Exact covering systems for groups

by

M. M. Parmenter * (St. John's, Newfoundland)

Abstract. A system of cosets a_iG_i of subgroups of a group G is said to be an (exact) covering system if every element of G belongs to (exactly) one of the cosets. When G is cyclic, we are really talking about arithmetic sequences, and problems concerned with such system have been studied by Erdös and others.

In this note, we prove that if $[a_iG_i]_{i=1}^n$ is a covering system of a finite solvable group, then either G has a subgroup of index $<n$ or all of the G_i are conjugate. As a corollary, we extend some earlier results of Curzio.

A system of cosets a_iG_i of subgroups of a group G is said to be an (*exact*) *covering system* if every element of G belongs to (exactly) one of the cosets. For the case where G is cyclic, we are really talking about arithmetic sequences, and problems concerned with such systems were first stated by Erdös. One of the nicest results is the following:

THEOREM [2]. *Let $a_1+n_1Z, \ldots, a_s+n_sZ$ be an exact covering system of the integers where $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_s$. Then $n_{s-1} = n_s$.*

There are a number of easily stated but seemingly intractable open problems in the cyclic case, and these can be found in [3].

A major paper in this area was written by Stein [5], and he was the first to suggest that investigations should be carried out on groups more general than infinite cyclic and to observe that partial results of this type had been obtained by Curzio [1]. Work on the more general case has since been done by Korec and Znám [4].

Curzio's theorems apply only to a few special cases. In this note, we show that an analogue of his result can be obtained for all solvable groups, and use these techniques to generalize and extend the cases which he discussed.

Our main result is the following:

THEOREM 1. *Let G be finite solvable and say $G = a_1G_1 \cup a_2G_2 \cup \ldots \cup a_nG_n$, where G_i is a proper subgroup of G. Then either*

(i) *G has a subgroup of index $<n$ or*

(ii) *all of the G_i are equal.*

Proof. Assume (i) is not true. Then $[G\colon G_i] = n$ for each i and we must have $a_iG_i \cap a_jG_j = \emptyset$ if $i \neq j$.

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Note that we can assume $\bigcap_{\substack{1\leqslant i\leqslant n\\ x\in G}} G_i^x = 1$, for if we put $N = \bigcap_{\substack{1\leqslant i\leqslant n\\ x\in G}} G_i^x$, and consider $\bar{G} = G/N$, then (i) will still not be true in $\bar{G}$ and we will still have $\bar{G}$ being finite solvable and $\bar{G} = \bar{a}_1\bar{G}_1 \cup \ldots \cup \bar{a}_n\bar{G}_n$ with $[\bar{G} : \bar{G}_i] = n$ for each i (since $N \subseteq G_i$ for each i). If we could prove that the $\bar{G}_i$ were all equal, then it would follow that the G_i would all be equal, again since $N \subseteq G_i$ for each i.

Hence we assume from now on that $\bigcap_{\substack{1\leqslant i\leqslant n\\ x\in G}} G_i^x = 1$. This implies that for a subgroup C of prime order p there exists a subgroup G_i^x such that $C \cap G_i^x = \{1\}$, thus $p \leqslant n$. Since G is solvable $|G/G'|$ is divisible by some nontrivial prime which is less than or equal to n, and hence G has a subgroup of prime index $\leqslant n$. Since (i) is not true, we are left with case where n is a prime.

Hence we are assuming that $n = p$ is prime and that $\bigcap_{\substack{1\leqslant i\leqslant n\\ x\in G}} G_i^x = 1$.

Now consider the case where $G_i \lhd G$ for some i. We may assume that $a_i = 1$ (multiplying by a_i^{-1}). If $G_j \neq G_i$, we must have $G_iG_j = G$ since G_iG_j is a subgroup of smaller index than G_i. Hence we have $a_j = xy$ where $x \in G_i$ and $y \in G_j$.

$$(*) \qquad a_jG_j = xyG_j = xG_j .$$

But then $x \in a_jG_j \cap G_i = a_jG_j \cap a_iG_i$ which is a contradiction. Thus we must have $G_j = G_i$ for all $j \neq i$ and (ii) holds.

We are now reduced to the case where $N_G(G_i) = G_i$ for each i.

Let P be a p-Sylow subgroup of G. Since $([G : P], p) = 1$, we must have $[P : G_i \cap P] = p$ for each i.

Now $P = (P \cap a_1G_1) \cup (P \cap a_2G_2) \cup \ldots \cup (P \cap a_pG_p)$ and, for each i, either $P \cap a_iG_i = \emptyset$ or $P \cap a_iG_i = x_i(P \cap G_i)$ for some $x_i \in P$. Since $[P : P \cap G_i] = p$ for each i, and there are only p sets in the union, we must have $P \cap a_iG_i = x_i(P \cap G_i)$ for each i.

$$(*) \qquad P = x_1(P \cap G_1) \cup x_2(P \cap G_2) \cup \ldots \cup x_p(P \cap G_p) .$$

But P is a p-group and $P \cap G_i$ is maximal in P for each i, so we must have $P \cap G_i \lhd P$ for each i. Hence, we are in the case discussed earlier and we must have $P \cap G_1 = P \cap G_2 = \ldots = P \cap G_p$.

Now the above argument holds for any covering of the type

$$G = a_1G_1 \cup \ldots \cup a_pG_p$$

where $[G : G_i] = p$ for each i. In particular, if we start with a right coset decomposition of G with respect to one of the G_i,

$$G = G_i \cup G_ib_2 \cup \ldots \cup G_ib_p = G_i \cup b_2(n_2^{-1}G_ib_2) \cup \ldots \cup b_p(b_p^{-1}G_ib_p)$$

and observe that any conjugate xG_ix^{-1} is of the form $b_k^{-1}G_ib_k$ for some k, we then conclude that $P \cap G_i = P \cap xG_ix^{-1}$ for any $x \in G$ and any i. Let $T_p = P \cap G_i$. Then $T_p \subseteq \bigcap_{\substack{1\leqslant i\leqslant p\\ x\in G}} G_i^x = 1$ and therefore we must have $|P| = p$.

But this means that $G_1, \ldots, G_p$ are Sylow-p complements in the solvable group G, and hence $G_1, G_2, \ldots, G_p$ are all conjugate, hence equal since p divides $|G/G'|$.

Remark. It is easy to find examples of exact covering systems where the subgroups are not of minimal index and are not all conjugate. For instance, let G be the dihedral group on eight elements, i.e.

$$G = \langle a, b \mid a^4 = b^2 = 1,\ ba = a^3b\rangle .$$

Then

$$G = \{1, b\} \cup a\{1, ab\} \cup a^2\{1, a^3b\} \cup a^3\{1, b\}$$

but $\{1, b\}$ and $\{1, ab\}$ are not conjugate.

We now show how to apply the type of argument seen in Theorem 1 to obtain Curzio's results, and to extend these results to the case $n = 5$.

THEOREM 2 [1]. *Let $G = a_1G_1 \cup a_2G_2 \cup \ldots \cup a_nG_n$ where the G_i are proper subgroups of G, are not all equal and $n = 3$ or 4. Then G has a subgroup of index 2 (if $n = 3$) or of index 2 or 3 (if $n = 4$).*

Proof. As before, we can assume that $\bigcap_{\substack{x\in G\\ 1\leqslant i\leqslant n}} G_i^x = 1$. Hence we have $|G| = 2^a3^b$ and thus G is solvable.

THEOREM 3. *Let $G = a_1G_1 \cup \ldots \cup a_5G_5$, where G_i is a proper subgroup of G. Then either all the G_i are conjugates of each other or G has a subgroup of index <5.*

Proof. Let us assume that G has no subgroup of index <5.

As before, we can assume that $\bigcap_{\substack{x\in G\\ 1\leqslant i\leqslant 5}} G_i^x = 1$, and we note that $|G| = 2^a\cdot 3^b\cdot 5$.

If G were solvable, we would be done by Theorem 1, so assume G is not solvable.

Say we have $N \lhd G$. If 5 divides $|N|$, then G/N would be solvable of order 2^x3^y and we could find a subgroup of index 2 or 3. Hence we can assume $|N| = 2^s3^t$, in which case N would be solvable. Therefore, N has a non trivial characteristic abelian subgroup T of order 2^l or 3^m, so we have $T \lhd G$ and $|T| = 2^l$ or 3^m. Hence T is contained in all 2-Sylow or 3-Sylow subgroups of G.

$$(*) \qquad T \subseteq \bigcap_{\substack{x\in G\\ 1\leqslant i\leqslant 5}} G_i^x = 1 \text{ and this is a contradiction.}$$

Hence we conclude that G is a simple group of order $2^a\cdot 3^b\cdot 5$. However, $\bigcap_{x\in G} G_1^x$ is a normal subgroup of G and hence $\bigcap_{x\in G} G_1^x = 1$. But, letting G act on the conjugates of G_1 by conjugation, we have that $G/\bigcap_{x\in G} G_1^x$ is isomorphic to a subgroup of S_5.

Hence we must have $G \simeq A_5$, but the subgroups of index 5 in A_5 are all conjugate.

Remarks. 1. It is not true that, in Theorem 3, the G_i have to be equal. As seen in the proof of Theorem 1, whenever you have $N_G(H) = H$, you can re-express a right coset decomposition of G in terms of H as a left coset decomposition of G in terms of the conjugates of H. In the case of A_5, if $|H| = 12$ then $N_{A_5}(H) = H$, but A_5 has no subgroup of index <5.

2. A natural place to look for an example showing that these results cannot be extended would be PSL(2, 7), which has two conjugacy classes of subgroups of index 7. Unfortunately, this group does not yield a counterexample, but I will not inflict my unpleasant calculations upon the reader. For a nice listing of the properties of this group, see [6].

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DEPARTMENT OF MATHEMATICS AND STATISTICS
MEMORIAL UNIVERSITY OF NEWFOUNDLAND
St. John's, Newfoundland
Canada

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On span and chainable continua

by

Lex G. Oversteegen (Birmingham, Ala.)
and **E. D. Tymchatyn** (Saskatoon, Sas.) (*)

Abstract. In 1964 Lelek [8] defined the notion of the span of a metric continuum and proved that chainable continua have span zero. He asked if the converse is also true, i.e., if continua with span zero are chainable. Recently, (see [10] and [13]) Lelek proved that continua with span zero are atriodic and tree-like. In [13] the authors gave some new characterization of continua with span zero and proved that continua with span zero are continuous images of the pseudo-arc. In this paper we prove that if a hereditarily indecomposable metric continuum has span zero and is an inverse limit of finite graphs with in some sense not too many branch-points or simple closed curves, then X is a pseudo-arc. In particular, it follows that if X is a continuum which is the continuous image of the pseudo-arc and such that all proper subcontinua of X are pseudo-arcs, then X itself is a pseudo-arc.

1. Introduction. All spaces considered in this paper are metric. A *compactum* is a compact metric space. A *continuum* is a connected compactum. We write $f\colon X \twoheadrightarrow Y$ to indicate that f is a mapping of X onto Y. We let I denote the closed unit interval and Q the Hilbert cube with a fixed but arbitrary metric d. Every continuum is a subspace of Q.

If $A \subset X$ and $\varepsilon > 0$ we let $S(A, \varepsilon)$ denote the open ε-ball around A in X. We let $\mathrm{Cl}(A)$ denote the closure of A in X.

If X and Y are continua we let $\pi_1\colon X\times Y \to X$ and $\pi_2\colon X\times Y \to Y$ denote the first and second coordinate projections, respectively. We let ΔX denote the diagonal in $X\times X$. We define (see [9]) the *surjective span* of X, $\sigma^*(X)$, (resp. the *surjective semi-span*, $\sigma_0^*(X)$) to be the least upper bound of all real numbers ε for which there exists a subcontinuum $Z \subset X\times X$ such that $\pi_1(Z) = X = \pi_2(Z)$ (resp. $\pi_1(Z) = X$) and $d(x, y) \geqslant \varepsilon$ for each $(x, y) \in Z$. The *span* of X

$$\sigma(X) = \sup\{\sigma^*(A) \mid A \text{ is a subcontinuum of } X\}$$

and the *semi-span* of X

$$\sigma_0(X) = \sup\{\sigma_0^*(A) \mid A \text{ is a subcontinuum of } X\}.$$

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