Effective cofinalities and admissibility in E-recursion

by

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Abstract. In this paper we study the interplay of Σ_1-admissibility and E-recursion theory. If $\alpha \in ON$ and $E(\alpha)$ is its E-closure, we show that the Σ_1-admissibility of $E(\alpha)$ implies that its greatest cardinal has RE $\wedge$ co-RE *cofinality* ω. Let γ denote $E(\alpha)$-cofinality of its greatest cardinal. A dynamic proof of *selection* on any $\delta < \gamma$ is given, which can therefore be relativized to recursion in an arbitrary relation on $E(\alpha)$. Among the applications of this selection result are: the consistency of the *extended plus-one hypothesis* with $\neg$CH, co-RE cofinality of γ is γ, and an *effective covering property* for co-RE subsets of γ. Further, we show that for $\alpha, \beta \in ON$ with $\alpha < \beta$: if $\mathrm{cf}(\beta) \leqslant \alpha$ by a function f recursive in α, β and some $\delta < \alpha$, then $\mathrm{cf}(\beta) \leqslant \alpha$ via some f recursive in α, β. Finally, let Γ be monotone inductive over $x \in V$. We prove that if $\varphi(x, \cdot)$ is Δ_0 and always has a solution in Γ_x^∞, then the function giving the least level of such is E-recursive. Van de Wiele's *characterization of the E-recursive functions* follows as a corollary.

§ 0. Introduction. E-recursion was introduced by D. Normann [7] as a natural generalization of normal Kleene recursion in objects of finite type. Unless otherwise stated the E-closed sets we shall consider shall be of the form $E(\alpha)$ for some $\alpha \in$ OR.

In § 1 we introduce the RE $\wedge$ co-RE cofinality and show that Σ_1-admissibility of $E(\alpha)$ implies that its greatest cardinal has RE $\wedge$ co-RE cofinality ω. In addition we show that RE-cofinality ω does not imply admissibility.

Section 2 is devoted to a dynamic proof of selection (i.e. $\gamma = \mathrm{cf}^{E(\alpha)}(\alpha)$ then we have uniform selection over RE subsets of any $\delta < \gamma$ on $E(\alpha)$), which can therefore be relativized. This selection theorem thus has among its corollaries the consistency of the extended plus one hypothesis at the type three level with $\neg$CH.

Applications of the proof of selection given in § 2 are presented in § 3. We show that if γ is the cofinality of α in $E(\alpha)$, then the co-RE cofinality of γ is γ. The proof of this gives rise to an effective covering property, namely, any co-RE subset of γ can be covered by a REC set of the same order type. The final application makes clear the connection between selection and singularities. We show that for $\alpha < \beta$ such that $\mathrm{cf}(\beta) \leqslant \alpha$ by a function f recursive in α, β and some $\delta < \alpha$, then $\mathrm{cf}(\beta) \leqslant \alpha$ by some f recursive in α, β.

The last section (§ 4) treats the interplay between monotone inductive definitions and E-recursive set functions using methods from Girard's β-logic [1], without introducing β-logic or its proof theory. If a $\Delta_0\, \varphi(x, \cdot)$ always has a solution in Γ_x^∞

(the least fixed point of monotone inductive Γ over x), then the function giving that solution is E-recursive in x. As a corollary we have an elementary proof of a theorem of Van de Wiele [14]:

If F: $V \to V$ is uniformly Σ_1-definable and total over all admissible sets, then F is E-recursive.

Outside of § 4, RE, co-RE etc. are the boldface notions.

§ 1. **Effective cofinalities.** Much attention has been given to various notions of definable cofinality, particularly in connection with priority arguments in E-recursion. We shall not attempt to give a complete picture and so the interested reader is directed to Griffor [2], Sacks [10] or Slaman [13]. The first question we address here was asked by Sacks, namely, is there a cofinality condition on α which characterizes when $E(\alpha)$ is Σ_1-admissible. The question was motivated by a result of Kirousis that: if $E(\alpha) \models \mathrm{cf}(\bar{\bar{\alpha}}) = \omega$, then $E(\alpha)$ is Σ_1-admissible. Thus an attractive conjecture was that: $E(\alpha)$ is Σ_1-admissible if and only if $E(\alpha) \models \mathrm{cf}(\bar{\bar{\alpha}}) = \omega$. However, Slaman noticed that if γ is the least ordinal where $E(\gamma) \models \mathrm{cf}(\bar{\bar{\gamma}}) > \omega$, then $E(\gamma)$ is Σ_1-admissible. If $E(\alpha)$ is Σ_1-admissible Sacks [10] showed that there is a divergent computation without a Moschovakis witness in $E(\alpha)$. This witness induces an ω-sequence through $\bar{\bar{\alpha}}$ and we will first analyse the level of definability of one such sequence.

DEFINITION. Consider $E(\alpha)$, $\alpha \in \mathrm{OR}$, and without loss of generality assume that α is the greatest cardinal in $E(\alpha)$. Define the RE join co-RE *cofinality* of α as:

RE$\wedge$co-RE-cf(α) = least $\tau \leqslant \alpha$ such that there exists an $R \leqslant \alpha$ of order type τ unbounded in α and R is RE$\wedge$co-RE, i.e. R is the intersection of an RE and a co-RE set.

THEOREM 1.0. *Suppose $E(\alpha)$ is Σ_1-admissible, then*

$$\mathrm{RE}\wedge\text{co-RE-cf}(\alpha) = \omega .$$

Proof. As above we assume that α is the greatest cardinal in $E(\alpha)$ (which is $L_\varkappa$ for some $\varkappa > \alpha$). If $e \in \omega$, $a \in E(\alpha)$, then associated with the computation tuple $\langle e, a\rangle$ is the tree of subcomputations $T_{\langle e,a\rangle}$ (which is recursive in $\langle e, a\rangle$ if $\{e\}(a)\downarrow$, but is in general only RE in $\langle e, a\rangle$). Assume that $E(\alpha)$ is Σ_1-admissible.

By Sacks [10] there exists an $e \in \omega$ and $a \in E(\alpha)$ such that $T_{\langle e,a\rangle}$ is not well-founded, but

$$L_\varkappa \models T_{\langle e,a\rangle} \text{ is well-founded} .$$

CLAIM 1. *The leftmost path in $T_{\langle e,a\rangle}$ is in* RE$\wedge$co-RE.

Proof. We say that σ is on the leftmost path if

(i) $\sigma \in T_{\langle e,a\rangle}$ (RE),

(ii) $\sigma\uparrow$ (co-RE),

(iii) If $\tau < \sigma$ in the lexicographical ordering and n is minimal such that $\tau(n) < \sigma(n)$, then $\bar{\tau}(n+1)\downarrow$ (RE).

This proves Claim 1.

Now assume that we have an effective coding of all finite sequences from α by α such that

$$\langle\sigma^\frown\tau\rangle > \langle\sigma\rangle, \quad \text{where} \quad \tau \neq \langle\ \rangle .$$

Let $\langle\beta_1, \ldots, \beta_n\rangle \in A$ if β_i is the index for the ith sequence of the leftmost path through $T_{\langle e,a\rangle}$. Then A is the intersection of an RE set A_1 and a co-RE set A_2.

CLAIM 2. *A is unbounded in α.*

Proof. If A is bounded by $\lambda < \alpha$, then use standard properties of the Σ_1-projectum on admissible ordinals to show that $A_1 \cap \lambda \in E(\alpha)$, $A_2 \cap \lambda \in E(\alpha)$ and so $A \in E(\alpha)$, which is impossible.

This completes the proof of the theorem.

DEFINITION. With $E(\alpha)$ as above let

(i) REC-cf(α) = $\mu\tau \leqslant \alpha$ such that there exists REC $R \subseteq \alpha$ of order type τ unbounded in α;

(ii) RE-cf(α) = $\mu\tau \leqslant \alpha$ such that there exists RE $R \subseteq \alpha$ of order type τ unbounded in α.

As one might expect the recursive cofinality is no stronger, on ordinals less than $\varkappa$, than the cofinality in the sense of $E(\alpha)$.

PROPOSITION 1.1. *If $\gamma < \varkappa$, then*

$$\text{REC-cf}(\gamma) = \mathrm{cf}^{L_\varkappa}(\gamma) .$$

Proof $\leqslant$: Let f: $\mathrm{cf}^{L_\varkappa}(\gamma) \to \gamma$, $f \in L_\varkappa$ witness $\mathrm{cf}^{L_\varkappa}(\gamma)$ and without loss of generality we may assume that f is strictly increasing. Let $R = \mathrm{im}(f)$, then R witnesses

$$\text{REC-cf}(\gamma) \leqslant \mathrm{cf}^{L_\varkappa}(\gamma) .$$

$\geqslant$: Let $R \subseteq \gamma$ witness the REC-cf$(\gamma) = \tau$, then $R \in L_\varkappa$ by the bounding principle and the function f: $\tau \to \gamma$ given by $\sigma < \tau$.

$$f(\sigma) = \sigma\text{th element of } R$$

is in $L_\varkappa$ and witnesses $\mathrm{cf}^{L_\varkappa}(\gamma) \leqslant$ REC-cf(γ).

COROLLARY 1.2. *If* REC-cf$(\alpha) = \omega$, *then $E(\alpha)$ is Σ_1-admissible.*

Proof. Use the proposition and the selection-theorem of Kirousis [4] stating

$$E(\alpha) \models \mathrm{cf}(\bar{\bar{\alpha}}) = \omega \Rightarrow E(\alpha) \text{ is } \Sigma_1\text{-admissible} .$$

We shall see now that RE-cf$(\alpha) = \omega$ is not enough to guarantee admissibility.

THEOREM 1.3. RE-cf$(\alpha) = \omega$ non $\Rightarrow$ *$E(\alpha)$ is Σ_1-admissible.*

Proof. Begin with $E(\aleph_1)$ (which is not Σ_1-admissible) and define the following $\varkappa_r$-sequence:

$$\varkappa_r(0) = \varkappa_r ;$$
$$\varkappa_r(n+1) = \varkappa_r^{\varkappa_r(n)} .$$

Now consider $\{x|\ x\in E(\aleph_1)$ and $x\leqslant_E \varkappa_r(n)$ for some $n\in\omega\} = M$. Let $\overline{M}$ be the Mostowski collapse of M, then $\overline{M}$ is E-closed and satisfies the Moschovakis Phenomenon (use the MP in $E(\aleph_1)$ and the definition of $\varkappa_r$) and $\overline{M}$ is an E-closure of one of its elements.

But $\overline{M}$ has an ω-sequence of $\varkappa_r$'s. Let $\alpha = (\aleph_1)_{\overline{M}}$ and let

$$R = \{x<\alpha|\ x \text{ is the index for an ordinal } \beta \text{ such that } \beta = \varkappa_r^a \text{ for some } a<\alpha\}.$$

R is RE and unbounded in α and clearly of order type ω. Thus $\overline{M}$ is not Σ_1-admissible, while over $\overline{M}$ RE-cf$(\alpha) = \omega$, where $\alpha = (\aleph_1)_{\overline{M}}$.

§ 2. **Dynamic selection.** We shall give a dynamic proof of the following theorem:

Let α be the greatest cardinal in $E(\alpha)$ and let γ be the $E(\alpha)$-cofinality of α. Then we have uniform selection for RE *subsets of any $\delta<\gamma$.*

As it stands, the theorem was proven by Kirousis [4], but the "dynamic" proof we shall give can be relativized, whereas Kirousis made use of a Skolem Hull-collapsing argument. A similar proof using a collapsing argument was given by Normann [8] for the case $\gamma = \alpha$, i.e. α is a regular cardinal in $E(\alpha)$. We now give the dynamic proof.

Let δ be fixed as in the theorem and let f be a δ-sequence of computations. Let R be the Moschovakis [6] subcomputation relation which is RE and, finally, let R_β denote the βth approximation to R. The relation R is such that for a given computation, the set of immediate subcomputations can uniformly be indexed by a finite set or by α (the case of an α-branching). In the case of composition we let the *innermost computation* be the leftmost one. If this one is convergent, then we know the other subcomputations.

Following Harrington–MacQueen [3] we let

$$\min(f) = \inf\{\|f(y)\|:\ y<\delta\},$$

where $\|\cdot\|$ denotes the function giving the height of a computation, if convergent, and equals ∞ otherwise. If $\min(f)<\infty$, i.e. one of the $f(y)$'s is convergent, we shall show that $\min(f)$ is uniformly recursive in f for $f\in E(\alpha)$. The situation $\min(f)<\infty$ corresponds to the non-emptiness of the associated RE subset of δ and, thus, we have shown selection over δ.

The proof proceeds by transfinite induction on $\min(f)$. An application of the recursion theorem yields the required uniformity.

The relation $\min(f) = 0$ is recursive, so assume that $\min(f)>0$ and that we have computed $\min(g)$ for all g such that $\min(g)<\min(f)$.

If $\min(f)>\beta$ (which is recursive in β) we let

$$g_\beta(y) = \text{leftmost subcomputation } z \text{ of } f(y) \text{ such that } \|z\|\geqslant\beta;$$

and otherwise we let $g_\beta = f$. Clearly g_β is recursive in f, β and if $\min(f)>\beta$, then

$$\beta\leqslant\min(g_\beta)<\min(f).$$

Let τ be a recursive function defined by:

$$\tau(0) = 1;$$
$$\tau(\lambda) = \sup\{\tau(\beta)|\ \beta<\lambda\}$$

if λ is a limit ordinal;

$$\tau(\beta+1) = \min(g_{\tau(\beta)+1}).$$

Claim. $\tau(\alpha)\geqslant\min(f)$.

Proof. Otherwise for each $\beta<\alpha$ let $h_\beta = g_{\tau(\beta)+1}$, then if $\beta_1<\beta_2$, there is a $y<\delta$ such that

$$h_{\beta_1}(y)<h_{\beta_2}(y).$$

Let $\beta_y = h_\beta(y)$, then if for some y, $\{\beta_y\colon\ \beta<\alpha\}$ is unbounded, we have $\|f(y)\|\leqslant\tau(\alpha)$, so this cannot be the case. Let $\beta_y^* = \sup\{\beta_y|\ \beta<\alpha\}$. Since

$$\delta<\gamma = \mathrm{cf}^{E(\alpha)}(\alpha),$$

we have that

$$\sigma = \sup\{\beta_y^*|\ y<\delta\}<\alpha.$$

But for each $\beta<\alpha$ there is one minimal y such that $(\beta+1)_y>\beta_y$. This gives a one-to-one map of α into $\delta\times\sigma$, which is impossible and gives the claim.

Since $\tau(\alpha)$ is recursive, we have computed $\min(f)$ from f giving the theorem.

Corollary 2.0. *We have selection over $\gamma = \mathrm{cf}^{E(\alpha)}(\alpha)$ if and only if we have selection over α.*

Proof. Selection over α clearly implies selection over γ. The other direction follows from the theorem and the dynamic proof of selection due to Sacks–Slaman (Theorem 2.8 in Slaman [13]) which inspired this proof.

Now assume that $E(\alpha)$ is not Σ_1-admissible and, hence, we do not have selection over α. The above corollary tells us we do not have selection over γ, however the theorem tells us:

Corollary 2.1. *Let $\delta<\gamma$, $C\subseteq\delta$ be* RE, *then $C\in E(\alpha)$.*

Proof. Since we have selection over δ, it follows that

$$\sup\{\varkappa_0^y|\ y<\delta\}<\varkappa$$

and C can be defined this level in $E(\alpha)$.

Corollary 2.2 (Further reflection). *Let δ, C be as above, then*

(a) $\varkappa_0^{C,\delta}<\varkappa_r^\delta$;

(b) *if $B\subseteq E(\alpha)$ is* RE *and $B(C)$ holds, then there exists a δ-recursive β such that $B(C_\delta)$ holds.*

Proof. Immediate.

Corollary 2.3. *Suppose $\overline{2}^\omega = \varkappa$, $\varkappa$ is a regular cardinal and there is a well-ordering of 2^ω of height $\varkappa$ recursive in ${}^4\mathbf{E}$ and a real. Then the extended plus one hypothesis is true at the type 3 level.*

This last corollary was pointed out to us by T. Slaman. The extended plus-one hypothesis (for reals) states: if F is a normal type $n+2$ object and $n \geqslant 1$, then there exists a normal type 3 object G such that

$$\tfrac{1}{2}\mathrm{sc}(G) = \tfrac{1}{2}\mathrm{sc}(F),$$

where $\frac{1}{2}\mathrm{sc}(F)$ is the collection of sets of reals recursive in F and some real.

For background and further results on the extended plus-one hypothesis see Sacks [9] or Slaman [13].

§ 3. Applications: co-RE cofinality, effective covering and uniform computation of cofinality. We turn first to an application of the above selection result which will yield a covering property for many co-RE sets "preserving cofinality" and characterize what will call co-RE cofinality. Let α be an ordinal and consider again $E(\alpha) = L_\varkappa$ for some $\varkappa > \alpha$. Without loss of generality we assume α is the greatest cardinal in $L_\varkappa$ and we let $\gamma = \mathrm{cf}^{L_\varkappa}(\alpha)$.

DEFINITION. Let $\beta \leqslant \varkappa$ and define the co-RE *cofinality* of β by:

co-RE-cf(β) = least δ such that there is a co-RE subset A of β of order type δ and unbounded in β.

LEMMA 3.0. co-RE-cf(α) = co-RE-cf(γ).

Proof. Let $f\colon \gamma \to \alpha$ be increasing and witness that $\mathrm{cf}^{L_\varkappa}(\alpha) = \gamma$.

$\leqslant$: If $A \subseteq \gamma$ is co-RE and of order type δ, then $A_f = \{f(y) \mid y \in A\}$ is the same order type through α. If A is unbounded in γ, then A_f is unbounded in α.

$\geqslant$: Let $A \subseteq \alpha$ be co-RE, unbounded and of order type δ. Let $y \in A^*$, if there exists $z \in [f(y), f(y+1)\rangle \cap A$. The RE sets are closed under the quantifiers $\forall z \in u$, so the co-RE sets are closed under $\exists z \in u$. Thus A^* is co-RE and clearly unbounded in γ. In addition o.t.$(A^*) \leqslant$ o.t.(A).

We shall show that co-RE-cf$(\gamma) = \gamma$. By the above selection theorem, $\beta < \gamma$ implies that the RE predicates are uniformly closed under $\exists y < \beta$ and, in addition, that

$$L_\varkappa \cap \mathrm{WF}(\beta) \in L_\varkappa,$$

where WF(β) denotes the set of well-founded relations as $\beta \times \beta$ (the latter cannot in general be relativized).

THEOREM 3.1. co-RE-cf$(\gamma) = \gamma$.

Proof. Let $A \subseteq \gamma$ be co-RE, cofinal in γ of order type β. Let A_δ be the δth approximation to A from the outside, i.e.

$$A_\delta = \{y \mid L_\delta \nvDash y \notin A\}.$$

We will show that there is a recursive δ such that o.t.(A) = o.t.(A_δ).

Let $y < \gamma$; then o.t.$(A \cap y) < \beta$ and by further reflection applied to cA, there is a δ recursive in y such that

$$\mathrm{o.t.}(A_\delta \cap y) < \beta.$$

Using this we construct a recursive increasing function $g\colon \gamma \to \varkappa$ such that

$$\forall y < \gamma (\mathrm{o.t.}(A_{g(y)} \cap y) < \beta).$$

Let $\delta = \sup\{g(y) \mid y < \gamma\}$, then δ is recursive so let $C = A_\delta$. Thus C is recursive and $A \subseteq C$. If o.t.$(C) > \beta$, then there exists a $y < \gamma$ such that o.t.$(C \cap y) = \beta$. But $C \cap y \subseteq A_{g(y)} \cap y$ since $g(y) < \delta$. Since o.t.$(A_{g(y)} \cap y) < \beta$, we have a contradiction.

COROLLARY 3.2. (Covering Property). *Any* co-RE *subset A of γ can be covered by a* REC *set of the same order type.*

The corollary is proven in the proof of the theorem and we used the ordinal β as a parameter. This lack of uniformity makes extension of the result in the corollary to ordinals other than γ difficult, however we offer:

PROBLEM. Is there a bounded co-RE set that cannot be covered by a REC set of the same order type?

If $L_\varkappa$ is Σ_1-admissible, then co-RE-cf$(\varkappa) = \omega$ (recall that $L_\varkappa = E(\alpha)$), but the converse is not true.

As far as the questions of § 1 go these results show that

$$\text{co-RE-cf}(\alpha) = \omega \Rightarrow E(\alpha) \text{ is } \Sigma_1\text{-admissible},$$

however

$$E(\alpha)\ \Sigma_1\text{-admissible non} \Rightarrow \text{co-RE-cf}(\alpha) = \omega.$$

Together with the results of § 2 this shows that there is no natural cofinality-assumption that will characterize when $E(\alpha)$ is admissible, the best seems to be the one implicit in the lack of certain Moschovakis witnesses.

Our next application makes clear the interplay between selection and singularities.

THEOREM 3.3. *Let $\alpha < \beta$ be ordinals such that $\mathrm{cf}(\beta) \leqslant \alpha$ by some function f recursive in α, β and some $\delta < \alpha$. Then $\mathrm{cf}(\beta) \leqslant \alpha$ by some function recursive in α, β.*

Proof. Let $g\colon \alpha \to \beta$ be a list of "computation tuples" over β such that $(\exists \delta < \alpha)[g(\delta)\downarrow]$. The intuition here is that we attempt to carry out a search for the $\delta < \alpha$ in question and we either compute it effectively, and hence the witness to $\mathrm{cf}(\beta) \leqslant \alpha$, or we do not and in so doing (not doing) obtain a witness to $\mathrm{cf}(\beta) \leqslant \alpha$. Let

$$\min(g) = \min\{\|g(\delta)\| \mid \delta < \alpha\}.$$

By the selection theorem in § 2: if $E(\beta) \models \mathrm{cf}(\beta) \geqslant \alpha$, we know that $\min(g)$ is computable by some recursive function $M(g)$. In general it is sufficient for $M(g)$ to be defined that $\min(g)$ exists. If $M(g) < \min(g)$ this means that we have

$$E_{M(g)+1}(\alpha) \models \mathrm{cf}(\beta) \leqslant \alpha;$$

where for $\gamma < \mathrm{OR} \cap E(\alpha)$

$$E_\gamma(\alpha) = \{x \in E(\alpha) \mid x \text{ computed by a computation of height } < \gamma\}.$$

Now let $g(\delta)$ be an index for f recursive in δ, α, β witnessing that $\mathrm{cf}(\beta)\leqslant\alpha$. Since $\min(g)$ exists we have that the selection algorithm $M(g)$ satisfies $M(g)\downarrow$.

If $\min(g) = M(g)$ we have computed the level at which the cofinality map is constructed. If $M(g)<\min(g)$, this is because we know at that ordinal that $\mathrm{cf}(\beta)\leqslant\alpha$. Thus in both cases we can find from $M(g)$ an f collapsing the cofinality of β below $\alpha+1$.

If $L_\varkappa = E(\alpha)$ then for all γ such that $\alpha<\gamma<\varkappa$ we can find effectively in α, γ a map in $L_\varkappa$ witnessing

$$\bar{\bar{\gamma}}^{L_\varkappa} = \bar{\bar{\alpha}}^{L_\varkappa}.$$

The above theorem will enable us to do this in many more cases. Suppose $L_\varkappa$ is E-closed and has a greatest cardinal ($\mathrm{gc}(\varkappa)$).

COROLLARY 3.4. *If $\gamma>\mathrm{gc}(\varkappa)$, let f_γ be the least (in the sense of $<_L$) collapse of γ to $\mathrm{gc}(\varkappa)$. If for some a, $\gamma_0<\varkappa$ we have that*

$$(*)\qquad (\forall\gamma>\gamma_0)(\exists z<\mathrm{gc}(\varkappa))[f_\gamma\leqslant_E a, \gamma_0, \mathrm{gc}(\varkappa), \gamma, z],$$

then the function $\gamma\to f_\gamma$ is uniformly computable in γ_0, a, $\mathrm{gc}(\varkappa)$ and a $\mathrm{gc}(\varkappa)$-enumeration of γ_0.

Proof. We proceed by induction on $\gamma>\gamma_0$. $\gamma = \gamma_0$ is trivial. If $\gamma>\gamma_0$, let a_γ be so large that all $\gamma'<\gamma$ are collapsed to $\mathrm{gc}(\varkappa)$ by level α_γ. Let $\alpha\geqslant\alpha_\gamma$ such that:

$$\text{if}\quad L_{\alpha_\gamma}\models\bar{\bar{\gamma}}>\mathrm{gc}(\varkappa),\quad\text{then}\quad L_\alpha\models\gamma = (\mathrm{gc}(\varkappa))^+,$$

where τ^+ is the successor cardinal of τ. By the theorem there is an α recursive in $\gamma, a, \gamma_0, \mathrm{gc}(\varkappa)$ and the collapse of γ_0 such that

$$L_\alpha\models\mathrm{cf}(\gamma)\leqslant\mathrm{gc}(\varkappa).$$

But a successor cardinal is regular, so this singularity will demonstrate that $\gamma = \mathrm{gc}(\varkappa)$ and the collapsing map can be computed.

Corollary 3.4 can be used to show that under $(*)$ we have

COROLLARY 3.5. *Let $L_\varkappa$ be E-closed and let $\alpha = \mathrm{gc}(L_\varkappa)$. Assume that $L_\varkappa\models(*)$. Then the following are equivalent*

(i) *$L_\varkappa$ is* RE *in an element of $L_\varkappa$,*

(ii) *Both $L_\varkappa\cap(\alpha)$ and $\varkappa$ are* RE *in an element of $L_\varkappa$.*

Remark. Using forcing-methods of Sacks [11] we may show that if $(*)$ holds, then $L_\varkappa$ is not RE.

§ 4. **E-recursive functions and inductive definability.** In this section we shall give a treatment of monotone inductive definitions using methods from Girard's β-logic [1], but without introducing β-logic and its proof theory. Masseron [5] has used the proof theory of β-logic to show that every total ω_1^{CK}-recursive function on ω_1^{CK} is dominated by a primitive recursive dilator on infinite arguments. As a corollary we give a proof of Van de Wiele's theorem:

If F: $V\to V$ is total uniformly Σ_1-definable over every admissible set, then F is E-recursive.

The converse for E-recursive functions (lightface) is immediate. Slaman has given an alternate proof, but his proof uses the theory of reflection in E-recursion, whereas we will require only familiarity with the generating schemata of E-recursion.

Like the completeness theorem for β-logic this proof is based on the Henkin-type construction of term models, otherwise the proof is elementary. For each set x let Γ_x be a uniformly $\Delta_0(x)$ positive inductive definition on x. Let $\leqslant_x$ denote the stage comparison relation on x. The following lemma is valid for monotone inductive definitions in general.

LEMMA 4.0. *Let $Y\subseteq x$, $\leqslant$ be a relation on y such that*

(i) *$\Gamma(Y) = Y$; and*

(ii) *for each $y\in Y$*

$$\{y'|\ y'\leqslant y\} = \Gamma(\{y'|\ y'<y\}),$$

then $\Gamma_x^\infty\leqslant Y$ and $\leqslant_x$ is the well-founded initial segment of $\leqslant$ (Γ_x^∞ is the least fixed-point of Γ_x).

For each x, let τ_x be the closure ordinal of Γ_x and let φ be a Δ_0-formula such that

$$\forall x\exists\gamma<\tau_x\varphi(x, \Gamma_x^{\gamma+1}).$$

THEOREM 4.1. *There is an E-recursive function G such that*

$$\forall\alpha\forall x(\mathrm{rank}(x)\leqslant\alpha\Rightarrow\exists\gamma\leqslant\min(G(\alpha), \tau_x)\varphi(x, \Gamma_x^{\gamma+1}));$$

DEFINITION. Let $T = T_{\Gamma,\varphi}$ be the following first order theory:

unary predicates $\boldsymbol{x}$, $\boldsymbol{Y}$, $\boldsymbol{ON}$

binary predicates $\boldsymbol{P}$ (for $\leqslant_x$) and $\in$

unary function R (for rank)

constants $\boldsymbol{c}_0, \boldsymbol{c}_1, \ldots$

Take standard axioms like regularity, extensionality, etc. together with:

(i) $\boldsymbol{Y} = \Gamma(\boldsymbol{Y})$;

(ii) $\varphi(x, \{y|\ \boldsymbol{P}(y, \boldsymbol{c}_0)\})\to\forall z\in\boldsymbol{Y}(\varphi(x, \{y|\ \boldsymbol{P}(y, z)\})\to\boldsymbol{P}(\boldsymbol{c}_0, z))$;

(iii) $\boldsymbol{P}(\boldsymbol{c}_{i+1}, \boldsymbol{c}_i)\wedge\neg\boldsymbol{P}(\boldsymbol{c}_i, \boldsymbol{c}_{i+1})$; and

(iv) $\forall z\in\boldsymbol{Y}(\{y|\ \boldsymbol{P}(y, z)\} = \Gamma(\{y|\ \boldsymbol{P}(y, z)\wedge\neg\boldsymbol{P}(z, y)\}))$.

DEFINITION. (a) Let T_n denote the part of T that does not contain any $\boldsymbol{c}_i$ for $i\geqslant n$;

(b) Let T^*, T_n^* denote the respective Henkin-extensions;

(c) Let $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots$ be a recursive enumeration of the terms of T^* such that $\forall i(\boldsymbol{e}_i\in T_i^*)$.

Now if f: $N\to ON$, let T^f be T^* extended with the following axioms:

$$\{R(\boldsymbol{e}_i)\leqslant R(\boldsymbol{e}_j)|\ f(i)\leqslant f(j)\}.$$

LEMMA 4.2. *Let f: $N\to ON$ and T^f be as above, then T^f is inconsistent.*

Proof. Assume T^f is consistent for a contradiction and let T^f denote a consistent completion of $\overline{T}^f$. The term model for $\overline{T}^f$ will then be a model of T and since the rank-relation is well-founded, the model will be isomorphic to a set z where $\boldsymbol{x}$ is interpreted as a subset of z. Let $\gamma<\tau_x$ be such that $\varphi(x, \Gamma_x^{\gamma+1})$. By lemma the interpretation c_0 of $\boldsymbol{c}_0$ must be in Γ_x^{∞} and have rank $\leqslant\gamma+1$. But then interpretations of $\boldsymbol{c}_i$ will form an $\leqslant$-infinite descending sequence, which is absurd.

If σ is a finite sequence of ordinals we define T^{σ} as an extension of $T^*_{\mathrm{lh}(\sigma)}$ as before. Thus we have

$$\forall f\colon N\to \mathrm{OR}\ \exists r\in N[T^{\overline{f(n)}} \text{ is inconsistent}].$$

Definition. Let σ be a sequence of ordinals of length n, then we say σ is *good* if we cannot prove a contradiction from T^{σ} using a proof of length $\leqslant n$ and at most the n first axioms of T^{σ} (in some uniform enumeration of T^f's).

For $\alpha\in\mathrm{OR}$ we let

$$S_{\alpha} = \{\sigma \mid \sigma \text{ is good and } \forall i<\mathrm{lh}(\sigma)(\sigma(i)<\alpha)\}$$

and set $G(\alpha)$ = height of S_{α}. Then G is E-recursive since we can uniformly compute the height of any well-founded relation in E-recursion.

Lemma 4.3. *Let* $\mathrm{rank}(x)\leqslant\alpha$, *then we can find* $\gamma\leqslant G(\alpha)$ *such that* $\varphi(x, \Gamma_x^{\gamma+1})$ *holds.*

Proof. Fix x and let γ be minimal such that $\varphi(x, \Gamma_x^{\gamma+1})$ and choose $y\in\Gamma_x^{\gamma+1}-\Gamma_x^{\gamma}$. Let p denote the ordinal norm function on Γ_x^{∞} induced by Γ_x. Then we have $p(y)=\gamma$. Assume that $y_0, \ldots, y_{n-1}$ is a sequence from Γ_x^{∞} such that $y_0 = y$ and $p(y_i)<p(y_{i-1})$ for $1\leqslant i<n$.

We shall construct a model for T_n using $\mathrm{TC}(x)$ as the domain, x for $\boldsymbol{x}$, Γ_x^{∞} for $\boldsymbol{Y}$, $\leqslant_x$ for $\boldsymbol{P}$ and $y_0, \ldots, y_{n-1}$ for $\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{n-1}$. This model can be extended to a model for T_n^* since T_n^* is a conservative extension of T_n and we do not change the domain. For $i<n$ let $\sigma(i) = \mathrm{rank}(e_i)$ (e_i is the interpretation of $\boldsymbol{e}_i$). Note that if we extend $\vec{y}$ in a consistent way, then we may extend σ (i.e. we cannot choose σ such that it is inconsistent with the construction based on extensions of $\vec{y}$).

If $\alpha = \mathrm{rank}(x)$, then $\mathrm{rank}(e_i)<\alpha$ by our choice of domain as $\mathrm{TC}(x)$ and so $\sigma\in S_{\alpha}$. By induction on $p(y_{n-1})$ we can show that $p(y_{n-1})\leqslant\|\sigma\|_{S_{\alpha}}$. The induction is trivial by the above remark on the consistency-considerations and, hence, the lemma follows. The theorem follows from the lemma.

Remark. The theory T in the proof asserts that x is a relation on a transitive set y; $\langle Y, P\rangle$ is the prewellordering induced by Γ over x and there is no $z\in\Gamma_x^{\infty}$ satisfying φ. If T' is a primitive recursive theory in the language of set theory, then the same proof gives:

Corollary 4.4. *Let* Γ, φ *and* τ_x *be as above. If*

$$\forall x(x \Vdash T' \Rightarrow \exists\gamma<\tau_x\varphi(x, \Gamma_x^{\gamma+1})),$$

then there is an E-recursive function G such that

$$\forall x(x\vDash T' \Rightarrow \exists\gamma<\min\{\tau_x, G(\mathrm{rank}(x))\}\varphi(x, \Gamma_x^{\gamma+1})).$$

Examples of such theories are:

(i) x is transitive, infinite and closed under finite subsets;

(ii) x is rudimentarily closed.

Now if x is transitive, infinite and closed under finite subsets, then we have a notation system for the next admissible ($\mathrm{HYP}(x)$) and that notation system is defined by a monotone inductive definition. If $\exists y\in\mathrm{HYP}(x)\varphi(x, y)$, then there is a Δ_0 formula φ' such that $\varphi'(x, \Gamma_x^{\gamma})$ for the least γ such that $\exists y\in L_{\gamma}[x]\varphi(x, y)$ where Γ defines that notation system.

Using this we have proven the following theorem of J. Van de Wiele:

Corollary 4.5 (Van de Wiele). *Let* $F\colon V\to V$ *be uniformly* Σ_1*-definable and total over all admissible sets, then* F *is* E*-recursive.*

Proof. Follows immediately from the theorem and the above remarks on the inductive generation of $\mathrm{HYP}(x)$.

Note that we actually show that F is computable in a weaker system than E-recursion, since we use elementary functions together with the operator which computes the height of a well-founded relation.

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