ordinal $\beta$ as a parameter. This lack of uniformity makes extension of the result in the corollary to ordinals other than $\gamma$ difficult, however, we offer:

**Problem.** Is there a bounded co-RE set that cannot be covered by a REC set of the same order type?

If $L_\alpha$ is $\Sigma_1$-admissible, then co-RE-cf($\alpha$) = $\omega$ (recall that $L_\alpha = E(\alpha)$), but the converse is not true.

As far as the questions of § 1 go these results show that
\[
\text{co-RE-cf}(\alpha) = \omega \Rightarrow E(\alpha) = \Sigma_1-
\]

admissible, however
\[
E(\alpha) \Sigma_1-
\]
admissible non $\Rightarrow$ co-RE-cf($\alpha$) = $\omega$.

Together with the results of § 1 this shows that there is no natural cofinality-assumption that will characterize when $E(\alpha)$ is admissible, the best seems to be the one implicit in the lack of certain Moschovakis witnesses.

Our next application makes clear the interplay between selection and singularities.

**Theorem 3.3.** Let $\alpha < \beta$ be ordinals such that $\text{cf}(\beta) \leq \alpha$ by some function $f$ recursive in $\alpha$ and some $\delta < \alpha$. Then $\text{cf}(\beta) \leq \alpha$ by some function $f$ recursive in $\alpha$.

**Proof.** Let $\alpha = \beta$ be a list of "computation tuples" over $\beta$ such that $\exists \delta < \alpha [g(\delta)]$. The intuition here is that we attempt to carry out a search for the $\delta < \alpha$ by question and we either compute it effectively, and hence the witness to $\text{cf}(\beta) \leq \alpha$, or we do not and in so doing (not doing) obtain a witness to $\text{cf}(\beta) \leq \alpha$. Let
\[
\text{min}(\beta) = \min \{ ||g(\beta)|| \mid \delta < \alpha \}.
\]

By the selection theorem in § 2: if $E(\beta) \neq \text{cf}(\beta) = \alpha$, we know that $\text{min}(\beta)$ is computable by some recursive function $M(g)$. In general it is sufficient for $M(g)$ to be defined that $\text{min}(\beta)$ exists. If $M(\beta) < \text{min}(\beta)$ this means that we have
\[
E_{M(g)}(\alpha) = \text{cf}(\beta) \leq \alpha;
\]

where for $\gamma < \text{OR} \cap E(\alpha)$
\[
E(\alpha) = \{ x \in E(\alpha) \mid \text{or computed by a computation of height } < \gamma \}.
\]
Now let \( g(\beta) \) be an index for \( f \) recursive in \( \delta, \alpha, \beta \) witnessing that \( cf(\beta) \leq \alpha \). Since \( \min(g) \) exists we have that the selection algorithm \( M(g) \) satisfies \( M(g) \).

If \( \min(g) = M(g) \) we have computed the level at which the cofinality map is constructed. If \( M(g) < \min(g) \), this is because we know at that ordinal that \( cf(\beta) \leq \alpha \).

Thus in both cases we can find from \( M(g) \) an \( f \) collapsing the cofinality of \( \beta \) below \( \alpha+1 \).

If \( L_{\alpha} = E(\gamma) \) then for all \( \gamma \) such that \( \alpha < \gamma < \kappa \) we can find effectively in \( \alpha, \gamma \) a map in \( L_{\alpha} \) witnessing

\[ \gamma^\gamma = \alpha^\alpha. \]

The above theorem will enable us to do this in many more cases. Suppose \( L_{\alpha} \) is \( E \)-closed and has a greatest cardinal \( gc(\alpha) \).

**Corollary 3.4.** If \( \gamma > gc(\alpha) \), let \( f_{\gamma} \) be the least (in the sense of \( < \)) collapse of \( \gamma \) to \( gc(\alpha) \). If for some \( \alpha, \beta < \kappa \) we have that

\[ (\forall \gamma > \gamma_0)(\exists \mathcal{Z} < gc(\alpha)) [f_\gamma \leq \mathcal{Z}, \gamma_0, gc(\alpha), \gamma, \mathcal{Z}], \]

then the function \( \gamma \rightarrow f_{\gamma} \) is uniformly computable in \( \gamma_0, \alpha, gc(\alpha) \) and a gc(\alpha)-enumeration of \( \eta_0 \).

Proof. We proceed by induction on \( \gamma > \gamma_0 \). \( \gamma_0 \) is trivial to show. If \( \gamma > \gamma_0 \), let \( a_\beta \) be a so large that all \( \gamma < \gamma \) are collapsed to \( gc(\alpha) \) by level \( a_\beta \). Let \( a_\beta \) such that:

\[ L_{\alpha} \uparrow \gamma \rightarrow gc(\alpha), \quad \text{then} \quad L_{\alpha} \uparrow \gamma \rightarrow (gc(\alpha))^+, \]

where \( \tau^+ \) is the successor cardinal of \( \tau \). By the theorem there is an \( \alpha \) recursive in \( \gamma, \alpha, \gamma_0, gc(\alpha) \) and the collapse of \( \gamma_0 \) such that

\[ L_{\alpha} \uparrow cf(\gamma) \leq gc(\alpha). \]

But a successor cardinal is regular, so this singularity will demonstrate that \( \gamma = gc(\alpha) \) and the collapsing map can be computed.

**Corollary 3.5.** Let \( L_{\alpha} \) be \( E \)-closed and let \( \alpha = gc(L_{\alpha}) \). Assume that \( L_{\alpha} \uparrow (\ast) \).

Then the following are equivalent

(i) \( L_{\alpha} \) is RE in an element of \( L_{\alpha} \).

(ii) Both \( L_{\alpha} \cap (\alpha) \) and \( \alpha \) are RE in an element of \( L_{\alpha} \).

Remark. Using forcing-methods of Sacks [11] we may show that if \( (\ast) \) holds, then \( L_{\alpha} \) is not RE.

§ 4. \( E \)-recursive functions and inductive definability. In this section we shall give a treatment of monotone inductive definitions using methods from Girard's \( \beta \)-logic [1], but without introducing \( \beta \)-logic and its proof theory. Masseran [5] has used the proof theory of \( \beta \)-logic to show that every total \( \alpha_{\alpha_0}^{\alpha} \)-recursive function on \( \alpha_{\alpha_0}^{\alpha} \) is dominated by a primitive recursive dilator on infinite arguments. As a corollary we give a proof of Van de Wiele's theorem:

If \( F : V \rightarrow V \) is total uniformly \( \Sigma_1 \)-definable over every admissible set, then \( F \) is \( E \)-recursive.

The converse for \( E \)-recursive functions (lightface) is immediate. Slaman has given an alternate proof, but his proof uses the theory of reflection in \( E \)-recursion, whereas we will require only familiarity with the generating schemata of \( E \)-recursion.

Like the completeness theorem for \( \beta \)-logic this proof is based on the Henkin-type construction of term models, otherwise the proof is elementary. For such \( x \), let \( \Gamma_x \) be a uniformly \( \Delta_0(x) \) positive inductive definition on \( x \). Let \( \leq_x \) denote the stage comparison relation on \( x \). The following lemma is valid for monotone inductive definitions in general.

**Lemma 4.0.** Let \( \forall x \leq x, \leq \) be a relation on \( y \) such that

(i) \( \Gamma(x) = \Gamma \); and

(ii) for each \( y \in \exists \{ y \mid y \leq y \} = \Gamma \}

then \( \Gamma_x \leq \forall x \) and \( \leq_x \) is the well-founded initial segment of \( \leq \) (\( \Gamma_x \leq \) is the least fixed-point of \( \Gamma \)).

For each \( x \), let \( \tau_x \) be the closure ordinal of \( \Gamma_x \) and let \( \varphi \) be a \( \Delta_\alpha \)-formula such that

\[ \forall x \exists y < \tau_x \varphi(x, \Gamma_x). \]

**Theorem 4.1.** There is an \( E \)-recursive function \( G \) such that

\[ \forall x \exists y (\text{rank}(x) \leq \tau_x = \exists y \min(G(x), \tau_x) \varphi(x, \Gamma_x)). \]

**Definition.** Let \( T = T_{\varphi} \) be the following first order theory:

unary predicates \( x, y \), \( ON \)

binary predicates \( P \) (for \( \leq_x \) and \( \in \))

unary function \( R \) (for rank)

constants \( c_i, e_i, \ldots \)

Take standard axioms like regularity, extensionality, etc. together with:

(i) \( \Gamma(x) \); \( \Gamma(y, z) \rightarrow \Gamma \)

(ii) \( \varphi(x, \Gamma(y, z)) \rightarrow \forall z \in \exists y (\varphi(x, \Gamma(y, z)) \rightarrow P(c_\varphi, z)) \); \( \Gamma(y, z) \rightarrow \Gamma \)

(iii) \( \exists \Gamma \exists y (\Gamma(y, z) \rightarrow \Gamma \)

(iv) \( \exists y \in \Gamma \Gamma(y, z) = \Gamma \)

**Definition.** (a) Let \( T^* \) denote the part of \( T \) that does not contain any \( c_i \) for \( i \neq n \).

(b) Let \( T^{*}, T^{**} \) denote the respective Henkin-extensions;

(c) Let \( e_\varphi, e_1, \ldots \) be a recursive enumeration of the terms of \( T^* \) such that

\[ \Psi(e_\varphi, e_i). \]

Now if \( f : N \rightarrow ON \), let \( T \) be \( T^* \) extended with the following axioms:

\[ \{ \exists R \leq R(q_i), f(i) \leq R(f(i)) \} \]

**Lemma 4.2.** Let \( f : N \rightarrow ON \) and \( T^* \) be as above, then \( T^* \) is inconsistent.

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Proof. Assume $T'$ is consistent for a contradiction and let $T'$ denote a consistent completion of $T$. The term model for $T'$ will then be a model of $T$ and since the rank-relation is well-founded, the model will be isomorphic to a set $z$ where $x$ is interpreted as a subset of $z$. Let $\gamma < \tau$ be such that $\varphi (x, \Gamma^{\tau+1}_z)$. By lemma the interpretation $c_0$ of $c_0$ must be in $\Gamma^{\tau}_z$ and have rank $\leq \gamma + 1$. But then interpretations of $c_1$ will form a $\leq$-infinite descending sequence, which is absurd.

If $x$ is a finite sequence of $\sigma$-ordinals we define $T^\sigma$ as an extension of $T_{\sigma}$ as before. Thus we have

$$\forall \gamma: \exists \sigma \in \mathbb{N} (\gamma \in T_{\sigma})$$

DEFINITION. Let $\sigma$ be a sequence of $\sigma$-ordinals of length $n$, then we say $\sigma$ is good if we cannot prove a contradiction from $T^\sigma$ using a proof of length $\leq n$ and at most the $n$ first values of $\sigma$ (in some uniform enumeration of $T^\sigma$).

For $\sigma \in \mathbb{N}$ we let

$$S_\sigma = \{ \sigma \mid \sigma \text{ is good and } \forall \gamma \in \mathbb{N} \forall \phi (\gamma) (\phi (\gamma), \sigma(\gamma)) \}$$

and set $G(\sigma) = \text{height of } S_\sigma$. Then $G$ is $E$-recursive since we can uniformly compute the height of any well-founded relation in $E$-recursion.

LEMMA 3.4. Let $\text{rank}(x) < \alpha$, then we can find $\gamma \in G(\alpha)$ such that $\varphi (x, \Gamma^{\alpha+1}_z)$ holds.

Proof. Fix $x$ and $\gamma$ be minimal such that $\varphi (x, \Gamma^{\gamma+1}_z)$ and choose $\gamma \in \Gamma^{\gamma+1}_z - \Gamma^\gamma_z$. Let $\rho$ denote the ordinal norm function on $\Gamma^\gamma_z$ induced by $\Gamma_z$. Then we have $p(\rho) = \gamma$. Assume that $\rho_0, \ldots, \rho_{n-1}$ is a sequence from $\Gamma^\gamma_z$ such that $\rho_0 = \gamma$ and $p(\rho_i) = \varphi (\rho_{i+1})$ for $1 \leq i < n$.

We shall construct a model for $T_z$ using $\text{TC}(x)$ as the domain, $x$ for $\Gamma^\gamma_z$ for $Y$, $\leq_x$ for $P$ and $\varphi_0, \ldots, \varphi_{n-1}$ for $c_0, \ldots, c_{n-1}$. This model can be extended to a model for $T_z$ since $T_z$ is a conservative extension of $T_z$ and we do not change the domain. For $i < n$ let $\sigma(i) = \text{rank}(\rho_i) = \sigma(i)$ the interpretation of $\sigma(i)$. Note that if we extend $\text{TC}(x)$ in a consistent way, then we may extend $\sigma$ (i.e. we cannot choose $\sigma$ such that it is inconsistent with the construction based on extensions of $\rho$).

If $\alpha = \text{rank}(x)$, then $\text{rank}(c_0, c_1, \ldots, c_{n-1})$ by our choice of domain as $\text{TC}(x)$ and so $\sigma \in S_\alpha$. By induction on $p(\rho_{n-1})$ we can show that $\rho_{n-1} < \text{length}(c_0, c_1, \ldots, c_{n-1})$. The induction is trivial by the above remark on the consistency-considerations and, hence, the lemma follows. The theorem follows from the lemma.

Remark. If $x$ is a relation on a transitive set $\gamma$: $(Y, P)$ is the prewellordering induced by $\Gamma$ over $x$ and there is no $\delta \in \Gamma_z$ satisfying $\varphi$. If $T$ is a primitive recursive language in the theory of $x$, then the same proof gives:

COROLLARY 4.4. Let $\Gamma$, $\varphi$ and $\tau$ be as above. If $\forall \chi (x \in \Gamma^\tau \Rightarrow \exists \gamma \in \text{rank}(x), \varphi (x, \Gamma^{\tau+1}_z))$, then there is an $E$-recursive function $G$ such that $\forall \chi (x \in \Gamma^\tau \Rightarrow \exists \gamma < \text{rank}(x) \forall \text{rank}(x) (\varphi (x, \Gamma^{\tau+1}_z))$.