terms of the conjugates of $H$. In the case of $A_5$, if $|H| = 12$ then $N_{A_5}(H) = H$, but $A_5$ has no subgroup of index $<5$.

2. A natural place to look for an example showing that these results cannot be extended would be PSL$(2, 7)$, which has two conjugacy classes of subgroups of index $7$. Unfortunately, this group does not yield a counterexample, but I will not inflict my unpleasant calculations upon the reader. For a nice listing of the properties of this group, see [6].

References


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On span and chainable continua

by

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Abstract. In 1964 Lelek [8] defined the notion of the span of a metric continuum and proved that chainable continua have span zero. He asked if the converse is also true, i.e., if continua with span zero are chainable. Recently, (see [10] and [11]) Lelek proved that continua with span zero are arc-like and tree-like. In [13] the authors gave some new characterization of continua with span zero and proved that continua with span zero are continuous images of the pseudo-arc. In this paper we prove that if a hereditarily indecomposable metric continuum has span zero and is an inverse limit of finite graphs with in some sense not too many branch-points or simple closed curves, then $X$ is a pseudo-arc. In particular, it follows that if $X$ is a continuum which is the continuous image of the pseudo-arc and such that all proper subcontinua of $X$ arc pseudo-arcs, then $X$ itself is a pseudo-arc.

1. Introduction. All spaces considered in this paper are metric. A compactum is a compact metric space. A continuum is a connected compactum. We write $f: X \rightarrow Y$ to indicate that $f$ is a mapping of $X$ onto $Y$. We let $I$ denote the closed unit interval and $Q$ the Hilbert cube with a fixed but arbitrary metric $d$. Every continuum is a subspace of $Q$.

If $A \subset X$ and $\varepsilon > 0$ we let $S(A, \varepsilon)$ denote the open $\varepsilon$-ball around $A$ in $X$. We let $Cl(A)$ denote the closure of $A$ in $X$.

If $X$ and $Y$ are continua we let $\pi_1: X \times Y \rightarrow X$ and $\pi_2: X \times Y \rightarrow Y$ denote the first and second coordinate projections, respectively. We let $d(X)$ denote the diagonal in $X \times X$. We define (see [9]) the surjective span of $X$, $\sigma^*(X)$ (resp. the surjective semi-span, $\sigma^s(X)$) to be the least upper bound of all real numbers $\varepsilon$ for which there exists a subcontinuum $Z \subset X \times X$ such that $\pi_1(Z) = X = \pi_2(Z)$ (resp. $\pi_1(Z) = X$) and $d((x, y)) \geq \varepsilon$ for each $(x, y) \in Z$. The span of $X$

$$\sigma(X) = \sup \{\sigma^*(A) | A \text{ is a subcontinuum of } X\}$$

and the semi-span of $X$

$$\sigma^s(X) = \sup \{\sigma^s(A) | A \text{ is a subcontinuum of } X\}.$$ (*) The first author was supported in part by NSF grant number MCS-8104866 and the second author was supported in part by NSERC grant number A5616.
A continuum is tree-like if it can be obtained as an inverse limit of trees, i.e., connected, simply connected, finite graphs. A continuum is chainable if it is an inverse limit of arcs. A continuum is indecomposable if it cannot be written as the union of two of its proper subcontinua. A compactum is hereditarily indecomposable if every subcontinuum is indecomposable. The pseudo-arc is the unique (up to homomorphism) hereditarily indecomposable chainable continuum (see [2]).

If $A$ is a set we let $|A|$ denote the cardinality of $A$. If $X \subset Q$ is a continuum such that $X = \lim_{i \to \infty}(X_i, f_{i+1}^i)$ where the $X_i$ are graphs we may suppose that the spaces $X_i$ are embedded in $Q$ such that their projections $f_i : X \to X_i$ converge to the identity map on $X$.

2. Preliminaries. A cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of a space $X$ is called a chain-cover provided $U_1 \cap U_2 \neq \emptyset \Rightarrow |i-j| \leq 1$. A cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of a space $X$ is said to be near provided $Cl(U_i) \cap Cl(U_j) \neq \emptyset \Rightarrow U_i \cap U_j \neq \emptyset$. If $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a cover of a space $X$ we denote $i(U_1, \mathcal{U}) = U_1 \setminus Cl(U_{i+1}, \mathcal{U})$. The following theorem (see [14, Theorem 3]) will be used:

**Theorem 1.** If $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a taut, open, chain-cover of a hereditarily indecomposable compactum $X$ then there exists a continuum $Z \subset X$ such that $Z \cap i(U_i, \mathcal{U}) \supset \emptyset \Rightarrow Z \cap i(U_i, \mathcal{U}) \supset \emptyset$. If $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a function such that $|f(i)-f(i+1)| \leq 1$, $i = 1, \ldots, n-1$, then there exists an open chain cover $\mathcal{V} = \{V_1, \ldots, V_m\}$ of $X$ such that $\mathcal{V}$ follows pattern $f$ in $\mathcal{U}$ (i.e., $V_i \subset U_{f(i)}$ for $i = 1, \ldots, n$).

**Corollary 2.** Let $g : X \to I$ be an $\varepsilon$-map of a hereditarily indecomposable compactum $X$ onto $I$ such that there exists a subcontinuum $Z \subset X$ such that $g(Z) = \{1, 2\}$. Let $\eta > 0$ be such that $4\eta < \varepsilon$. If $\delta = \min\{\varepsilon/4, \eta\}$ for each $i \in \{1, \ldots, n\}$ and $f : f^{-1}(\{1, 2\}) = \{1, 2\}$ is one-to-one. Let $U_i = g^{-1}(\{1, 2\} \times (x_{i-1} + \delta, x_{i-1} + \delta))$ and $U_i = g^{-1}(\{1, 2\} \times (x_{i-1} + \delta, x_{i-1} + \delta))$ for $i < n$. If $\mathcal{U} = \{U_1, \ldots, U_n\}$, then $\mathcal{U}$ is a taut, open, chain-cover of $X$ of mesh less than $\varepsilon$. By taking the partition $[x_0, x_1, \ldots, x_n]$ fine enough we may assume that the piecewise linear map $f$ determines a function $f : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ such that $f(i) - f(i+1) \leq \delta$ for $i \in \{0, \ldots, m-1\}$, $f^{-1}(0) = 0$, $f^{-1}(m) = m$ where $\{y_0 = y_1 = \ldots = y_m = 1\}$ is a partition of $I$ such that $f$ is linear or constant on each $[x_{i-1} + \delta, x_i]$ and $f(y_i) = x_i$.

By Theorem 1, there exists a taut open chain cover $\mathcal{V}$ of $X$ such that $\mathcal{V}$ follows pattern $f$ in $\mathcal{U}$. Note that $(U_i, \mathcal{U}) \subset V_i \subset U_i$ and $(U_i, \mathcal{U}) \subset V_i \subset U_i$. Define $h : X \to I$ such that $h^{-1}(0) = 0$, $h^{-1}(1) = 1$, $h(V_i \cap V_j) = x_{i}$ if $i < n$ using the Tietze extension theorem such that the conditions in the statement of the corollary are satisfied.

3. Continua with span zero. In [13] the authors characterized surjective semiparametric zero spaces using uniformizations of two sequences of arcs converging onto a continuum $X$. The next lemma is, in some sense, a generalization of these results.

**Lemma 3.** Let $X$ be a continuum in $Q$ with $\sigma(X) > 0$. Let $I$ be a sequence of arcs in $Q$ such that $\lim I = X$. For each $i > 0$ there exists a $\delta > 0$ and an integer $n_i$ such that $n_i \geq n_0$ and $G \subset X \setminus I$ is any Peano continuum, then a component of $\{(x, y) : (x, y) \subset G \times I\}$ separates $G \times \{0\}$ from $G \times \{1\}$ where $0 \leq 1$ denote the endpoints of $I$.

**Proof.** Suppose that for some $i > 0$ there exists a sequence $G_j$ of Peano continua and integers $m_j$ with $m_j < m_{j+1}$, $G_j \subset X \setminus I$ and continua $K_j \subset \{(x, y) : (x, y) \subset G_j \times I\}$ such that $K_j$ meets both $G_j \times \{0\}$ and $G_j \times \{1\}$. Without loss of generality the sequence of continua $K_j$ converges to a continuum $K \subset X \setminus I$. Then $\sigma_2(X) = \lim \sigma_2(G_j) = \lim \sigma_2(K_j) = \sigma(K) \subset X$. Also $\sigma(K, d, X) > \sigma$. This contradicts the assumption that $\sigma_2(X) = 0$. Hence, there exists an integer $n_0$ and $\delta > 0$ such that if $n > n_0$ and $G \subset X \setminus I$ is a Peano continuum then $\{(x, y) : (x, y) \subset G \times I\}$ separates $G \times \{0\}$ from $G \times \{1\}$.

Let $n > n_0$ and let $\sigma G$ be the suspension of $G$.

$$\sigma G = \{x \in [-1, 2] : (x, y) \subset G \times \{1\}, x \in \{2\} \}

Note that $\sigma G$ is a locally connected continuum. By the Mayer-Vietoris theorem $\sigma G$ has trivial first cohomology and, hence, is unicoherent. Also, $\{(x, y) : (x, y) \subset G \times I\}$ separates $G \times \{0\}$ from $G \times \{1\}$ in $\sigma G$ so some component $K$ does. Since $K \subset G \times I \subset G \subset X \setminus I$ is a component of $G \times \{0\}$ and $G \times \{1\}$.

**Note 4.** The restriction to a continuum $S \subset \{(x, y) : (x, y) \subset G \times I\}$ of the second coordinate projection $\pi_2 : G \times I \to \{0\}$ is a 2-sp-map (i.e., $\pi_2^{-1}(1) \subset S \subset \{1\}$ for each $i \in I$).

**Lemma 5.** Let $X \subset Q$ be a continuum with $\sigma_2(X) = 0$. Let $G_1$ and $G_2$ be two
sequences of graphs in $Q$ such that $\text{Lim}G_n = X = \text{Lim}F_n$. Let $\varepsilon > 0$ be given. There exists an integer $n_0$ such that if $n \geq n_0$, then no component $K$ of $\{(x, y) \in G_n \times F_n \mid d(x, y) \geq \varepsilon\}$ has $\pi_1(K) \neq G_n$.

Proof. The proof is similar to the first part of Lemma 3 and is omitted.

Lemma 6. Let $X \subset Q$ be a hereditarily indecomposable continuum with $\pi_0(X) = 0$. Let $I_n$ be a sequence of arcs in $Q$ such that $\text{Lim}I_n = X$ and let $\varepsilon > 0$. There exists a $\delta > 0$ and an integer $n_0$ such that if $G \subset S(X, \delta)$ is a finite connected graph, $n \geq n_0$, $(a, b) \in G \times I_n$ with $d(a, b) < \delta$ and $K$ is the component of $\{(x, y) \in G \times I_n \mid d(x, y) < \varepsilon\}$ which contains $(a, b)$, then no component of $G \times I_n$ meets both $(a) \times I_n$ and $G \times \{b\}$.

Proof. Suppose there exists a sequence $m_1 < m_2 < \ldots$ of positive integers and a sequence $G_n$ of finite connected graphs with $G_n \subset S(X, 1/n)$ such that for each $n$ there exist $(a_n, b_n) \in G_n \times I_n$ with $d(a_n, b_n) < 1/n$ and a continuum

$$K_n = \{(x, y) \in G_n \times I_n \mid d(x, y) < \varepsilon\}$$

such that $K_n$ meets both $(a_n) \times I_n$ and $G_n \times \{b_n\}$. Without loss of generality the sequence $K_n$ converges to a continuum $K \subset X \times I_n$. Then $d(K, X \times I_n) \geq \varepsilon$ and

$$d(x, y) = d(x, y) = \lim d(x_n, y_n) \leq \lim d(a_n, b_n) = 1/n < \varepsilon$$

Hence $\pi_1(K) \cap \pi_1(G_n) \neq \emptyset$. Since $X$ is hereditarily indecomposable either $\pi_1(K) = \pi_1(G_n)$ or $\pi_1(K) \supseteq \pi_1(G_n)$. Thus $\pi_1(K) \supseteq \pi_1(G_n) \supseteq \pi_1(G_n) \supseteq \emptyset$ which is a contradiction.

Thus, there exists $\delta > 0$ and an integer $n_0$ such that if $G \subset S(X, 2\delta)$ is a finite connected graph, $n \geq n_0$, $(a, b) \in G \times I_n$, then no component of $\{(x, y) \in G \times I_n \mid d(x, y) < \varepsilon\}$ meets both $(a) \times I_n$ and $G \times \{b\}$. Hence, if $G$ is the component of $\{(x, y) \in G \times I_n \mid d(x, y) < \varepsilon\}$ containing the point $(a, b)$ separates $(a) \times I_n$ from $G \times \{b\}$ in $G \times I_n$.

Suppose $G \subset S(X, \delta)$ is a finite connected non-acyclic graph. Let $G$ be the universal covering space of $G$ and $\overline{G} \to G$ be the covering projection. We may suppose $G$ is embedded in $S(X, \delta)$ such that $G \cup G$ is a compactification of $G$ by $G$ and the natural extension $\overline{G}$ of $G$ to $G$ is a $\delta$-retraction of $G \cup G$ to $G$.

Let $n \geq n_0$ and let $A$ be an arc in $G \times I_n$ which is irreducible with respect to intersecting $(a) \times I_n$ and $G \times \{b\}$. Then $\pi_2(G \times I_n) = A = A \times \{b\} \cup \ldots$ where the $A_i$ are pairwise disjoint arcs in $G \times I_n$ which map homeomorphically onto $A$ under $\pi_2 | A \times I_n$. Let $\pi_2: G \times I_n \to G$ be the first coordinate projection and let $\pi_2(A_0) = C$. Let $a_0 \in C$ such that $\pi_2(a_0) = a_0 \in G$. Then $\pi_2(a_0, b) < 2\delta$ and if $K$ is the component of $(a_0, b) \in (a) \times I_n \mid d(x, y) < \varepsilon\}$, then the second paragraph of this proof of $K$ separates $(a_0) \times I_n$ from $C \times \{b\}$ in $C \times I_n$ since $C$ is unicollinear. Hence $K \cap A \neq \emptyset$. Clearly

$$(a, b) \in (p \times \text{id}_I)(K) \subset (x, y) \in G \times I_n \mid d(x, y) < \varepsilon$$

and $(p \times \text{id}_I)(K) \cap A \neq \emptyset$.

This completes the proof of the lemma.

Lemma 7. Let $T_1$ and $T_2$ be connected graphs and let $x$ be a point such that $T_1 \cap T_2 = \{x\}$. Let $y \in I$ and let $K_1$ and $K_2$ be graphs with $K_1 \subset T_1 \times I$ such that $K_1 \cap \{(x, y) \mid y \in I\} = \{(x, y) \mid y \in I\}$ and $K_2$ separates $T_1 \times \{0\}$ from $T_1 \times \{1\}$ for each $i = 1, 2$. Then $K_1 \cup K_2$ separates $(T_1 \cup T_2) \times \{0\}$ from $(T_1 \cup T_2) \times \{1\}$ in $(T_1 \cup T_2) \times I$.

Proof. We may suppose that $K_1$ is irreducible with respect to separating $T_1 \times \{0\}$ from $T_1 \times \{1\}$ in $T_1 \times I$ for $i = 1, 2$. If $K_1 \cup K_2$ does not separate $(T_1 \cup T_2) \times \{0\}$ from $(T_1 \cup T_2) \times \{1\}$ in $(T_1 \cup T_2) \times I$, then there exists a polygonal arc $A \subset (T_1 \cup T_2) \times I \setminus (K_1 \cup K_2)$ such that $A$ meets both $(T_1 \cup T_2) \times \{0\}$ and $(T_1 \cup T_2) \times \{1\}$. Since $(K_1 \cup K_2) \cap \{(x, y) \mid y \in I\} = \{(x, y) \mid y \in I\}$, it is easy to see that there exists an arc $B$ such that either $B \subset [A \cup \{(y, y) \mid y \in I\}] \cap (T_1 \times K_1)$, or $B \subset [A \cup \{(y, y) \mid y \in I\}] \cap (T_2 \times K_2)$ and $B$ meets both $(T_1 \cup T_2) \times \{0\}$ and $(T_1 \cup T_2) \times \{1\}$ which is a contradiction.

4. Some span zero type conditions for chainability. Let $X \subset Q$ be a continuum. We give five conditions that $X$ may satisfy:

(i) $X$ is chainable.

(ii) For each sequence of arcs $I_n$ in $Q$ such that $\text{Lim}I_n = X$ there exists an inverse sequence of graphs $(G_n, f_n)$ such that:

(a) $X = \text{Lim}(G_n, f_n)$.

(b) $G_n \subset Q$ such that $\text{Lim}G_n = X$ and the projection $f_n: X \to G_n$ is a $1/n$-map such that $d(x, f_n(x)) < 1/n$.

(c) for each $x \neq 0$ and for each integer $n$ there exists $n \geq n_0$ and a graph $G_n \subset C_n \times I_n \mid d(x, y) < \varepsilon\}$ such that $G_n$ is homeomorphic to $G_n$ and $G_n$ separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$.

(iii) For each sequence $G_n$ of graphs in $Q$ such that $X = \text{Lim}G_n$, $d(x, f_n(x)) < 1/n$ and $X = \text{Lim}(G_n, f_n)$: $X \to G_n$ is a $1/n$-map which exists a sequence of arcs $I_n$ in $Q$ such that

(a) $X = \text{Lim}I_n$.

(b) for each $e > 0$ and for each integer $n$ there exists $n \geq n_0$ and a graph $G_n \subset C_n \times I_n \mid d(x, y) < \varepsilon\}$ such that $G_n$ is homeomorphic to $G_n$ and $G_n$ separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$.

(iv) There exist sequences $I_n$ of arcs and $G_n$ of graphs in $Q$ such that

(a) $X = \text{Lim}I_n = X$.

(b) $X = \text{Lim}(G_n, f_n)$: $X \to G_n$ is a $1/n$-map such that $d(x, f_n(x)) < 1/n$.

(c) for each $x \neq 0$ and for each integer $n$ there exists $n \geq n_0$ and a graph $G_n \subset C_n \times I_n \mid d(x, y) < \varepsilon\}$ such that $G_n$ is homeomorphic to $G_n$ and $G_n$ separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$.

(v) For every pair of sequences $I_n$ of arcs and $G_n$ of graphs in $Q$ such that $X = \text{Lim}I_n = X$ and for each $e > 0$ there exists an integer $n_0$ such that for
each \( n \geq n_0 \) there exists a graph \( K_n \subset C(n, \epsilon) = \{(x, y) \in G_n \times I_n : d(x, y) < \epsilon\} \) such that \( K_n \) separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \) in \( G_n \times I_n \).

**Theorem 8.** If \( X \subset Q \) is a continuum then we have the following relations among these conditions:

\[(i) \iff (ii) \iff (iv) \]

\[ (i) \iff (\sigma(X) \neq 0) \iff (\sigma_0(X) \neq 0) \iff (\sigma_0(X) = 0) \iff (v) \]

Moreover, if \( X \) is hereditarily indecomposable then \( (iv) \implies (i) \).

Proof. (i) \(\iff\) (ii). Suppose \( X \) is chainable and let \( I_0 \) be a sequence of arcs in \( Q \) converging onto \( X \). Since \( X \) is chainable, there exists a sequence of arcs \( G_n \) in \( Q \) such that \( X = \lim_{n \to \infty} (G_n \cup I_0) \), \( X = \lim_{n \to \infty} (G_n \cup I_0) \), \( d(x, f(x)) < 1/n \) and \( f_n : X \to G_n \) is a \( 1/n \)-map. Let \( \epsilon > 0 \) be given. Since \( X \) is chainable, \( \sigma_0(X) = 0 \). Hence by Lemma 3 there exists an integer \( n_0 \) such that for \( n > n_0 \) a component \( K_n \) of

\[ \{(x, y) \in G_n \times I_n : d(x, y) < \epsilon\} \]

separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \). By [6, p. 438] there exists a locally connected continuum \( H \subset K_n \) which separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \). Hence there exists an arc \( M \subset H \) which separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \).

\[ (i) \iff (ii) \iff (iv) \]

(i) \( \iff \) (ii). Let \( G_n \) be a sequence of graphs in \( Q \) such that \( \lim_{n \to \infty} G_n = X \) and \( \lim_{n \to \infty} G_n = X \). Let \( I_n \) be a \( 1/n \)-map and let \( \epsilon > 0 \) be given. Let \( G_{n_0} \) be a nested sequence of open covers of \( X \) in \( Q \) such that \( \mathcal{U}_{n_0} \subset \mathcal{U}_{n_1} \). Let \( G_n \) be the closure of \((G_{n_0}, \mathcal{U}_{n_0})\). Then \( G_n \) is chainable and \( G_n \) separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \).

\[ (i) \iff (\sigma(X) = 0) \iff (\sigma_0(X) = 0) \iff (\sigma_0(X) = 0) \iff (v) \]

\[ (i) \iff (ii) \iff (iv) \]

(iii) \(\iff\) (iv). Trivial.

Suppose \( X \) is hereditarily indecomposable and satisfies (iv). Let \( I_0 \) and \( G_n \) be sequences in \( Q \) which satisfy (iv). We will show that \( X \) is chainable. Let \( \epsilon > 0 \) be given. Without loss of generality we may assume that \( I_0 \) and \( G_n \) are piecewise linear \((n = 1, 2, \ldots)\) in \( Q \). Let \( n \) be an integer so large that the projection \( f : X \to G_n \) moves points less than \( \epsilon \) and there exists a graph \( G^*_n \subset C(n, \epsilon) \) such that \( G^*_n \) separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \) in \( G_n \times I_n \) and such that \( G^*_n \) is homeomorphic to \( G_n \). We may suppose, by compressing \( G^*_n \) slightly in the second coordinate, that \( G^*_n \cap (G_n \times \{0\}) = \emptyset \), since \( G^*_n \subset C(n, \epsilon) \) is open in \( G_n \times I_n \). Let \( \epsilon_1 < \epsilon \) such that \( f_n \) is an \( \epsilon_1 \)-map. Let \( \eta > 0 \) be such that \( \delta \eta > \epsilon - \epsilon_1 \) and the diameter of \( f_n^{-1}(\delta \eta \times \mathbb{R}) \) is less than \( \epsilon \) for each \( x \in G_n \). Let \( G^*_n \) be irreducible with respect to separating \( G_n \times \{0\} \) from \( G_n \times \{1\} \) in \( G_n \times I_n \).

Let \( \{x_1, \ldots, x_k\} \) be the branch-points and endpoints of \( G_n \). It is not difficult to see that if \( x_k \) is a branch-point of \( G_n \), then there exists a \( G^*_n \) at least one branch-point of the form \( (x_k, y) \) for some \( y \in I_n \) since \( G^*_n \) separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \).

In fact the order of \( G_n \) at \((x_k, y)\) is at least as great as the order of \( G_n \) at \( x_k \). Since \( G^*_n \subset G_n \), it follows that \( G^*_n \) has exactly one branch point in \( (x_k, y) \) for some \( y \in I_n \) since \( G^*_n \) separates \( G_n \times \{0\} \) from \( G_n \times \{1\} \).

Let \( U \) be a connected neighbourhood of \( x_k \) in \( G_n \) such that \( G_n \subset U \) contains only one vertex of the graph \( G_n \). If \( (x_k, z) \in G^*_n \) such that the component \( C \) of \((x_k, z) \) in \( G^*_n \) does not contain \((x_k, y)\), then there exists an arc \( A \subset U \) having \( x_k \) as an endpoint such that some neighborhood of \( C \) in \( G^*_n \) is contained in \( A \times L \).

By adjusting \( G^*_n \) slightly in the open set \( C(n, \epsilon) \) we may suppose \( G^*_n \cap (x_k \times I_n) \) is connected for each branch-point \( x_k \) of \( G_n \). By a further small adjustment of \( G^*_n \) we may assume \( G^*_n \cap (x_k \times I_n) \) is connected for each branch-point \( x_k \) of \( G_n \) and that \( G^*_n \) is piecewise linear in \( G_n \times I_n \).

It follows that \( G_n \) is homeomorphic to \( G^*_n \) under a homeomorphism which takes \( x_k \) to \((x_k, y)\) for each \( k \in \{1, \ldots, m\} \).

If \( G^*_n \) does not contain any branch-point, then \( G_n \) is either an arc or a circle. The proof is complete if \( G_n \) is an arc. If \( G_n \) is a circle, let \( (G_n, p) \) be the universal covering space of \( G_n \). Since \( \sigma_0(X) = 0 \), \( X \) is tree-like. Hence the projection \( f_n : X \to G_n \) has a lifting \( \tilde{f}_n : X \to G^*_n \) such that \( f_n = p \circ \tilde{f}_n \).

It follows that \( \tilde{f}_n \) is an \( \epsilon \)-map of \( X \) onto an arc. Hence we may assume that \( G_n \) contains at least one branch-point.

Let \( A \) be an arc in \( G_n \) with end points \( x_0 \) and \( x_1 \) such that \( A \) contains no other points of \( \{x_0, \ldots, x_k\} \). Then \( \pi_1(A \cap G_n) \subset G_n \) is an arc \( A' \). By Corollary 2 there exists an \( \epsilon \)-map \( g : f_n^{-1}(A') \to A' \) such that

\[ g^{-1}(\delta \eta \times \mathbb{R}) = f_n^{-1}(\delta \eta \times \mathbb{R}) \quad \text{and} \quad d(f_n(x), \pi_1 \circ g(x)) < \eta \]

or each \( x \in f_n^{-1}(A') \).

If \( C \) is a simple closed curve in \( G_n \) such that \( Bd(C) \subset G_n \) at most a single point \( x \), then \( \pi_1(C \cap G_n) \) is a simple closed curve \( C' \). Choose an arc \( K \subset C \cap G_n \) with end point \( x \) and \( b \) such that \( diam(f_n^{-1}(C \cap G_n)) < \epsilon_1 \) and \( (x_k \times I_n) \setminus G_n \) is a singleton for each \( x \in C \cap G_n \). Since \( \pi_1(C \cap G_n) \subset G_n \) is an \( \epsilon \)-map \( C' \subset G_n \) there exists an \( \epsilon \)-map \( g : f_n^{-1}(C') \to C' \) such that for each \( x \in C \cap G_n \)

\[ f_n^{-1}(x) = g^{-1}(\delta \eta \times \mathbb{R}) \subset G_n \quad \text{and} \quad d(f_n(x), \pi_1 \circ g(x)) < \eta \]

for each \( x \in f_n^{-1}(C') \).

It follows that there exists an \( \epsilon \)-map \( h : X \to G_n \) such that \( d(f_n(x), \pi_1 \circ h(x)) < \eta \) for \( x \in X \).
We claim that $p_2 \circ h : X \to I$ is a $4\varepsilon$-map. To see this let $x, y \in (p_2 \circ h)^{-1}(f)$. By Note 4, $p_2 \circ G^*_2$ is a $2\varepsilon$-map, it follows $d(h(x), h(y)) < 2\varepsilon$ and hence

$$d(x, p_1 h(x), p_1 h(y)) < 2\varepsilon .$$

Also

$$d(x, p_1 h(x)) = d(f_1(x), p_1 h(x)) < \varepsilon + \eta < \varepsilon \quad \text{for each } x \in X .$$

Hence

$$d(x, y) \leq d(x, p_1 h(x)) + d(p_1 h(x), p_1 h(y)) + d(p_1 h(y), y) < 4\varepsilon .$$

Since $\varepsilon$ was arbitrary $X$ is chainable.

**Problem 9.** Suppose $X$ is a hereditarily indecomposable continuum such that $q(X) = 0$. Does $X$ satisfy condition (iv) and, as a consequence, is $X$ chainable?

**S. Applications.** In this section we will give some partial solutions to Problem 9. It is known ([10] and [11]) that continua $X$ with $q(X) = 0$ are atrioc and tree-like. The reason for allowing graphs (rather than trees) in the inverse limit description of $X$ in the following theorems is that it makes this easier to satisfy the condition concerning the number of branch-points on arcs in $G_1$ (cf. the proof of Corollary 14).

**Theorem 10.** Let $X = \lim (G_n, f_n^*)$ be a hereditarily indecomposable continuum in $Q$ with $q(X) = 0$ where each $G_n$ is a graph with the property that each single closed curve in $G_n$ has at most one point in its boundary in $G_n$. Suppose also there exists an integer $N$ such that for each integer $n$ and each arc $A \subset G_n$, $A$ contains at most $N$ branch points of $G_n$. Then $X$ is a pseudo-arc.

**Proof.** We may suppose by the remark at the end of Section 1 that the graphs $G_n$ are embedded in $Q$ such that $X = \lim G_n$ and $f_n^*$ moves no point of $G_n$ more than $1/n$ for each $m \geq n$.

Let $I_j$ be a sequence of arcs in $Q$ such that $\lim I_n = X$.

For each $d > 0$ and each positive integer $n$ let

$$C(n, d) = \{ (x, y) \in G_n \times I_d \mid d(x, y) < \delta \} .$$

Let $e > 0$ be given. By Lemma 6, let $e_0 > e_1 > \ldots > e_{m_0} > 0$ and let $m_0 < m_1 < \ldots < m_n$ be integers such that if $(x, y) \in C(n, e_{m_i})$ then $(x, y) \in C(n, e_{m_i+1})$. By Lemma 8 we may suppose there is a component $K(n, e_0)$ of $C(n, e_0)$ such that $K(n, e_0)$ separates $G_n \times [0]$ from $G_n \times [1]$ in $G_n \times I_n$ for $n \geq m_0$. By Lemma 8 we may suppose no component $L$ of $G_n \times I_n$ which contains $x \times I_n$ for $n \geq m_0$. If $m > m_0$ and $G_n$ has no branch-points, then $G_n$ is an arc or a simple closed curve.

In either case $K(n, e_0)$ contains a continuum homeomorphic to $G_n$ which separates $G_n \times [0]$ from $G_n \times [1]$.

Now, suppose $m \geq m_1$ and $x_1$ is a branch-point of $G_n$. Let $(x_1, t_1) \in C(m, e_0)$ and let $H_1, \ldots, H_n$ be the closures of the components of $G_n \times \{x_1\}$. After reindexing if necessary we may suppose $H_1, \ldots, H_n$ are not arcs or simple closed curves and each of $H_{n+1}, \ldots, H_m$ is an arc or a simple closed curve. Let $K(m, e_{m-1})$ be the component of $C(m, e_{m-1})$ which contains $(x_1, t_1)$. By the choice of $m$ and $e_{m-1}$ no component of $G_n \times I_n \setminus K(m, e_{m-1})$ meets both $\{x_1\} \times I_n$ and $G_n \times \{t_1\}$. We will show that $K(m, e_{m-1})$ separates $G_n \times [0]$ from $G_n \times [1]$ in $G_n \times I_n$.

If $K(m, e_{m-1}) \cap K(m, e_{m-1}) \neq \emptyset$, then $K(m, e_{m-1}) \subset K(m, e_{m-1})$ and there is nothing to prove. Hence suppose $K(m, e_{m-1}) \cap K(m, e_{m-1}) \neq \emptyset$. Then $\pi_1(K(m, e_{m-1})) = G_n$ since

$$\pi_1(K(m, e_{m-1})) \neq G_n ,$$

and this contradicts the fact that $K(m, e_{m-1})$ separates $G_n \times [0]$ from $G_n \times [1]$ in $G_n \times I_n$. We may suppose without loss of generality that $K(m, e_{m-1})$ separates $G_n \times [0]$ from $K(m, e_{m-1})$.

Let $\varphi : G_n \times I_n \to Y = G_n \times I_n / G_n \times [0]$ denote the natural projection, then $Y$ is a locally connected unicoherent continuum. Now $G_n \times I_n \setminus K(m, e_{m-1})$ separates $K(m, e_{m-1})$ from $K(m, e_{m-1})$ in $G_n \times I_n$. Hence $\varphi(K(m, e_{m-1}), G_n \times [0])$ separates $\varphi(K(m, e_{m-1}), G_n \times [0])$ from $\varphi(K(m, e_{m-1}), G_n \times [0])$ in $Y$. Hence a component $L$ of $\varphi(K(m, e_{m-1}), G_n \times [0])$ contains these sets. Then $L \cap \varphi(K(m, e_{m-1})) \neq \emptyset$ and $\varphi^{-1}(L)$ is a component of $G_n \times I_n \setminus K(m, e_{m-1})$ such that $\varphi(L) = G_n$. This contradicts the choice of $m_1$. We have proved that $K(m, e_{m-1})$ separates $G_n \times [0]$ from $G_n \times [1]$ in $G_n \times I_n$.

Let $M_n$ be a graph in the open set $K(m, e_{m-1})$ which is minimal with respect to separating $G_n \times [0]$ from $G_n \times [1]$ and such that $\{x_1\} \times I_n \cup M_n = (x_1, t_1)$. For each $i$, $t_1 \leq t_1$, $(H_i \times I_n) \cap M_n$ is homeomorphic to $H_i$.

For each $i$, $1 \leq i \leq k$, let $s_{i}$ be the unique branch-point of $H_i$ which separates every other branch-point of $H_i$ from $s_{i}$. Let $M_n$ be the closure of the component of $M_n \setminus \{s_{i} : 1 \leq i \leq k\} \times I_n$ which contains $(x_1, t_1)$. Let $M_n$ be a minimal subcontinuum of $M_n$ which contains $m_1 \in \{H_i : 1 \leq i \leq k\}$ and meets $\{x_1\} \times I_n$ for each $1 \leq i \leq k$. For each $i = 1, \ldots, k_1$ let $(s_{i}, t_{i}) \in M_n$. Notice that if $i \in \{1, \ldots, k\}$ and $s_{i}$ is the arc in $G_n$ with endpoints $x_{i}$ and $x_{j}$, then $A \times I_n \cap M_n$ is an arc. Also $M_n \cap (B_i \times I_n)$ is homeomorphic to $B_i$ where $B_i$ is the subcontinuum of $G_n$ which is minimal with respect to containing $\{x_{1}, \ldots, x_{m}\} \cup H_{s_{1}+1} \cup \ldots \cup H_{s_{n}}$.

For $i = 1, \ldots, k_1$ let $H_{s_{i}} \cup H_{s_{i+1}}$ be the closures of the components of $G_n \times \{x_{i}\}$ which do not contain $s_{i}$. After reindexing if necessary we may suppose that $H_{s_{i}} \cup H_{s_{i+1}}$ are not arcs or simple closed curves and $H_{s_{i}+1}, \ldots, H_{s_{n}}$ are arcs or simple closed curves. Let $K(m, e_{m-1})$ be the component of $C(m, e_{m-1})$ which contains $(s_{i}, t_{j})$. As above, $K(m, e_{m-1})$ separates $G_n \times [0]$ from $G_n \times [1]$ in $G_n \times I_n$.

Let $s_{i}$ be a graph in $K(m, e_{m-1}) \cap (\bigcup_{i=1}^{k_1} H_{i})$ which is minimal with...
The natural projection $\xi_n : K_n \to G_n$ moves points less than $n$, i.e., $\xi_n = \pi_1 \circ \delta_n$, where $\pi_1 : K_n \to K_1$ is a homeomorphism and $\pi_1 : G_n \times I_n \to G_1$ is the usual projection. Let $B_n$ be the maximal arc in $K_n$ such that $\xi_n(B_n) = A_n$. For each sufficiently small $n$, there exists an arc $L_n \subset \{(x,y) \in B_n \times I_n : (x,y) < \varepsilon_n + 1\}$ such that $L_n$ projects homeomorphically onto $B_n$ by $\pi_1 : K_n \times I_n \to K_1$.

We will show that there exists a $\frac{1}{2}\varepsilon_{n+2}$-map $\phi : X \to K_n$. Choose a taut open chain cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of $G_n$ such that:

1. Each $U_t$ is connected.
2. For each vertex $(v, t)$ of the piecewise linear arc $K_n$ there exists exactly one element $U_t \in \mathcal{U}$ such that $v \in U_t$.
3. For each element $U_t \in \mathcal{U}$ there is at most one point $v \in U_t$ such that $(v, t)$ is a vertex of $K_n$ for some $t \in I_n$.
4. For each $t = 1, \ldots, n$ there exists exactly one element $U_t \in \mathcal{U}$ such that $H_t \cap C(U_t) \neq \emptyset$.

Let $\mathcal{W} = \{W_1, \ldots, W_n\}$ be a taut open chain cover of $K_n$ such that $W_t$ is connected, $C(W_t)$ contains at most one vertex of $K_n$, and $\{\xi_n(W_t)\}_{t=1}^n$ refines $\mathcal{W}$. The map $\pi_1(K_n : K_1 \to A_n$ induces a (partial) map $\pi_1 : W_t \to \{v \in U_t : (v, t) \in X \}$ such that $\pi_1^{-1}(v) = \{t \in I_n : (v, t) \in X\}$ for $t = 1, \ldots, n$. By [14, Theorem 3], there exists a taut open chain cover $\mathcal{W} = \{W_1, \ldots, W_n\}$ of $X$ such that $X \subset U_t^{(1)} = \{v \in U_t : (v, t) \in X\}$ for $t = 1, \ldots, n$. It is now not difficult (cf. the proof of Corollary 2) to construct a $\frac{1}{2}\varepsilon_{n+2}$-map $\phi : X \to K_n$ (this map is not necessarily onto) such that $\phi(x, \sigma(x)) < \varepsilon_{n+2}$.

For each branch-point $x \in B_n$ of $K_n$, let $(x, y)_{x}$ be the unique point on $J_n$ and let $C_0$ be the closure of the union of the components of $K_n \times \{x\}$ disjoint from $B_n$. As in the proof of Theorem 10 (recall $d((x, y), (x, y)) < \varepsilon_{n+2}$), there exists a graph $M_n = \{(a, b) \in G_n \times I_n : d((a, b), (x, y)) < \varepsilon_{n+2}\}$ such that $M_n$ is homeomorphic to $C_0$, $M_n \cap (J_n \times L) \subset C_0 \times I_n$ and $M_n$ separates $C_0 \times \{0\}$ from $C_0 \times \{1\}$.

By Lemma 7, $K_n \cup \{M_n \} \times B_n$ is a branch point of $K_n$ is a graph in $K_n \times I_n$, which separates $K_n \times \{0\}$ from $K_n \times \{1\}$, is homeomorphic to $K_n$, and $d((a, b), e)$ for each $(a, b) \in K_n$. The theorem now follows by Theorem 8.

Theorem 12. Let $X$ be a hereditarily indecomposable continuum in $Q$ such that $\sigma(X) = 0$. Suppose $X = \lim G_n$ where each $G_n$ is a graph such that:

1. If $C$ is a simple closed curve in $G_n$ then $C$ has at most one boundary point in $G_n$.
2. There exists an integer $N$ and a sequence of arcs $A_n \subset G_n$ such that if $B \subset G_n$ and $A_n$ is an arc, then $B$ contains at most $N$ ramification points of $G_n$.

Then $X$ is a pseudo-arc.

Proof. Let $\varepsilon > 0$ be given. Let $I_n$ be a sequence of arcs in $Q$ such that $\lim I_n = X$. We may assume that the graphs $G_n$ are piecewise linearly embedded in $Q$ such that $\lim G_n = X$, $f_n : X \to G_n$, and $f_n : X \to G_n$ moves no point more than $1/n$. By Lemmas 3 and 6 choose $\varepsilon_n > \varepsilon_{n+1} > \cdots > \varepsilon_{n+n} > 0$ integers $m_n < \varepsilon_{n+n}$ such that if $n > m_n$, $G_n \times \{0\} \subset \{x \in Q : d((x, y), (x, y)) < \varepsilon_{n+n}\}$ is a connected graph and $(x, y) \in G_n \times I_n$ such that $d((x, y), (x, y)) < \varepsilon_{n+n}$, then a component $C_n(x, y) = \{(a, b) \in G_n \times I_n : d((a, b), (x, y)) < \varepsilon_{n+n}\}$ separates $(x, y) \times I_n$ from $G_n \times \{0\}$ and $G_n \times \{1\}$.

Let $n > m_n$, $A_n \subset G_n \times \{(x, y) \in Q : d((x, y), (x, y)) < \varepsilon_{n+n}\}$. Let $K_0$ be a piecewise linear arc in $\{x \in A_n \times I_n : d((x, y), (x, y)) < \varepsilon_{n+n}\}$ which is irreducible with respect to $A_n \times \{0\}$ from $A_n \times \{1\}$ in $A_n \times I_n$. We may suppose that the natural projection $f_n : X \to G_n$ moves each point of $X$ a distance less than $\varepsilon_{n+n}$. Let $\{x_1, x_2, \ldots, x_i\}$ be all the branch-points of $G_n$ on $A_n$.

We may assume that $K_0 \cap \{(x, y) \in L) \subset \{x \in \lim G_n \}$ is a finite set for $i = 1, \ldots, e$. Let $H_i$ be the closure of the union of the components of $G_n \times \{x_i\}$ which are disjoint from $A_n$. Then $H_i \cap H_j = \emptyset$ if $i \neq j$. Let $(x_{i_1}, t_{j_1}), \ldots, (x_{i_k}, t_{j_k}) = \{(x, y) \in \lim G_n \times \{x_i\} \cap K_0\}$ and define $K_n = K_0 \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{e} (H_i \times \{t_{j_i}\}) \subset G_n \times I_n$.

Notice that $\xi_n$, restricted to every component of $K_n \times K_n$, is a homeomorphism.

For each $n > 0$, let $K_n$ be a homeomorphic copy of $K_n$ embedded in $Q$ such that
Proof. The case $M = 1$ was done in Theorem 11. The proof is similar to that of Theorem 11 and is omitted.

Corollary 13. Let $X$ be a hereditarily indecomposable continuum with $\sigma(X) = 0$. If there exist an integer $N$ and a sequence $T_n$ of trees with at most $N$ branch-points such that $X = \lim_{n \to \infty} (T_n, \sigma_n)$, then $X$ is a pseudo-arc.

Proof. This follows immediately from Theorem 10.

Corollary 14. If $X$ is a continuous image of the pseudo-arc such that every proper subcontinuum is a pseudo-arc, then $X$ is a pseudo-arc.

Proof. It follows from the Boundary Bumping Theorem [6], p. 172 that $X$ is indecomposable and hence hereditarily indecomposable. By [13], Theorem 15, $\sigma(X) = 0$. Let $x \in X$ and let $\varepsilon > 0$ be given. Let $U$ be an open neighbourhood of $x$ of diameter less than $\varepsilon$. If $C$ is any component of $X \setminus U$, then $C$ is either a point or a pseudo-arc. Hence there exists an open chain cover $\mathcal{C}$ of $C$ such that $\bigcup \mathcal{C}$ is open and closed in $X \setminus U$. Since $X \setminus U$ is compact there exists an integer $n$ and $C_1, \ldots, C_n$ components of $X \setminus U$ such that $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$ cover $X \setminus U$. Let $\mathcal{C}_i = \{ W \in \mathcal{C}_i | W \cap U = \emptyset \}$ and for $i \in \{2, \ldots, n\}$ let $\mathcal{C}_{i-1} = \{ V \setminus \bigcup_{j \neq i} W \in \mathcal{C}_i | V \setminus U = \emptyset \}$. Then $(\bigcup \mathcal{C}_1) \cap (\bigcup \mathcal{C}_j) = \emptyset$ for $i \neq j$.

Let $\mathcal{C} = \text{St}(U, \{ U \} \cup \mathcal{C}_1 \cup \ldots \cup \mathcal{C}_n) \cup \mathcal{C}_{1-1} \cup \ldots \cup \mathcal{C}_{n-1}$. Then $\mathcal{C}$ is an open cover of $X$ of mesh less than $3\varepsilon$ such that the nerve of $\mathcal{C}$ has at most one branch-point. The corollary now follows from Theorem 10.

A continuum $X$ is said to be almost chainable if for every $\varepsilon > 0$ there exists an open cover $\mathcal{C}$ of $X$ such that mesh$\mathcal{C} < \varepsilon$ and a chain $\mathcal{G} = \{ C_1, \ldots, C_n \}$ in $\mathcal{C}$ with $X = S(\mathcal{C}, \varepsilon)$ and $\text{Cl}(C_1) \cap \text{Cl}(C_2) \neq \emptyset$ if and only if $|i-j| \leq 1$. Then $\text{Cl}(C_1) \cup \ldots \cup \text{Cl}(C_n) = C_1 \cup \ldots \cup C_{n-1} \cup \text{Cl}(C_n)$.

Corollary 15 (Lewis [11]). If $X$ is an almost chainable homogeneous continuum, then $X$ is a pseudo-arc.

Proof. By [3], all proper subcontinua of $X$ are pseudo-arcs. By the proof of [12, I, Theorem 3.6], $\sigma(X) = 0$ and hence by [13] $X$ is the continuous image of the pseudo-arc. The result follows from Corollary 14.

Problem 16. Suppose $X$ is a homogeneous hereditarily indecomposable continuum such that $\sigma(X) = 0$. Does $X$ satisfy the conditions of Theorem 127?

References


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