Proof. We follow the ideas used in the proof of the Main Theorem. Most of the argument is the same, and so we shall not repeat it here. An analysis of that proof shows that the property of M contained in our Lemma 3.7 is all we need to extend the proof to the general case. This completes the proof.

Now we make a comment on the possibility of extending the properties of internal composants established in [3] (for indecomposable continua lying in S^2) to the general case. An easy inspection shows that the properties listed in Section 3 of [3], except 3.6, follow from property 3.6, from the Main Theorem [2] (4.1 extends it to all surfaces) and from the general results on indecomposable continua.

Now we are going to state a property of indecomposable subcontinua of surfaces which is a generalization of 3.6 from [3] (proved only for the sphere S^2). We can prove the generalization proceeding in the same way as in the proof of 3.6 in [3], applying our Lemma 3.7 in place of 2.2 of [3].

4.2. Lemma. *Let $L \subset M$ be a continuum intersecting all composants of X and let C be an arbitrary composant of X. If $X \not\subset L$, then there exist an open neighborhood U of L in M and a continuum $A \subset C$ which separates U between two points of L.*

Having this lemma, we can extend all the properties 3.1–3.9 from [3] to the general case. In particular, we have the following generalization of 3.2 from [3]:

4.3. Theorem. *The set $\bigcup \mathbf{E}$ is an F_σ-set in X. Consequently, the union of internal composants of X is a G_δ-set dense in X.*

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$L_{\infty\omega_1}$-elementary equivalence of ω_1-like models of PA

by

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Abstract. We show that two recursively saturated, ω_1-like models of PA are $L_{\infty\omega_1}$-elementarily equivalent iff they are elementarily equivalent and have the same standard systems. On the other hand, for every countable model M of PA we construct a continuum of pairwise non-isomorphic, ω_1-like, recursively saturated, elementary end extensions of M.

Any two recursively saturated models of PA (Peano Arithmetic) are $L_{\infty\omega_1}$-elementarily equivalent iff they are elementarily equivalent and have the same standard systems. This result was first noted by Craig Smoryński in [8] but its roots go back to some earlier works of George Wilmers and Alex Wilkie (see [8] for historical remarks).

For the purpose of this paper let us call elementary equivalent models with the same standard systems *similar*.

The result mentioned above is particularly important in the case of countable models of PA since it says that any two recursively saturated countable similar models are isomorphic. The situation is different when we consider uncountable models and, as usual, the first counterexamples can be found among ω_1-like models. Take for instance a "rather classless" model M of M. Kaufmann [3] (the existence of a rather classless model is provable in ZFC, cf [7]) and a recursively saturated ω_1-like model similar to M but not "rather classless", hence not isomorphic to M. In the last section of this paper we show that there is at least a continuum of pairwise non-isomorphic ω_1-like recursively saturated similar models. But our main theorem says that ω_1-like recursively saturated similar models are still very similar, namely they are $L_{\infty\omega_1}$-elementarily equivalent.

1. Preliminaries. We assume the readers' acquaintance with the basic properties of recursively saturated models of PA (Smoryński [8] is a perfect survey of this subject). With some minor changes we are going to follow the terminology and notation of [8] and [10].

Models of PA are called shortly models. As usual, $x \in D_y$ is an abbreviation for an arithmetical formula expressing that x is an element of a set coded by y and $(y)_i$ denotes the ith element of this set (in increasing order).

If b is an element of a model M then

$$D_b^M = \{x \in M\colon\ M \models x \in D_b\}.$$

If $I \subseteq_e M$ (I is an initial segment of M) then a subset A of I is coded in M if $A = I \cap D_b^M$ for some b in M. The set of subsets of ω coded in M is called the standard system of M and is denoted by $\mathrm{SSy}(M)$. For information about $L_{\infty\omega_1}$-logic we refer to [1]. We will use the following characterization of $L_{\infty\omega_1}$-elementary equivalence.

THEOREM (Kueker [6]). *Assume M and N are models (of any theory in a countable language) of power ω_1; then $M \equiv_{\infty\omega_1} N$ iff M and N can be written as the unions of chains $\{M_\xi\}_{\xi<\omega_1}$ and $\{N_\xi\}_{\xi<\omega_1}$ of countable submodels such that*

$$(M_\xi, M_\nu) \cong (N_\xi, N_\nu) \quad \text{whenever } \nu<\xi<\omega_1$$

and in fact there are non-empty sets G_ξ of isomorphisms of M_ξ with N_ξ, for all $\xi<\omega_1$, such that any isomorphism in G_ν extends to one in G_ξ, for any $\nu<\xi<\omega_1$.

We are going to use only the easy part of the above equivalence, proved by the standard back and forth criterion for $L_{\infty\omega_1}$-elementary equivalence (cf [6]).

Suppose we have models M and N, $I \subseteq_e M$, $J \subseteq_e N$, and f is an isomorphism of I with J. If there is an isomorphism $g: M \cong N$ extending f then the image of every subset A of I coded in M, written $f * A$, must be coded in N. This leads to the following definition:

DEFINITION. If f is an isomorphism of I with J and $I \subseteq_e M$ and $J \subseteq_e N$, then we say that f is an *(M, N)-isomorphism* if, for any $A \subseteq I$ coded in M and $B \subseteq J$ coded in N, $f * A$ and $f^{-1} * B$ are coded in N and M, respectively.

Observe that if f is an (M, N)-isomorphism and $M \subseteq_e M_1$ and $N \subseteq_e N_1$ then f is an (M_1, N_1)-isomorphism.

We will be interested in the following problem. Let M and N be countable, recursively saturated similar models and let $M_0 \prec_e M$ and $N_0 \prec_e N$ be recursively saturated. In this situation all four models are pairwise isomorphic. Suppose we have an isomorphism $f: M_0 \cong N_0$. When is it possible to extend f to an isomorphism of M with N? There are two necessary conditions:

1. f must be an (M, N)-isomorphism,
2. the structures (M, M_0) and (N, N_0) must be isomorphic.

So now the question is: for which pairs of models are these conditions also sufficient? A partial answer is given in the next section.

2. Models coded by ascending sequences of skies. First we recall some terminology from Smoryński [10]. If a and b are elements of a model M then we write $a \ll b$ if, for every parameter-free Skolem function F, $F(a)<b$. We say that b codes an ascending sequence of skies, $b \in \mathrm{ASS}(M)$, if b codes a sequence of a non-standard length and, for every $i<j<\mathrm{lh}(b)$, $(b)_i \ll (b)_j$.

If $b \in \mathrm{ASS}(M)$ then

$$M(\omega, b) = \sup_{n\in\omega}(b)_n = \{x \in M: \exists n \in \omega \ x<(b)_n\}.$$

It is not difficult to check that if M is recursively saturated then, for every $b \in \mathrm{ASS}(M)$, $M(\omega, b) \prec_e M$ and $M(\omega, b)$ is recursively saturated.

2.1. THEOREM (Smoryński [10]). *If M is countable and recursively saturated then for every $a, b \in \mathrm{ASS}(M)$*

$$(M, M(\omega, a)) \cong (M, M(\omega, b)). \quad \blacksquare$$

As a corollary we have the following theorem.

2.2. THEOREM. *If M and N are countable recursively saturated similar models, $a \in \mathrm{ASS}(M)$, $b \in \mathrm{ASS}(N)$, then*

$$(M, M(\omega, a)) \cong (N, N(\omega, b)).$$

Proof. Let f be an isomorphism of M with N and let $c = f(a)$. Then by 2.1 $(N, N(\omega, b)) \cong (N, N(\omega, c))$ since $c \in \mathrm{ASS}(N)$, but $(M, M(\omega, a)) \cong (N, N(\omega, c))$ and the result follows. $\blacksquare$

2.3. COROLLARY. *If M, N, a and b are as in 2.2 then the set of (M, N)-isomorphisms of $M(\omega, a)$ with $N(\omega, b)$ is non-empty.* $\blacksquare$

Now we will refine Smoryński's arguments (cf [10] Theorem 2.4) to prove a stronger result. First we will prove a basic back and forth lemma. Recall that if $I \subseteq_e M$ then we say that *I has cofinality ω in M* if, for some $b \in M$, $I = \sup_{n\in\omega}(b)_n$ (b is not assumed to be in $\mathrm{ASS}(M)$).

2.4. LEMMA. *Let M and N be recursively saturated similar models and let $M_0 \prec_e M$ and $N_0 \prec_e N$ have cofinality ω. Suppose f is an (M, N)-isomorphism of M_0 with N_0 and for two sequences of parameters $\bar a \in M$, $\bar b \in N$ and for all formulae φ with an appropriate number of free variables we have*

$$\forall x \in M_0,\ M \models \varphi(\bar a, x) \text{ iff } N \models \varphi(\bar b, f(x)).$$

Then for every $a \in M$ there exists a $b \in N$ such that for all formulae φ with an appropriate number of free variables we have

$$\forall x \in M_0,\ M \models \varphi(\bar a, a, x) \text{ iff } N \models \varphi(\bar b, b, f(x)).$$

Proof. Let $\alpha \in M$ and $\beta \in N$ be such that $M_0 = \sup_{n\in\omega}(\alpha)_n$ and $N_0 = \sup_{n\in\omega}(\beta)_n$. Since f is an (M, N)-isomorphism, we may assume that $f((\alpha)_n) = (\beta)_n$ (this assumption is not essential for the proof: we make it for notational convenience).

For a given $a \in M$ let $s(v)$ be the set of formulae of the form

$$\forall x<(\alpha)_n \varphi(\bar a, a, x) \leftrightarrow \langle \ulcorner\varphi\urcorner, x\rangle \in D_v,$$

where φ is a formula with an appropriate number of free variables, $\ulcorner\varphi\urcorner$ is a Gödel number of φ and $n \in \omega$. It is clear that $s(v)$ is recursive and finitely realizable in M; let γ be its realization. Since f is an (M, N)-isomorphism, the set $f * (D_\gamma^M \cap M_0)$ must be coded in N; let δ be its code.

Now we consider the set $t(v)$ of foımulae of the form

$$\forall x<(\beta)_n \varphi(\bar b, v, x) \leftrightarrow \langle \ulcorner\varphi\urcorner, x\rangle \in D_\delta,$$

where φ has an appropriate number of free variables and $n \in \omega$. We claim that $t(v)$ is finitely realizable in N. Take φ and $n \in \omega$. There are $\gamma_1 \in M_0$ and $\delta_1 \in N_0$ such that

$$M \models \forall x<(\alpha)_n \langle \ulcorner\varphi\urcorner, x\rangle \in D_\gamma \leftrightarrow \langle \ulcorner\varphi\urcorner, x\rangle \in D_{\gamma_1},$$
$$N \models \forall x<(\beta)_n \langle \ulcorner\varphi\urcorner, x\rangle \in D_\delta \leftrightarrow \langle \ulcorner\varphi\urcorner, x\rangle \in D_{\delta_1}.$$

By the definition of δ we can choose γ_1, δ_1 such that $f(\gamma_1) = \delta_1$. Observe that the following two statements are equivalent:

$$N \models \exists v \forall x<(\beta)_n \varphi(\bar{b}, v, x) \leftrightarrow \langle \ulcorner\varphi\urcorner, x\rangle \in D_{\delta_1},$$
$$M \models \exists v \forall x<(\alpha)_n \varphi(\bar{a}, v, x) \leftrightarrow \langle \ulcorner\varphi\urcorner, x\rangle \in D_{\gamma_1},$$

and the latter is obviously true, which proves our claim.

To finish the proof notice that any element b realizing $t(v)$ in N satisfies the condition of our lemma. ■

2.5. Corollary. *Let M, N, M_0, N_0 be as in 2.4. If M and N are countable, then every (M, N)-isomorphism of M_0 with N_0 can be extended to an isomorphism of M with N.*

Proof. By the standard back and forth construction. ■

The above corollary is still not sufficient for our subsequent applications. What we need is the following theorem:

2.6. Theorem. *Let M, N, M_0, N_0 be as in 2.4 with M and N countable. If $a \in \mathrm{ASS}(M)$ and $b \in \mathrm{ASS}(N)$ are such that $M_0<(a)_0$ and $N_0<(b)_0$ then every (M,N)-isomorphism of M_0 with N_0 can be extended to an isomorphism of $(M, M(\omega, a))$ with $(N, N(\omega, b))$.*

Proof. First we will construct a sequence d_n: $n \in \omega$ of elements of N such that, for every formula φ and every $n \in \omega$,

$$\forall x \in M_0,\ M \models \varphi((a)_0, \ldots, (a)_n, x) \text{ iff } N \models \varphi(d_0, \ldots, d_n, f(x))$$

and $(b)_n<d_n<(b)_{n+1}$.

If we have found $d_0, \ldots, d_n$ with the above properties then we proceed as follows. Let $t(v)$ be the type from the proof of 2.4 corresponding to $\bar{a} = (a)_0, \ldots, (a)_n$, $\bar{b} = d_0, \ldots, d_n$ and $a = (a)_{n+1}$. Parameters β and δ can be chosen arbitrarily low in $N-N_0$, and so we may assume that $\beta, \delta<d_0$ (For the 0-step in our construction observe that we can change β and δ after d_0 realizing $t(v)$ has been found.)

Now, since $(b)_{n+1} \ll (b)_{n+2}$, we can find a recursively saturated model $K \prec_e N$ such that $(b)_{n+1} \in K<(b)_{n+2}$. So the type $t(v)$ can be realized also in K and, since $(a)_n \ll (a)_{n+1}$, for any element d' realizing $t(v)$ in K we must also have $d_n \ll d'$. But then it is not difficult to show that $t(v)$ can be realized arbitrarily high in K (cf [10]), and so finally we can get d_{n+1} realizing $t(v)$ such that $(b)_{n+1}<d_{n+1}<(b)_{n+2}$.

To continue the proof of our theorem let us denote a code of a set

$$\{z\colon \exists i<y\, z = (x)_i\}$$

by $x \restriction y$ and for every formula $\varphi(v)$ let $\varphi^*(v, w)$ be $\varphi(v \restriction w)$.

Again let $\beta \in N$ be such that $N_0 = \sup_{n \in \omega} (\beta)_n$ and let $\delta \in N$ be as in the proof of 2.4 corresponding to a with $\bar{a} = \bar{b} = \emptyset$. Let $u(v, w)$ be the set of formulae of the form:

1. $\forall x<(\beta)_n\ \varphi^*(v, w, z) \leftrightarrow \langle \ulcorner\varphi^*\urcorner, \langle x, w\rangle\rangle \in D_\delta$, where $\varphi(v)$ is a formula with an appropriate number of free variables and $n \in \omega$.
2. $n<\omega<c_1$, where $n \in \omega$ and c_1 is a fixed non-standard element of N_0.
3. $(b)_n<(v)_n<(b)_{n+1}$, where $n \in \omega$.

By the existence of d_n's, $u(v, w)$ is finitely realizable in N: let d_1 and e realize it. Now if we put $c = f^{-1}(e)$ and $d = d_1 \restriction e$ then we have

$$\forall x \in M_0,\ M \models \varphi(a \restriction c, x) \text{ iff } N \models \varphi(d, f(x)).$$

To finish the proof we enumerate $M-(M_0 \cup \{a \restriction c\})$ and $N-(N_0 \cup \{d\})$, and using the standard back and forth procedure together with Lemma 2.4 we construct an isomorphism g of M with N extending f such that $g(a \restriction c) = d$.

Since $M(\omega, a) = \sup_{n \in \omega} (a \restriction c)_n$ and $N(\omega, b) = \sup_{n \in \omega} (d)_n$, it also follows that for any $x \in M$

$$x \in M(\omega, a) \quad \text{iff} \quad g(x) \in N(\omega, b),$$

which completes the proof. ■

2.7. Corollary. *Let M and N be countable recursively saturated similar models. Suppose $a_n \in \mathrm{ASS}(M)$, $b_n \in \mathrm{ASS}(N)$, for $n \in \omega$ are such that*

$$M = \sup_{n \in \omega} a_n, \quad N = \sup_{n \in \omega} b_n$$
$$M(\omega, a_n) \subseteq M(\omega, a_{n+1}) \quad \text{and} \quad N(\omega, b_n) \subseteq N(\omega, b_{n+1}).$$

Then

$$(M, \{M(\omega, a_n)\}_{n \in \omega}) \cong (N, \{N(\omega, b_n)\}_{n \in \omega}).$$

It would be nice if in the above results models of the form $M(\omega, a)$ could be replaced by recursively saturated models with cofinality ω. As has been pointed out to me by Henryk Kotlarski, this can be done at least when ω is strong in M, since in this situation if $M_0 \prec_e M$ is recursively saturated and has cofinality ω in M then there is an $a \in \mathrm{ASS}(M)$ such that $M_0 = M(\omega, a)$. ([1])

Briefly, the proof of this goes as follows. With the help of a suitable satisfaction class on M (see Section 4 for remarks about satisfaction classes) we may formalize a definition of the following function $f\colon M^2 \to M$:

$f(x, y)$ = the maximal number i such that for all Skolem functions $F<i$, $F(x)<y$.

([1]) In fact it turns out that the last statement is true only when ω is strong in M.

Now assume that $b \in M$ is such that $M_0 = \sup_{n\in\omega} (b)_n$. Since ω is strong in M it is possible to find a non-standard $d \in M$ such that for all $n \in \omega, f((b)_n, (b)_{n+1})$ is standard iff $f((b)_n, (b)_{n+1}) < d$. This allows us to define a subsequence c_n: $n \in \omega$ of $(b)_n$: $n \in \omega$, coded in M such that, for all $n \in \omega$, $c_n \ll c_{n+1}$. Then, for $c \in \mathrm{ASS}(M)$ such that for all $n \in \omega$, $(c)_n = c_n$ we have $M_0 = M(\omega, c)$.

3. Main theorem and an open problem. Now we are ready to give a quick proof of the main theorem. First let us mention a lemma which shows that for any recursively saturated model M the set $\mathrm{ASS}(M)$ is non-empty.

3.1. Lemma. *If M is a recursively saturated model then for every $a \in M$ there is a $b \in \mathrm{ASS}(M)$ such that $a < (b)_0$.*

Proof. The lemma follows from the fact that a recursive type

$$t(v) = \{(v)_n \ll (v)_{n+1}\colon n \in \omega\} \cup \{(v)_0 > a\}$$

is finite) -lizable in M. ■

Recall that a model M is said to be ω_1-like if M is of power ω_1 but every initial segment of M is countable.

3.2. Theorem. *Let M and N be ω_1-like recursively saturated models. The following are equivalent:*

1. *M and N are similar,*
2. *$M \equiv_{\infty\omega_1} N$.*

Proof. Since every two $L_{\infty\omega_1}$-elementarily equivalent models are similar, all we have to do is to show that 1 implies 2.

Let M and N be ω_1-like recursively saturated and similar. According to Lemma 3.1 we can find two sequences $\{a_\xi\}_{\xi<\omega_1}$ and $\{b_\xi\}_{\xi<\omega_1}$ such that, for every $\xi < \omega_1$, $a_\xi \in \mathrm{ASS}(M)$, $b_\xi \in \mathrm{ASS}(N)$, and if $M_\xi = M(\omega, a_\xi)$, $N_\xi = N(\omega, b_\xi)$ then $M_\xi \prec_e M_{\xi+1}$, $N_\xi \prec_e N_{\xi+1}$ and $M = \bigcup_{\xi<\omega_1} M_\xi$, $N = \bigcup_{\xi<\omega_1} N_\xi$.

For $\xi < \omega_1$ let G_ξ be the set of all (M, N)-isomorphisms of M_ξ with N_ξ. All M_ξ's and N_ξ's are countable; hence, by the results of the preceding section, M_ξ's, N_ξ's and G_ξ's satisfy the conditions of Kuekers theorem, and so we are done. ■

A few words should be said about arbitrary models of power ω_1. Here we only have one negative result. Smoryński observed in [9] that for any countable recursively saturated model M and any initial segment I of M which is closed under addition and multiplication there is a (countable) recursively saturated model N such that $M \prec_c N$ (N is a cofinal extension of M) and I is the greatest common initial segment of M and N. Starting from M and I and repeating this ω_1-times, we will obtain a recursively saturated model N of power ω_1 such that $M \prec_c N$ and I is the greatest countable segment of N. Starting from two different non-standard initial segments I_1 and I_2 we will obtain two similar models N_1 and N_2. Now it is fairly easy to write down a formula φ of $L_{\infty\omega_1}$ such that in every model N φ defines either the greatest countable segment of N, if such a segment exists, or the ω_1-like part of N.

So if $N_1 \equiv_{\infty\omega_1} N_2$ then $I_1 \equiv_{\infty\omega_1} I_2$ and, since I_1 and I_2 are countable, $I_1 \cong I_2$. But it is well known (cf [2]) that in any countable model M there is a continuum of pairwise non-elementary equivalent initial segments which are models of PA.

Summing this up, we get the following theorem

3.3. Theorem. *For any countable model M of PA there is a continuum of pairwise non-$L_{\infty\omega_1}$-elementarily equivalent, recursively saturated similar elementary extensions of M of power ω_1.*

To end this section let us mention the following open problem.

Befoге I noticed that models coded by ascending sequences of skies fit perfectly the proof of 3.2. I had been considering another natural class of isomorphic pairs of recursively saturated models, namely recursively saturated pairs. By the resplendency argument, in every countable recursively saturated model M there are pairs of submodels, say M_0 and M_1, such that $M_0 \prec_e M_1 \prec_e M$ and the structure (M_1, M_0) is recursively saturated.

If M and N are recursively saturated, ω_1-like similar models, then it is possible to write M and N as sums of elementary chains of their recursively saturated submodels $\{M_\xi\}_{\xi<\omega_1}$ and $\{N_\xi\}_{\xi<\omega_1}$ such that for every $\nu < \xi < \omega_1$:

1. $M_\xi \prec_e M_{\xi+1} \prec_e M$, $N_\xi \prec_e N_{\xi+1} \prec_e N$,
2. (M_ξ, M_ν), (N_ξ, N_ν) are recursively saturated,
3. $(M_\xi, M_\nu) \equiv (N_\xi, N_\nu)$;

hence

4. $(M_\xi, M_\nu) \cong (N_\xi, N_\nu)$.

A proof similar to the proofs of 2.4 and 2.5 gives the following theorem.

3.4. Theorem. *If M and N are countable, recursively saturated similar models, $M_0 \prec_e M$, $N_0 \prec_e N$, and structures (M, M_0), (N, N_0) are recursively saturated, then every (M, N)-isomorphism of M_0 with N_0 can be extended to an isomorphism of M with N.*

The existence of an (M, N)-isomorphism of M_0 with N_0 is equivalent to the elementary equivalence of (M, M_0) to (N, N_0). ■

Open Problem. Let M and N be ω_1-like recursively saturated similar models. Is it possible to write M and N as sums of elementary chains of submodels $\{M_\xi\}_{\xi<\omega_1}$ and $\{N_\xi\}_{\xi<\omega_1}$ such that for all $\nu < \xi < \omega_1$:

1. $M_\xi \prec_e M_{\xi+1} \prec_e M$, $N_\xi \prec_e N_{\xi+1} \prec_e N$,
2. (M_ξ, M_ν), (N_ξ, N_ν) are recursively saturated,
3. $(M_\xi, M_\nu) \equiv (N_\xi, N_\nu)$ and every (M, N)-isomorphism of M_ν with N_ν can be extended to an (M, N)-isomorphism of M_ξ with N_ξ.

Observe that structures of the form $(M, M(\omega, b))$ are obviously not recursively saturated.

4. Non-isomorphic similar models. As was noted in the introduction, there are non-isomorphic ω_1-like recursively saturated similar models. Now we show how to construct a continuum of such models. For this purpose we will apply some results concerning satisfaction classes on models of arithmetic. We will just list the relevant results omitting their proofs, since they can be found in the literature and also are rather easy, and so may be treated as exercises.

A subset X of a model M is said to be *inductive in* M if the structure (M, X) satisfies the induction schema in the language of M with an additional predicate symbol to be interpreted in (M, X) as X. A subset X of M is called *piecewise definable* if, for every a in M, $X \cap \{x \in M: x < a\}$ is definable (with parameters) in M. Clearly every inductive set is piecewise definable.

4.1. PROPOSITION. *Every non-standard model possessing an inductive satisfaction class is recursively saturated.* ■

4.2. PROPOSITION. *If S_1, S_2 are satisfaction classes over M and the induction schema holds in the structure (M, S_1, S_2) then $S_1 = S_2$.* (2) ■

4.3. THEOREM. *If M is a countable model and S is an inductive satisfaction class over M, then there is an ω_1-like model N and an inductive satisfaction class $\bar{S}$ over N such that $M \prec_e N$, $(M, S) \prec (N, \bar{S})$ and every piecewise definable subset of N is definable in $(N, \bar{S})$.* ■

The next theorem was suggested to me by Henryk Kotlarski.

4.4. THEOREM. *For every countable recursively saturated model M there is a continuum of inductive satisfaction classes over M such that for any two such classes S_1 and S_2*

$$(M, S_1) \not\equiv (M, S_2)\,. \blacksquare$$

The proof of the above theorem follows the main line of the proof of the well-known Jensen–Ehrenfeucht theorem and will appear in [4]. Finally we have the main result of this section.

4.5. THEOREM. *For every countable recursively saturated model M there is a continuum of pairwise non-isomorphic recursively saturated ω_1-like elementary and extensions of M.*

Proof. Let A be the family of satisfaction classes over M given by 4.4. For any $S \in A$ we produce $(N, \bar{S})$ as in 4.3. We claim that, for different $S_1, S_2 \in A$, the models N_1 and N_2 corresponding to them are non-isomorphic (but it is clear that they are similar). Suppose on the contrary that f is an isomorphism of N_1 with N_2. Consequently $f * \bar{S}_1 = \bar{S}_3$ is an inductive satisfaction class on N_2, but then, by 4.3, $\bar{S}_3$ is definable in $(N_2, \bar{S}_2)$; hence the structure $(N_2, \bar{S}_2, \bar{S}_3)$ satisfies the induction schema. So by 4.2 we have $\bar{S}_2 = \bar{S}_3$, which is impossible. ■

(2) This proposition is true only when S is a full satisfaction class on M, thus the proof of Theorem 4.5 given below is valid for M's which have full inductive satisfaction classes. A full proof of even a more general version of 4.5 will appear in [5].

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