terms of the conjugates of H. In the case of A_5, if $|H| = 12$ then $N_{A_5}(H) = H$, but A_5 has no subgroup of index <5.

2. A natural place to look for an example showing that these results cannot be extended would be PSL(2, 7), which has two conjugacy classes of subgroups of index 7. Unfortunately, this group does not yield a counterexample, but I will not inflict my unpleasant calculations upon the reader. For a nice listing of the properties of this group, see [6].

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On span and chainable continua

by

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Abstract. In 1964 Lelek [8] defined the notion of the span of a metric continuum and proved that chainable continua have span zero. He asked if the converse is also true, i.e., if continua with span zero are chainable. Recently, (see [10] and [13]) Lelek proved that continua with span zero are atriodic and tree-like. In [13] the authors gave some new characterization of continua with span zero and proved that continua with span zero are continuous images of the pseudo-arc. In this paper we prove that if a hereditarily indecomposable metric continuum has span zero and is an inverse limit of finite graphs with in some sense not too many branch-points or simple closed curves, then X is a pseudo-arc. In particular, it follows that if X is a continuum which is the continuous image of the pseudo-arc and such that all proper subcontinua of X are pseudo-arcs, then X itself is a pseudo-arc.

1. Introduction. All spaces considered in this paper are metric. A *compactum* is a compact metric space. A *continuum* is a connected compactum. We write $f\colon X \twoheadrightarrow Y$ to indicate that f is a mapping of X onto Y. We let I denote the closed unit interval and Q the Hilbert cube with a fixed but arbitrary metric d. Every continuum is a subspace of Q.

If $A \subset X$ and $\varepsilon > 0$ we let $S(A, \varepsilon)$ denote the open ε-ball around A in X. We let $\mathrm{Cl}(A)$ denote the closure of A in X.

If X and Y are continua we let $\pi_1\colon X\times Y \to X$ and $\pi_2\colon X\times Y \to Y$ denote the first and second coordinate projections, respectively. We let ΔX denote the diagonal in $X\times X$. We define (see [9]) the *surjective span* of X, $\sigma^*(X)$, (resp. the *surjective semi-span*, $\sigma_0^*(X)$) to be the least upper bound of all real numbers ε for which there exists a subcontinuum $Z \subset X\times X$ such that $\pi_1(Z) = X = \pi_2(Z)$ (resp. $\pi_1(Z) = X$) and $d(x, y) \geqslant \varepsilon$ for each $(x, y) \in Z$. The *span* of X

$$\sigma(X) = \sup\{\sigma^*(A) \mid A \text{ is a subcontinuum of } X\}$$

and the *semi-span* of X

$$\sigma_0(X) = \sup\{\sigma_0^*(A) \mid A \text{ is a subcontinuum of } X\}.$$

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A continuum is *tree-like* if it can be obtained as an inverse limit of trees, i.e., connected, simply connected, finite graphs. A continuum is *chainable* if it is an inverse limit of arcs. A continuum is *indecomposable* provided it cannot be written as the union of two of its proper subcontinua. A compactum is *hereditarily indecomposable* provided every subcontinuum is indecomposable. The *pseudo-arc* is the unique (up to homeomorphism) hereditarily indecomposable chainable continuum (see [2]).

If A is a set we let $|A|$ denote the cardinality of A. If $X\subset Q$ is a continuum such that $X=\underleftarrow{\lim}(X_n,f_n^m)$ where the X_n are graphs we may suppose that the spaces X_n are embedded in Q such that their projections $f_n\colon X\to X_n$ converge to the identity map on X.

2. Preliminaries. A cover $\mathcal{U}=\{U_1,\ldots,U_n\}$ of a space X is called a *chain-cover* provided $U_i\cap U_j\neq\emptyset \Leftrightarrow |i-j|\leqslant 1$. A cover $\mathcal{U}=\{U_1,\ldots,U_n\}$ of a space X is said to be *taut* provided $\mathrm{Cl}(U_i)\cap\mathrm{Cl}(U_j)\neq\emptyset \Rightarrow U_i\cap U_j\neq\emptyset$. If

$$\mathcal{U}=\{U_1,\ldots,U_n\}$$

is a cover of a space X we denote $i(U_j,\mathcal{U})=U_j\setminus\mathrm{Cl}(\bigcup_{i\neq j}\{U_i\in\mathcal{U}\})$. The following theorem (see [14, Theorem 3]) will be used:

THEOREM 1. *Let $\mathcal{U}=\{U_1,\ldots,U_n\}$ be a taut, open, chain-cover of a hereditarily indecomposable compactum X such that there exists a continuum $Z\subset X$ such that $Z\cap i(U_1,\mathcal{U})\neq\emptyset\neq Z\cap i(U_n,\mathcal{U})$. Let $f\colon\{1,\ldots,m\}\twoheadrightarrow\{1,\ldots,n\}$ be a function such that $|f(i)-f(i+1)|\leqslant 1$, $i=1,\ldots,m-1$. Then there exists a taut open chain cover $\mathcal{V}=\{V_1,\ldots,V_m\}$ of X such that $\mathcal{V}$ follows pattern f in $\mathcal{U}$* (i.e., *$V_i\subset U_{f(i)}$ for $i=1,\ldots,m$*).

COROLLARY 2. *Let $g\colon X\twoheadrightarrow I$ be an ε-map of a hereditarily indecomposable compactum X onto I such that there exists a subcontinuum $Z\subset X$ such that $g|Z\colon Z\twoheadrightarrow I$. Let $\eta>0$ be such that $4\eta<\varepsilon-\varepsilon_1$ where $\varepsilon_1=\max(\mathrm{diam}\{g^{-1}(t)|\ t\in I\})$. Let $f\colon I\twoheadrightarrow I$ be a piecewise linear map such that $f^{-1}(0)=0$ and $f^{-1}(1)=1$. Then there exists an ε-map $h\colon X\twoheadrightarrow I$ such that $h^{-1}(0)=g^{-1}(0)$, $h^{-1}(1)=g^{-1}(1)$ and*

$$d\big(g(x),f\circ h(x)\big)<\eta$$

for each $x\in X$.

Proof. Let $\{0=x_0<x_1<\ldots<x_n=1\}$ be a partition of I of mesh less than $\frac{1}{3}\eta$ such that $\mathrm{diam}\big(g^{-1}([x_{i-2},x_{i+1}])\big)<\varepsilon_1+\eta$ for each $i\in\{2,\ldots,m-1\}$ and $f|f^{-1}([0,x_2])$ and $f|f^{-1}([x_{m-2},1])$ are one-to-one.

Let

$$U_1=g^{-1}([0,\tfrac{3}{4}x_1+\tfrac{1}{4}x_2)),\qquad U_n=g^{-1}((\tfrac{3}{4}x_{n-1}+\tfrac{1}{4}x_{n-2},1])$$

and

$$U_i'=g^{-1}((\tfrac{3}{4}x_{i-1}+\tfrac{1}{4}x_{i-2},\tfrac{3}{4}x_i+\tfrac{1}{4}x_{i+1}))\quad\text{for}\quad 1<i<n.$$

Let $\mathcal{U}=\{U_1,\ldots,U_n\}$, then $\mathcal{U}$ is a taut, open, chain-cover of X of mesh less than ε. By taking the partition $\{x_0<x_1<\ldots<x_n\}$ fine enough we may assume that

the piecewise linear map f determines a function $\bar{f}\colon\{0,1,\ldots,m\}\twoheadrightarrow\{0,1,\ldots,n\}$ such that $|\bar{f}(i)-\bar{f}(i+1)|\leqslant 1$ for $i\in\{0,\ldots,m-1\}$, $\bar{f}^{-1}(0)=0$, $\bar{f}^{-1}(n)=m$ where $\{0=y_0<y_1<\ldots<y_m=1\}$ is a partition of I such that f is linear or constant on each $[y_{i-1},y_i]$ and $f(y_i)=x_{\bar{f}(i)}$.

By Theorem 1, there exists a taut open chain cover $\mathcal{V}$ of X such that $\mathcal{V}$ follows pattern $\bar{f}$ in $\mathcal{U}$. Note that $i(U_1,\mathcal{U})\subset V_1\subset U_1$ and $i(U_n,\mathcal{U})\subset V_m\subset U_n$. Define $h\colon X\twoheadrightarrow I$ such that $h^{-1}(0)=g^{-1}(0)$, $h^{-1}(1)=g^{-1}(1)$, $h(V_i\cap V_{i+1})=x_{\bar{f}(i)}$ if $i<n$ using the Tietze extension theorem such that the conditions in the statement of the corollary are satisfied.

3. Continua with span zero. In [13] the authors characterized surjective semi-span zero using uniformizations of two sequences of arcs converging onto a continuum X. The next lemma is, in some sense, a generalization of these results.

LEMMA 3. *Let X be a continuum in Q with $\sigma_0^*(X)=0$. Let $I_n)$ be a sequence of arcs in Q such that $\mathrm{Lim}\,I_n=X$. For each $\varepsilon>0$ there exists a $\delta>0$ and an integer n_0 such that if $n\geqslant n_0$ and $G\subset S(X,\delta)$ is any Peano continuum, then a component of $\{(x,y)\in G\times I_n|\ d(x,y)<\varepsilon\}$ separates $G\times\{0\}$ from $G\times\{1\}$ where 0 and 1 denote the endpoints of I_n.*

Proof. Suppose that for some $\varepsilon>0$ there exists a sequence $G_n)$ of Peano continua and integers $m_n)$ with $m_n<m_{n+1}$, $G_n\subset S(X,1/n)$ and continua

$$K_n\subset\{(x,y)\in G_n\times I_{m_n}|\ d(x,y)\geqslant\varepsilon\}$$

such that K_n meets both $G_n\times\{0\}$ and $G_n\times\{1\}$. Without loss of generality the sequence of continua K_n converges to a continuum $K\subset X\times X$. Then $\pi_2(K)=\mathrm{Lim}\,\pi_2(K_n)=\mathrm{Lim}\,I_{m_n}=X$. Also $d(K,\Delta X)\geqslant\varepsilon$. This contradicts the assumption that $\sigma_0^*(X)=0$. Hence, there exists an integer n_0 and $\delta>0$ such that if $n\geqslant n_0$ and $G\subset S(X,\delta)$ is a Peano continuum then $\{(x,y)\in G\times I_n|\ d(x,y)<\varepsilon\}$ separates $G\times\{0\}$ from $G\times\{1\}$.

Let $n\geqslant n_0$ and let sG be the suspension of G

$$sG=G\times[-1,2]/\{G\times\{-1\},G\times\{2\}\}.$$

Note that sG is a locally connected continuum. By the Mayer–Vietoris theorem sG has trivial fiıst cohomology and, hence, is unicoherent. Also,

$$\{(x,y)\in G\times I_n|\ d(x,y)<\varepsilon\}$$

separates $G\times\{-\frac{1}{2}\}$ from $G\times\{\frac{3}{2}\}$ in sG so some component K does. Since $K\subset G\times I_n\subset G\times[0,1]\subset sG$, no component of $G\times I_n\setminus K$ meets both $G\times\{0\}$ and $G\times\{1\}$.

Note 4. The restriction to a continuum $S\subset\{(x,y)\in G\times I_n|\ d(x,y)<\varepsilon\}$ of the second coordinate projection $\pi_2\colon G\times I_n\twoheadrightarrow I_n$ is a 2ε-map (i.e., $\mathrm{diam}(\pi_2^{-1}(t)\cap S)<2\varepsilon$ for each $t\in I_n$).

LEMMA 5. *Let $X\subset Q$ be a continuum with $\sigma_0^*(X)=0$. Let $G_n)$ and $F_n)$ be two* *sequences of graphs in* Q *such that* $\operatorname{Lim} G_n = X = \operatorname{Lim} F_n$. *Let* $\varepsilon>0$ *be given. There exists an integer* n_0 *such that if* $n\geqslant n_0$, *then no component* K *of*

$$\{(x, y)\in G_n\times F_n|\ d(x, y)\geqslant\varepsilon\}$$

has $\pi_2(K) = G_n$.

Proof. The proof is similar to the first part of Lemma 3 and is omitted.

LEMMA 6. *Let* $X\subset Q$ *be a hereditarily indecomposable continuum with* $\sigma_0(X) = 0$. *Let* $I_n)$ *be a sequence of arcs in* Q *such that* $\operatorname{Lim} I_n = X$ *and let* $\varepsilon>0$. *There exists a* $\delta>0$ *and an integer* n_0 *such that if* $G\subset S(X, \delta)$ *is a finite connected graph,* $n\geqslant n_0$, $(a, b)\in G\times I_n$ *with* $d(a, b)<\delta$ *and* L *is the component of* $\{(x, y)\in G\times I_n|\ d(x, y)<\varepsilon\}$ *which contains* (a, b), *then no component of* $G\times I_n\setminus L$ *meets both* $\{a\}\times I_n$ *and* $G\times\{b\}$.

Proof. Suppose there exists a sequence $m_1<m_2<\ldots$ of positive integers and a sequence $G_n)$ of finite connected graphs with $G_n\subset S(X, 1/n)$ such that for each n there exist $(a_n, b_n)\in G_n\times I_{m_n}$ with $d(a_n, b_n)<1/n$ and a continuum

$$K_n\subset\{(x, y)\in G_n\times I_{m_n}|\ d(x, y)\geqslant\varepsilon\}$$

such that K_n meets both $\{a_n\}\times I_{m_n}$ and $G_n\times\{b_n\}$. Without loss of generality the sequence $K_n)$ converges to a continuum $K\subset X\times X$. Then $d(K, \Delta X)\geqslant\varepsilon$ and

$$d(\pi_1(K), \pi_2(K)) = \lim d(\pi_1(K_n), \pi_2(K_n))\leqslant\lim d(a_n, b_n)\leqslant\lim 1/n = 0\,.$$

Hence $\pi_1(K)\cap\pi_2(K)\neq\emptyset$. Since X is hereditarily indecomposable either $\pi_1(K)\supset\pi_2(K)$ or $\pi_2(K)\supset\pi_1(K)$. Thus $\sigma_0(X)\geqslant\sigma_0(\pi_1(K)\cup\pi_2(K))\geqslant\varepsilon>0$ which is a contradiction.

Thus, there exists δ with $0<\delta<\varepsilon$ and an integer n_0 such that if $G\subset S(X, 2\delta)$ is a finite connected graph, $n\geqslant n_0$ and $(a, b)\in G\times I_n$ with $d(a, b)\leqslant 2\delta$, then no component of $\{(x, y)\in G\times I_n|\ d(x, y)\geqslant\frac{1}{2}\varepsilon\}$ meets both $\{a\}\times I_n$ and $G\times\{b\}$. Hence, if G is unicoherent the component of $\{(x, y)\in G\times I_n|\ d(x, y)<\frac{1}{2}\varepsilon\}$ containing the point (a, b) separates $\{a\}\times I_n$ from $G\times\{b\}$ in $G\times I_n$.

Suppose $G\subset S(X, \frac{1}{2}\delta)$ is a finite connected non-acyclic graph. Let $\hat{G}$ be the universal covering space of G and $p\colon \hat{G}\twoheadrightarrow G$ be the covering projection. We may suppose $\hat{G}$ is embedded in $S(X, \frac{1}{2}\delta)$ such that $\hat{G}\cup G$ is a compactification of $\hat{G}$ by G and the natural extension $\hat{p}$ of p to $\hat{G}\cup G$ is a $\frac{1}{2}\delta$-retraction of $\hat{G}\cup G$ to G.

Let $n\geqslant n_0$ and let A be an arc in $G\times I_n$ which is irreducible with respect to intersecting $\{a\}\times I_n$ and $G_n\times\{b\}$. Then $(p\times \mathrm{id}_{I_n})^{-1}(A) = A_1\cup A_2\cup\ldots$ where the A_i are pairwise disjoint arcs in $\hat{G}\times I_n$ which map homeomorphically onto A under $p\times\mathrm{id}_{I_n}$. Let $\pi_1\colon \hat{G}\times I_n\to\hat{G}$ be the first coordinate projection and let $\pi_1(A_1) = C$. Let $a_1\in C$ such that $p(a_1) = a\in G$. Then $d(a_1, b)<2\delta$ and if K is the component of (a_1, b) in $\{(x, y)\in C\times I_r|\ d(x, y)<\frac{1}{2}\varepsilon\}$, then by the second paragraph of this proof K separates $\{a_1\}\times I_n$ from $C\times\{b\}$ in $C\times I_n$ since C is unicoherent. Hence $K\cap A_1\neq\emptyset$. Clearly

$$(a, b)\in(p\times\mathrm{id}_{I_n})(K)\subset\{(x, y)\in G\times I_n|\ d(x, y)<\varepsilon\}\ \text{and}\ (p\times\mathrm{id}_{I_n})(K)\cap A\neq\emptyset\,.$$

This completes the proof of the lemma.

LEMMA 7. *Let* T_1 *and* T_2 *be connected graphs and let* x *be a point such that* $T_1\cap T_2 = \{x\}$. *Let* $y\in I$ *and let* K_1 *and* K_2 *be graphs with* $K_i\subset T_i\times I$ *such that* $K_i\cap[\{x\}\times I] = \{(x, y)\}$ *and* K_i *separates* $T_i\times\{0\}$ *from* $T_i\times\{1\}$ *for each* $i = 1, 2$. *Then* $K_1\cup K_2$ *separates* $(T_1\cup T_2)\times\{0\}$ *from* $(T_1\cup T_2)\times\{1\}$ *in* $(T_1\cup T_2)\times I$.

Proof. We may suppose that K_i is irreducible with respect to separating $T_i\times\{0\}$ from $T_i\times\{1\}$ in $T_i\times I$ for $i = 1, 2$. If $K_1\cup K_2$ does not separate $(T_1\cup T_2)\times\{0\}$ from $(T_1\cup T_2)\times\{1\}$ in $(T_1\cup T_2)\times I$, then there exists a polygonal arc

$$A\subset(T_1\cup T_2)\times I\setminus(K_1\cup K_2)$$

such that A meets both $(T_1\cup T_2)\times\{0\}$ and $(T_1\cup T_2)\times\{1\}$. Since $(K_1\cup K_2)\cap(\{x\}\times I) = \{(x, y)\}$, it is easy to see that there exists an arc B such that either

$$B\subset[A\cup(\{x\}\times([0, y)\cup(y, 1]))]\cap(T_1\times I\setminus K_1)\,,$$

or

$$B\subset[A\cup(\{x\}\times([0, y)\cup(y, 1]))]\cap(T_2\times I\setminus K_2)$$

and B meets both $(T_1\cup T_2)\times\{0\}$ and $(T_1\cup T_2)\times\{1\}$ which is a contradiction.

4. Some span zero type conditions for chainability. Let $X\subset Q$ be a continuum. We give five conditions that X may satisfy:

(i) X is chainable.

(ii) For each sequence of arcs $I_n)$ in Q such that $\operatorname{Lim} I_n = X$ there exists an inverse sequence of graphs (G_n, f_n^m) such that:

(a) $X = \varprojlim(G_n, f_n^m)$,

(b) $G_n\subset Q$ such that $\operatorname{Lim} G_n = X$ and the projection $f_n\colon X\twoheadrightarrow G_n$ is a $1/n$-map such that $d(x, f_n(x))<1/n$,

(c) for each $\varepsilon>0$ and for each integer n_0 there exists $n\geqslant n_0$ and a graph $G_n^*\subset C(n, \varepsilon) = \{(x, y)\in G_n\times I_n|\ d(x, y)<\varepsilon\}$ such that G_n^* is homeomorphic to G_n and G_n^* separates $G_n\times\{0\}$ from $G_n\times\{1\}$ in $G_n\times I_n$.

(iii) For each sequence $G_n)$ of graphs in Q such that $X = \operatorname{Lim} G_n$, $d(x, f_n(x))<1/n$ $X = \varprojlim(G_n, f_n^m)$ and $f_n\colon X\twoheadrightarrow G_n$ is a $1/n$-map there exists a sequence of arcs $I_n)$ in Q such that

(a) $\operatorname{Lim} I_n = X$,

(b) for each $\varepsilon>0$ and for each integer n_0 there exists $n\geqslant n_0$ and a graph $G_n^*\subset C(n, \varepsilon) = \{(x, y)\in G_n\times I_n|\ d(x, y)<\varepsilon\}$ such that G_n^* is homeomorphic to G_n and G_n separates $G_n\times\{0\}$ from $G_n\times\{1\}$ in $G_n\times I_n$.

(iv) There exist sequences $I_n)$ of arcs and $G_n)$ of graphs in Q such that

(a) $\operatorname{Lim} I_n = \operatorname{Lim} G_n = X$,

(b) $X = \varprojlim(G_n, f_n^m)$ and $f_n\colon X\to G_n$ is a $1/n$-map such that $d(x, f_n(x))<1/n$

(c) for each $\varepsilon>0$ and for each integer n_0 there exists $n\geqslant n_0$ and a graph $G_n^*\subset C(n, \varepsilon) = \{(x, y)\in G_n\times I_n|\ d(x, y)<\varepsilon\}$ such that G_n^* is homeomorphic to G_n and G_n^* separates $G_n\times\{0\}$ from $G_n\times\{1\}$ in $G_n\times I_n$.

(v) For every pair of sequences $I_n)$ of arcs and $G_n)$ of graphs in Q such that $\operatorname{Lim} I_n = \operatorname{Lim} G_n = X$ and for each $\varepsilon>0$ there exists an integer n_0 such that for each $n \geqslant n_0$ there exists a graph $K_n \subset C(n, \varepsilon) = \{(x, y) \in G_n \times I_n |\ d(x, y) < \varepsilon\}$ such that K_n separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$.

THEOREM 8. *If $X \subset Q$ is a continuum then we have the following relations among these conditions*:

$$\text{(i)} \begin{array}{c} \nearrow \text{(ii)} \searrow \\ \searrow \text{(iii)} \nearrow \end{array} \text{(iv)}$$

$$\text{(i)} \Rightarrow (\sigma(X) = 0) \Leftrightarrow (\sigma_0(X) = 0) \Rightarrow (\sigma_0^*(X) = 0) \Rightarrow \text{(v)}$$

Moreover, if X is hereditarily indecomposable then (iv) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). Suppose X is chainable and let $I_n)$ be a sequence of arcs in Q converging onto X. Since X is chainable, there exists a sequence of arcs $G_n)$ in Q such that $X = \varprojlim(G_n, f_n^m)$, $X = \operatorname{Lim} G_n$, $d(x, f_n(x_n)) < 1/n$ and $f_n\colon X \to G_n$ is a $1/n$-map. Let $\varepsilon > 0$ be given. Since X is chainable, $\sigma_0(X) = 0$. Hence by Lemma 3 there exists an integer n_0 such that for $n \geqslant n_0$ a component K_n of

$$\{(x, y) \in G_n \times I_n |\ d(x, y) < \varepsilon\}$$

separates $G_n \times \{0\}$ from $G_n \times \{1\}$. By [6, p. 438] there exists a locally connected continuum $H_n \subset K_n$ which separates $G_n \times \{0\}$ from $G_n \{1\}$. Hence there exists an arc $M_n \subset H_n$ which separates $G_n \times \{0\}$ from $G_n \times \{1\}$.

(i)$\Rightarrow$(iii). Let $G_n)$ be a sequence of graphs in Q such that $\operatorname{Lim} G_n = X = \varprojlim(G_n, f_n^m)$, $f_n\colon X \to G_n$ is a $1/n$-map and let $\varepsilon > 0$ be given. Let $\mathscr{U}_n$ be a nested sequence of open chain covers of X in Q such that mesh $\mathscr{U}_n < 1/n$ and let $I_n \subset \bigcup \mathscr{U}_n$ be the nerve of $\mathscr{U}_n$. Without loss of generality $G_n \subset \bigcup \mathscr{U}_n$. Let $r_n\colon \bigcup \mathscr{U}_n \to I_n$ be a $1/n$-retraction. Let $G_n^* \subset G_n \times I_n$ be the graph of $r_n | G_n$. Then $G_n^* \approx G_n$ and G_n^* separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$.

(i) $\Rightarrow \sigma(X) = 0$. See [8].

$\sigma(X) = 0 \Leftrightarrow \sigma_0(X) = 0$. See [5].

$\sigma_0^*(X) \Rightarrow$ (v). Let $\varepsilon > 0$ be given. By Lemma 3 there exists an integer n_0 such that for $n \geqslant n_0$ there exists a component K_n of $\{(x, y) \in G_n \times I_n |\ d(x, y) < \varepsilon\}$ which separates $G_n \times \{0\}$ from $G_n \times \{1\}$. Since $G_n \times I_n$ is embedded in the suspension of G_n (cf. the proof of Lemma 3) which is a locally connected unicoherent continuum, it follows that there exists a locally connected continuum $H_n \subset K_n$ which separates $G_n \times \{0\}$ from $G_n \times \{1\}$. ([6], p. 438). Since G_n is a finite graph and K_n is open in $G_n \times I_n$ it follows that there exists a graph $C_n \subset K_n$ which separates $G_n \times \{0\}$ from $G_n \times \{1\}$.

(ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) trivial.

Suppose X is hereditarily indecomposable and satisfies (iv). Let $I_n)$ and $G_n)$ be sequences in Q which satisfy (iv). We will show that X is chainable. Let $\varepsilon > 0$ be given. Without loss of generality we may assume that I_n and G_n are piecewise linear $(n = 1, 2, \ldots)$ in Q. Let n be an integer so large that the projection $f\colon X \twoheadrightarrow G_n$ moves points less than ε and there exists a graph

$$G_n^* \subset C(n, \varepsilon) = \{(x, y) \in G_n \times I_n |\ d(x, y) < \varepsilon\}$$

such that G_n^* separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$ and such that G_n^* is homeomorphic to G_n. We may suppose, by compressing G_n^* slightly in the second coordinate, that $G_n^* \cap (G_n \times \{0, 1\}) = \emptyset$, since $G_n^* \subset C(n, \varepsilon)$ which is open in $G_n \times I_n$. Let $\varepsilon_1 < \varepsilon$ such that f_n is an ε_1-map. Let $\eta > 0$ be such that $4\eta < \varepsilon - \varepsilon_1$ and the diameter of $f_n^{-1}(S(x, \eta)) < \varepsilon$ for each $x \in G_n$. Let $G_n' \subset G_n^*$ be irreducible with respect to separating $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$.

Let $\{x_1, \ldots, x_m\}$ be the branch-points and endpoints of G_n. It is not difficult to see that if x_t is a branch-point of G_n, then there exists in G_n' at least one branch-point of the form (x_t, y_t) for some $y_t' \in I_n$ since G_n' separates $G_n \times \{0\}$ from $G_n \times \{1\}$. In fact the order of G_n' at (x_t, y_t) is at least as great as the order of G_n at x_t. Since $G_n' \subset G_n^* \approx G_n$, it follows that G_n' has exactly one branch point in $\{x_t\} \times I_n$. Let U be a connected neighbourhood of x_t in G_n such that $\operatorname{Cl}(U)$ contains only one vertex of the graph G_n. If $(x_t, z) \in G_n'$ such that the component C of (x_t, z) in $G_n' \cap (\{x_t\} \times I_n)$ does not contain (x_t, y_t), then there exists an arc $A \subset U$ having x_t as an endpoint such that some neighbourhood of C in G_n' is contained in $A \times I_n$. By adjusting G_n' slightly in the open set $C(n, \varepsilon)$ we may suppose $G_n' \cap (\{x_t\} \times I_n)$ is connected for each branch-point x_t of G_n. By a further small adjustment of G_n' we may assume $G_n' \cap (\{x_t\} \times I_n) = \{(x_t, y_t)\}$ for each branch-point x_t of G_n and that G_n' is piecewise linear in $G_n \times I_n$. It follows that G_n is homeomorphic to G_n' under a homeomorphism which takes x_t to (x_t, y_t) for each $t \in \{1, \ldots, m\}$.

If G_n does not contain any branch-point, then G_n is either an arc or a circle. The proof is complete if G_n is an arc. If G_n is a circle, let $(\hat{G}_n, p)$ be the universal covering space of G_n. Since $\sigma(X) = 0$, X is tree-like. Hence the projection $f_n\colon X \to G_n$ has a lifting $\hat{f}_n\colon X \to \hat{G}_n$ such that $f_n = p \circ \hat{f}_n$. It follows that $\hat{f}_n$ is an ε-map of X onto an arc. Hence we may assume that G_n contains at least one branch-point.

Let A be an arc in G_n with end points x_i and x_j such that A contains no other points of $\{x_1, \ldots, x_m\}$. Then $\pi_1^{-1}(A) \cap G_n'$ is an arc A'. By Corollary 2 there exists an ε-map $g\colon f_n^{-1}(A) \to A'$ such that

$$g^{-1}((x_i, y_i)) = f_n^{-1}(x_i), \qquad g^{-1}((x_j, y_j)) = f_n^{-1}(x_j) \qquad \text{and} \qquad d(f_n(x), \pi_1 \circ g(x)) < \eta$$

or each $x \in f_n^{-1}(A)$.

If C is a simple closed curve in G_n such that $\operatorname{Bd}(C)$ in G_n is at most a single point x_k, then $\pi_1^{-1}(C) \cap G_n'$ is a simple closed curve C'. Choose an arc $K \subset C \setminus \{x_k\}$ with end point a_c and b_c such that $\operatorname{diam}(f_n^{-1}(\operatorname{Cl}(C \setminus K))) < \varepsilon$ and $(\{x\} \times I_n) \cap G_n'$ is a singleton for each $x \in \operatorname{Cl}(C \setminus K)$. Since $\pi_1^{-1}(K) \cap G_n'$ is an arc $K' \subset C'$ there exists as above an ε-map $g\colon f_n^{-1}(C) \twoheadrightarrow C'$ such that for each $x \in \operatorname{Cl}(C \setminus K)$

$$f_n^{-1}(x) = g^{-1}((\{x\} \times I_n) \cap G_n') \quad \text{and} \quad d(f_n(x), \pi_1 \circ g(x)) < \eta \quad \text{for each } x \in f_n^{-1}(C).$$

It follows that there exists an ε-map $h\colon X \twoheadrightarrow G_n'$ such that $d(f_n(x), \pi_1 \circ h(x)) < \eta$ for $x \in X$.

We claim that $\pi_2 \circ h\colon X \to I_n$ is a 4ε-map. To see this let $x, y \in (\pi_2 \circ h)^{-1}(t)$. By Note 4, $\pi_2|G_n'$ is a 2ε-map, it follows $d(h(x), h(y)) < 2\varepsilon$ and hence

$$d(\pi_1 h(x), \pi_1 h(y)) < 2\varepsilon .$$

Also

$$d(x, \pi_1 h(x)) \leqslant d(x, f_n(x)) + d(f_n(x), \pi_1 h(x)) \leqslant \varepsilon_1 + \eta < \varepsilon \quad \text{for each } x \in X .$$

Hence

$$d(x, y) \leqslant d(x, \pi_1 h(x)) + d(\pi_1 h(x), \pi_1 h(y)) + d(\pi_1 h(y), y) < 4\varepsilon .$$

Since ε was arbitrary X is chainable.

PROBLEM 9. Suppose X is a hereditarily indecomposable continuum such that $\sigma_0(X) = 0$. Does X satisfy condition (iv) and, as a consequence, is X chainable?

5. Applications. In this section we will give some partial solutions to Problem 9. It is known ([10] and [13]) that continua X with $\sigma(X) = 0$ are atriodic and tree-like. The reason for allowing graphs (rather than trees) in the inverse limit description of X in the following theorems is that this makes it easier to satisfy the condition concerning the number of branch-points on arcs in G_n (cf. the proof of Corollary 14).

THEOREM 10. *Let $X = \varprojlim(G_n, f_n^m)$ be a hereditarily indecomposable continuum in Q with $\sigma(X) = 0$ where each G_n is a graph with the property that each simple closed curve in G_n has at most one point in its boundary in G_n. Suppose also there exists an integer N such that for each integer n and each arc $A \subset G_n$, A contains at most N branch points of G_n. Then X is a pseudo-arc.*

Proof. We may suppose by the remark at the end of section 1 that the graphs G_n are embedded in Q such that $X = \operatorname{Lim} G_n$ and f_n^m moves no point of G_m more than $1/n$ for each $m \geqslant n$.

Let $I_n)$ be a sequence of arcs in Q such that $\operatorname{Lim} I_n = X$.

For each $\delta > 0$ and each positive integer n let

$$C(n, \delta) = \{(x, y) \in G_n \times I_n |\ d(x, y) < \delta\} .$$

Let $\varepsilon > 0$ be given. By Lemma 6, let $\varepsilon = \varepsilon_0 > \varepsilon_1 > \ldots > \varepsilon_N > 0$ and let $m_1 < m_2 < \ldots < m_N$ be integers such that if $(x, y) \in C(n, \varepsilon_{i+1})$ where $n \geqslant m_i$, then no component of $G_n \times I_n \setminus C(n, \varepsilon_i)$ meets both $\{x\} \times I_n$ and $G_n \times \{y\}$. By Lemma 3 we may suppose there is a component $K(n, \varepsilon_N)$ of $C(n, \varepsilon_N)$ such that $K(n, \varepsilon_N)$ separates $G_n \times \{0\}$ from $G_n \times \{1\}$ in $G_n \times I_n$ for $n \geqslant m_N$. By Lemma 5 we may suppose no component L of $G_n \times I_n \setminus C(n, \varepsilon_N)$ has $\pi_1(L) = G_n$ for $n \geqslant m_N$.

If $m \geqslant m_N$ and G_m has no branch-points, then G_m is an arc or a simple closed curve. In either case $K(m, \varepsilon_N)$ contains a continuum homeomorphic to G_m which separates $G_m \times \{0\}$ from $G_m \times \{1\}$.

Now, suppose $m \geqslant m_N$ and x_1 is a branch-point of G_m. Let $(x_1, t_1) \in C(m, \varepsilon_N)$ and let $H_1, \ldots, H_{n_1}$ be the closures of the components of $G_m \setminus \{x_1\}$. After reindexing if necessary we may suppose $H_1, \ldots, H_{k_1}$ are not arcs or simple closed curves and each of $H_{k_1+1}, \ldots, H_{n_1}$ is an arc or a simple closed curve. Let $K(m, \varepsilon_{N-1})$ be the component of $C(m, \varepsilon_{N-1})$ which contains (x_1, t_1). By the choice of m_N and ε_N no component of $G_m \times I_m \setminus K(m, \varepsilon_{N-1})$ meets both $\{x_1\} \times I_m$ and $G_m \times \{t_1\}$. We will show that $K(m, \varepsilon_{N-1})$ separates $G_m \times \{0\}$ from $G_m \times \{1\}$ in $G_m \times I_m$.

If $K(m, \varepsilon_N) \cap K(m, \varepsilon_{N-1}) \neq \emptyset$, then $K(m, \varepsilon_N) \subset K(m, \varepsilon_{N-1})$ and there is nothing to prove. Hence suppose $K(m, \varepsilon_N) \cap K(m, \varepsilon_{N-1}) = \emptyset$. Then $\pi_1(K(m, \varepsilon_{N-1})) = G_m$ since if

$$\pi_1(K(m, \varepsilon_{N-1})) \neq G_m ,$$

then

$$K(m, \varepsilon_{N-1}) \cap [G_m \times \{0\}] \neq \emptyset \neq K(m, \varepsilon_{N-1}) \cap [G_m \times \{1\}]$$

and this contradicts the fact that $K(m, \varepsilon_N)$ separates $G_m \times \{0\}$ from $G_m \times \{1\}$ in $G_m \times I_m$. We may suppose without loss of generality that $K(m, \varepsilon_N)$ separates $G_m \times \{0\}$ from $K(m, \varepsilon_{N-1})$. Let $\varphi\colon G_m \times I_m \twoheadrightarrow Y = G_m \times I_m / G_m \times \{0\}$ denote the natural projection, then Y is a locally connected unicoherent continuum. Now $G_m \times I_m \setminus C(m, \varepsilon_{N-1})$ separates $K(m, \varepsilon_N)$ from $K(m, \varepsilon_{N-1})$ in $G_m \times I_m$. Hence $\varphi(G_m \times I_m \setminus C(m, \varepsilon_{N-1}))$ separates $\varphi(K(m, \varepsilon_N))$ from $\varphi(K(m, \varepsilon_{N-1}))$ in Y. Hence a component L of $\varphi((G_m \times I_m \setminus C(m, \varepsilon_{N-1})))$ separates these sets. Then $L \cap \varphi((G_m \times \{0\})) = \emptyset$ and $\varphi^{-1}(L)$ is a component of $G_m \times I_m \setminus C(m, \varepsilon_{N-1})$ such that $\pi_1(L) = G_m$. This contradicts the choice of m_N. We have proved that $K(m, \varepsilon_{N-1})$ separates $G_m \times \{0\}$ from $G_m \times \{1\}$ in $G_m \times I_m$.

Let M_1 be a graph in the open set $K(m, \varepsilon_{N-1})$ which is minimal with respect to separating $G_m \times \{0\}$ from $G_m \times \{1\}$ and such that $(\{x_1\} \times I_m) \cap M_1 = \{(x_1, t_1)\}$. For each i, $k_1 + 1 \leqslant i \leqslant n_1$, $(H_i \times I_m) \cap M_1$ is homeomorphic to H_i.

For each i, $1 \leqslant i \leqslant k_1$, let $x_{2,i}$ be the unique branch-point of H_i which separates every other branch-point of H_i from x_1. Let M_1' be the closure of the component of $M_1 \setminus (\{x_{2,1}, \ldots, x_{2,k_1}\} \times I_m)$ which contains (x_1, t_1). Let M_1'' be a minimal subcontinuum of M_1' which contains $M_1 \cap (\bigcup_{i=k_1+1}^{n_1} H_i \times I_m)$ and meets $\{x_{2,i}\} \times I_m$ for each $i = 1, \ldots, k_1$. For each $i = 1, \ldots, k_1$ let $(x_{2,i}, t_{2,i}) \in M_1''$. Notice that if $i \in \{1, \ldots, k_1\}$ and A is the arc in G_m with end-points x_1 and $x_{2,i}$, then $(A_i \times I_m) \cap M_1''$ is an arc. Also $M_1'' \cap (B_1 \times I_m)$ is homeomorphic to B_1 where B_1 is the subcontinuum of G_m which is minimal with respect to containing

$$\{x_{2,1}, \ldots, x_{2,k_1}\} \cup H_{k_1+1} \cup \ldots \cup H_{n_1} .$$

For $i = 1, \ldots, k_1$ let $H_{i,1}, \ldots, H_{i,n_{2,i}}$ be the closures of the components of $G_m \setminus \{x_{2,i}\}$ which do not contain x_1. After reindexing if necessary we may suppose $H_{i,1}, \ldots, H_{i,k_{2,i}}$ are not arcs or simple closed curves and $H_{i,k_{2,i}+1}, \ldots, H_{i,n_{2,i}}$ are arcs or simple closed curves. Let $K(m, \varepsilon_{N-2}, i)$ be the component of $C(m, \varepsilon_{N-2})$ which contains $(x_{2,i}, t_{2,i})$. As above, $K(m, \varepsilon_{N-2}, i)$ separates $G_m \times \{0\}$ from $G_m \times \{1\}$ in $G_m \times I_m$.

Let $M_{2,i}$ be a graph in $K(m, \varepsilon_{N-2}, i) \cap (\bigcup_{j=1}^{n_{2,i}} H_{i,j} \times I_m)$ which is minimal with respect to separating $\bigcup_{j=1}^{n_{2,i}} H_{i,j}\times\{0\}$ from $\bigcup_{j=1}^{n_{2,i}} H_{i,j}\times\{1\}$ in $\bigcup_{j=1}^{n_{2,i}} H_{i,j}\times I_m$ and such that $M_{2,i}\cap(\{x_{2,i}\}\times I_m)=\{(x_{2,i},t_{2,i})\}$. For each $j=1,\dots,k_{2,i}$ let $x_{3,i,j}$ be the unique branch-point of $H_{i,j}$ which separates $x_{2,i}$ from every other branch-point of $H_{i,j}$. As above, let $M''_{2,i}$ be a minimal subcontinuum of $M_{2,i}$ which contains $M_{2,i}\cap[\bigcup_{j=k_{2,i}+1}^{n_{2,i}}(H_{i,j}\times I_m)]$ and meets $\{x_{3,i,j}\}\times I_m$ for each $j=1,\dots,k_{2,i}$.

Then $M''_2=M''_1\cup\bigcup_{i=1}^{k_1}M''_{2,i}$ is a continuum homeomorphic to $\pi_1(M''_2)$ which meets $\{x\}\times I_m$ in precisely one point for each branch-point of $\pi_1(M''_2)$ and which separates $\pi_1(M''_2)\times\{0\}$ from $\pi_1(M''_2)\times\{1\}$ in $\pi_1(M''_2)\times I_m$. One can continue this argument inductively through at most N stages to construct a graph $M=M''_N\subset C(m,\varepsilon)$ such that M is homeomorphic to G_m and M separates $G_m\times\{0\}$ from $G_m\times\{1\}$ in $G_m\times I_m$. The theorem follows from Theorem 8.

THEOREM 11. *Let X be a hereditarily indecomposable continuum in Q such that $\sigma(X)=0$. Suppose $X=\varprojlim(G_n,f_n^m)$ where each G_n is a graph such that:*

(1) *if C is a simple closed curve in G_n, then C has at most one boundary point in G_n;*

(2) *there exists an integer N and a sequence of arcs $A_n\subset G_n$ such that if $B\subset G_n\setminus A_n$ is an arc, then B contains at most N ramification points of G_n.*

Then X is a pseudo-arc.

Proof. Let $\varepsilon>0$ be given. Let I_n be a sequence of arcs in Q such that $\operatorname{Lim} I_n=X$. We may assume that the graphs G_n are piecewise linearly embedded in Q such that $\operatorname{Lim} G_n=X$ and $f_n\colon X\to G_n$ moves no point more than $1/n$. By Lemmas 3 and 6 choose $\varepsilon=\varepsilon_0>\varepsilon_1>\dots>\varepsilon_{N+2}>0$ and integers $m_1<\dots<m_{N+2}$ such that if $n\geqslant m_{i+1}$, $G\subset S(X,\varepsilon_{i+1})$ is a connected graph and $(x,y)\in G\times I_n$ such that $d(x,y)<\varepsilon_{i+1}$ then a component of $C(n,\varepsilon_i)=\{(a,b)\in G\times I_n|\ d(a,b)<\varepsilon_1\}$ separates $\{x\}\times I_n$ from $G\times\{y\}$ in $G\times I_n$ and a component of $C(n,\varepsilon_{N+2})$ separates $G_n\times\{0\}$ from $G_n\times\{1\}$ for $n\geqslant m_{N+2}$.

Let $n\geqslant m_{N+2}$, $A_n\subset G_n\subset S(X,\varepsilon_{N+2})$. Let K_0 be a piecewise linear arc in

$$\{(x,y)\in A_n\times I_n|\ d(x,y)<\varepsilon_{N+2}\}$$

which is irreducible with respect to separating $A_n\times\{0\}$ from $A_n\times\{1\}$ in $A_n\times I_n$. We may suppose that the natural projection $f_n\colon X\to G_n$ moves each point of X a distance less than $\frac{1}{2}\varepsilon_{N+2}$. Let $\{x_1,x_2,\dots,x_l\}$ be all the branch-points of G_n on A_n. We may assume that $K_0\cap[\{x_i\}\times I_n]$ is a finite set for $i=1,\dots,e$. Let H_i be the closure of the union of the components of $G_n\setminus\{x_i\}$ which are disjoint from A_n. Then $H_i\cap H_j=\emptyset$ if $i\neq j$. Let $\{(x_i,t_{i,1}),\dots,(x_i,t_{i,s_i})\}=\{(\{x_i\}\times I_n)\cap K_0\}$ and define

$$K_1=K_0\cup\bigcup_{i=1}^{l}\Big[\bigcup_{j=1}^{s_i}(H_i\times\{t_{i,j}\})\Big]\subset G_n\times I_n.$$

Notice that π_1 restricted to every component of $K_1\setminus K_0$ is a homeomorphism.

For each $\eta>0$, let K_η be a homeomorphic copy of K_1 embedded in Q such that the natural projection $\xi_\eta\colon K_\eta\to G_\eta$ moves points less than η, (i.e. $\xi_\eta=\pi_1\circ h_\eta$ where $h_\eta\colon K_\eta\twoheadrightarrow K_1$ is a homeomorphism and $\pi_1\colon G_n\times I_n\to G_n$ is the usual projection). Let B_η be the maximal arc in K_η such that $\xi_\eta(B_\eta)=A_n$. For each sufficiently small η there exists an arc $L_\eta\subset\{(x,y)\in B_\eta\times I_n|\ d(x,y)<\varepsilon_{N+2}\}$ such that L_η projects homeomorphically onto B_η by $\pi_1\colon K_\eta\times I_n\to K_\eta$.

We will show that there exists a $\frac{1}{2}\varepsilon_{N+2}$-map $\varphi\colon X\to K_\eta$. Choose a taut open chain cover $\mathcal{U}=\{U_1,\dots,U_n\}$ of G_n such that:

(1) each U_i is connected,

(2) for each vertex (v,t) of the piecewise linear arc K_0 there exists exactly one element $U_i\in\mathcal{U}$ such that $v\in U_i$,

(3) for each element $U_i\in\mathcal{U}$ there is at most one point $v\in U_i$ such that (v,t) is a vertex of K_0 for some $t\in I_n$,

(4) for each $t=1,\dots,l$ there exists exactly one element $U_t\in\mathcal{U}$ such that $H_t\cap\operatorname{Cl}(U_t)\neq\emptyset$.

Let $\mathcal{W}=\{W_1,\dots,W_p\}$ be a taut open chain cover of K_0 such that W_j is connected, $\operatorname{Cl}(W_j)$ contains at most one vertex of K_0 and $\{\pi_1(W_j)\}_{j=1}^{p}$ refines $\mathcal{U}$. The map $\pi_1|K_0\colon K_0\to A_0$ induces a (pattern) map $\pi_1^*\colon\{1,\dots,p\}\twoheadrightarrow\{1,\dots,n\}$ such that $|\pi_1^*(t)-\pi_1^*(t+1)|\leqslant 1$, $t=1,\dots,p-1$. By [14, Theorem 3], there exists a taut open chain cover $\mathcal{V}=\{V_1,\dots,V_p\}$ of X such that $V_t\subset U_{\pi_1^*(t)}$ for $t=1,\dots,p$. It is now not difficult (cf. the proof of Corollary 2) to construct a $\frac{1}{2}\varepsilon_{N+2}$-map $\varphi\colon X\to K_1\approx K_\eta$ (this map is not necessarily onto) such that $d(x,\varphi(x))<\varepsilon_{N+2}$.

For each branch-point $x\in B_\eta$ of K_η, let (x,y_x) be the unique point on L_η and let C_x be the closure of the union of the components of $K_\eta\setminus\{x\}$ disjoint from B_η. As in the proof of Theorem 10 (recall $d(x,y_x)<\varepsilon_{N+2}$), there exists a graph $M_x\subset\{(a,b)\in C_x\times I_n|\ d(a,b)<\varepsilon\}$ such that M_x is homeomorphic to C_x, $M_x\cap(\{x\}\times I_n)=\{(x,y_x)\}$ and M_x separates $C_x\times\{0\}$ from $C_x\times\{1\}$.

By Lemma 7, $K=L_\eta\cup\bigcup\{M_x|\ x\in B_\eta$ is a branch point of $K_\eta\}$ is a graph in $K_\eta\times I_n$ which separates $K_\eta\times\{0\}$ from $K_\eta\times\{1\}$, K is homeomorphic to K_η and $d(a,b)<\varepsilon$ for each $(a,b)\in K$. The theorem now follows by Theorem 8.

THEOREM 12. *Let X be a hereditarily indecomposable continuum in Q such that $\sigma(X)=0$. Suppose $X=\varprojlim(G_n,f_n^m)$ where each G_n is a graph such that:*

(1) *if C is a simple closed curve in G_n then C has at most one boundary point in G_n,*

(2) *there exist integers N and M and families of arcs $\mathcal{B}_{n,1},\dots,\mathcal{B}_{n,M}$ in G_n for each n such that:*

(a) *each arc in $G_n\setminus\bigcup_{i=1}^{M}(\bigcup\mathcal{B}_{n,i})$ contains at most N branch points,*

(b) *$\mathcal{B}_{n,1}$ consists of a single arc and for $i=2,\dots,M$ each arc in $\mathcal{B}_{n,i}$ intersects $\bigcup_{j=1}^{i-1}(\bigcup\mathcal{B}_{n,j})$ in exactly one point,*

(c) *each pair of elements of $\mathcal{B}_{n,i}$ intersects in at most one point.*

Then X is a pseudo-arc.

Proof. The case $M = 1$ was done in Theorem 11. The proof is similar to that of Theorem 11 and is omitted.

COROLLARY 13. *Let X be a hereditarily indecomposable continuum with $\sigma(X) = 0$. If there exist an integer N and a sequence T_n) of trees with at most N branch-points such that $X = \varprojlim(T_n, f_n^m)$, then X is a pseudo-arc.*

Proof. This follows immediately from Theorem 10.

COROLLARY 14. *If X is a continuous image of the pseudo-arc such that every proper subcontinuum is a pseudo-arc, then X is a pseudo-arc.*

Proof. It follows from the Boundary Bumping Theorem [6], p. 172 that X is indecomposable and hence hereditarily indecomposable. By [13], Theorem 15, $\sigma_0(X) = 0$. Let $x \in X$ and let $\varepsilon > 0$ be given. Let U be an open neighbourhood of x of diameter less than ε. If C is any component of $X \setminus U$, then C is either a point or a pseudo-arc. Hence there exists an open ε-chain cover $\mathscr{U}_C$ of C such that $\bigcup \mathscr{U}_C$ is open and closed in $X \setminus U$. Since $X \setminus U$ is compact there exists an integer n and $C_1, \ldots, C_n$ components of $X \setminus U$ such that $\mathscr{U}_{C_1} \cup \ldots \cup \mathscr{U}_{C_n}$ cover $X \setminus U$. Let $\mathscr{V}_1 = \{W \in \mathscr{U}_{C_1} |\ W \cap U = \emptyset\}$ and for $i \in \{2, \ldots, n\}$ let

$$\mathscr{V}_i = \{V \setminus \bigcup_{j<i} \cup \mathscr{U}_{C_j} |\ V \in \mathscr{U}_i, V \cap U = \emptyset\}.$$

Then $(\bigcup \mathscr{V}_i) \cap (\bigcup \mathscr{V}_j) = \emptyset$ for $i \neq j$.

Let

$$\mathscr{V} = \mathrm{St}(U, \{U\} \cup \mathscr{U}_{C_1} \cup \ldots \cup \mathscr{U}_{C_n}) \cup \mathscr{V}_1 \cup \ldots \cup \mathscr{V}_n.$$

Then $\mathscr{V}$ is an open cover of X of mesh less than 3ε such that the nerve of $\mathscr{V}$ has at most one branch-point. The corollary now follows from Theorem 10.

A continuum X is said to be *almost chainable* if for every $\varepsilon > 0$ there exists an open cover $\mathscr{U}$ of X such that $\mathrm{mesh}(\mathscr{U}) < \varepsilon$ and a chain $\mathscr{C} = \{C_1, \ldots, C_n\}$ in $\mathscr{U}$ with $X \subset S(\bigcup \mathscr{C}, \varepsilon)$, $\mathrm{Cl}(C_i) \cap \mathrm{Cl}(C_j) \neq \emptyset$ if and only if $|i-j| \leq 1$ and

$$\mathrm{Cl}(C_1 \cup \ldots \cup C_n) = C_1 \cup \ldots \cup C_{n-1} \cup \mathrm{Cl}(C_n).$$

COROLLARY 15 (Lewis [11]). *If X is an almost chainable homogeneous continuum, then X is a pseudo-arc.*

Proof. By [3], all proper subcontinua of X are pseudo-arcs. By the proof of [12, I, Theorem 3.6], $\sigma(X) = 0$ and hence by [13] X is the continuous image of the pseudo-arc. The result follows from Corollary 14.

PROBLEM 16. Suppose X is a homogeneous hereditarily indecomposable continuum such that $\sigma(X) = 0$. Does X satisfy the conditions of Theorem 12?

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