On indecomposable subcontinua of surfaces

by

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Abstract. Let $X$ be an indecomposable subcontinuum of a surface. It is proved that the union of all external components of $X$ is an $F_\sigma$-set of the first category in $X$. This result generalizes an analogous theorem of Krasinkiewicz.

1. By a surface we shall mean a compact connected 2-dimensional manifold (without boundary). Let $X$ be an indecomposable continuum. By a component of a point $x \in X$ we understand the union of all proper subcontinua of $X$ containing $x$. Consider $X$ as a subset of a surface $M$. A component $C$ of $X$ is called external if there exists a continuum $L \subseteq M$ such that $L \cap C \neq \emptyset$, $L \cap C$, and $L$ does not intersect all components of $X$. A component of $X$ which is not external is called internal [2].

If $M = S^2$, then by a theorem of Krasinkiewicz we have:

**Theorem ([3], 3.2).** The union of all external components of $X$ is an $F_\sigma$-set of the first category in $X$.

In this paper we generalize this result to all surfaces (see Section 4).

2. The homology (cohomology) theory used in this section is the singular one with coefficients in the $R$-module $Z_2$, where $R = Z_2$. The $Z_2$-module $H_q(X, A; Z_2)$ will be denoted briefly by $H_q(X, A)$. Similar notation applies to cohomology.

For a surface $M$ we have

$$H_q(M) = \begin{cases} Z_2 & \text{for } q = 0 \text{ and } q = 2, \\ Z_2^{2 - \chi(M)} & \text{for } q = 1, \\ 0 & \text{for } q > 2, \end{cases}$$

(*)

where $\chi(M)$ is the Euler characteristic of $M$ (see [1], p. 141). The number $c(M) = 2 - \chi(M)$ is called the connectivity of $M$.

A continuum homeomorphic to a 1-dimensional polyhedron is called a graph. A simply connected graph is called a tree.

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* This paper contains some results from the author's M. Sc. thesis, which was written during 1961/1962 under direction of Dr. J. Krasinkiewicz at the Division of the Warsaw University in Bialystok.
For a graph $G$ we have
\[
H^n(G) = \begin{cases} 
Z_2 & \text{for } q = 0, \\
Z_2^{-1} & \text{for } q = 1, \\
0 & \text{for } q > 1, 
\end{cases}
\]
(see [1], p. 141). Put $c(G) = 1 - x(G)$ and call $c(G)$ the connectivity of $G$. If $G$ is a tree then $c(G) = 0$.

Now suppose the graph $G$ is a subset of the surface $M$. Since $G \in \text{ANR}$, from the Lefschetz duality theorem (see [3], p. 290 and p. 297) we get
\[H^n(M \setminus G) = H^{n+1}(M, G),\]
for $q \geq 0$ (an isomorphism of $Z_2$-modules).

In particular $\dim_{Z_2}(H^q(M, G))$ is equal to the number of components of the set $M \setminus G$.

All results of this paper heavily depend on the following

2.2. LEMMA. Let $n$ be the number of components of the set $M \setminus G$. Then we have
\[c(G) - c(M) + 1 \leq n \leq c(G) + 1.\]

Proof. Let $i: G \hookrightarrow M$ and $j: M \setminus i(G) \rightarrow M$ be the exact cohomology sequence of $(M, G)$:
\[\begin{CD}
\cdots @>>> H^q(M, G) @>>> j^* H^q(M) @>>> H^q(M) @>>> H^{q+1}(M, G) @>>> j_* H^q(M) @>>> \cdots
\end{CD}
\]
(see [5], p. 240). By $(\ast)$ and $(\ast\ast)$ a portion of the sequence looks like this:
\[Z_2 \xrightarrow{j_*} Z_2 \xrightarrow{i^*} H^q(M) \xrightarrow{c} H^q(M) \xrightarrow{j_*} Z_2 \xrightarrow{i^*} 0.
\]
This is a sequence of $Z_2$-modules and $Z_2$-morphisms.

For a $Z_2$-module $H$ set $r(H) = \dim_{Z_2} H$. The exactness of the sequence implies the following relations:
\[r(H^q(M, G)) = 1 + r(i_* i^*) \leq 1 + c(G),
\]
\[r(i_* i^*) = c(G) - r(i_* i^*) \leq c(M) - c(G).
\]
Hence $1 + c(G) - c(M) \leq r(H^q(M, G)) \leq 1 + c(G)$. Applying 2.1, we get the conclusion of 2.2.

2.3. COROLLARY. If $c(G) > c(M)$, then $G$ separates $M$.

2.4. COROLLARY. No tree in $M$ separates $M$.

3. Separation of surfaces by certain collections of subcontinua. Throughout this section let $M$ denote a closed connected surface. If $D$ is a disk, then by $\overline{D}$ we denote its interior and by $\partial D$ its boundary. By an $n$-od we mean a continuum homeomorphic to the cone over an $n$-point set. We say that an $n$-od $T \subset D$ is regularly embedded in $D$ provided $T \cap \partial D$ is the set of the endpoints of $T$.

3.1. LEMMA. Let $D_1$ and $D_2$ be disks in $M$ with disjoint interiors. Let $q = c(M) + 2$ and let $T_i$ be a $q$-od regularly embedded in $D_i$, $i = 1, 2$. Let $(a_i)$ be the endpoints of $T_i$ and let $(b_i)$ be the endpoints of $T_2$. Let $A_1, \ldots, A_q$ be a collection of mutually disjoint subcontinua of $M \setminus (D_1 \cup D_2)$ such that $A_j \cap A_i = \emptyset$, $j = 1, \ldots, q$. Then $B = T_1 \cup T_2 \cup A_1 \cup \ldots \cup A_q$ separates $M$ between two points of $D_1 \cap \overline{D_2}$.

Proof. There is a finite set $E \subset D_1 \cap \overline{D_2}$ which has exactly one point in common with each component of $\overline{D_1 \cap \overline{D_2}}$. Let us observe that the lemma holds true in the special case where

1. $A_j$ is an arc $a_jb_j$ (or the point $a_j$ in case $a_j = b_j$, $j = 1, \ldots, q$.

In fact, under this assumption the set $B$ is homeomorphic to the suspension over a $q$-point set. It follows that each point $x \in M \setminus B$ can be joined with $e_j \in E$ by an arc lying in $M \setminus B$. Moreover, $B$ is a graph such that $c(B) = c(M) + 1$. By 2.3 there exist $x, y \in M \setminus B$ such that $B$ separates $M$ between $x$ and $y$. Then $e_i \neq e_j$, which completes the proof.

Let us return to the general case. Since $M \setminus (D_1 \cup D_2)$ is a locally connected continuum, there exist $q$ sequences $\{A_{j,n}\}_{n=1}^{\infty}$ are contained in this neighborhood,

2. $A_{j,n} \subset M \setminus (D_1 \cup D_2)$ is an arc joining $a_j$ and $b_j$ (and is the point $a_j$ in the case $a_j = b_j$),

3. for each neighborhood of $A_j$ almost all sets $\{A_{j,n}\}_{n=1}^{\infty}$ are contained in this neighborhood,

4. $A_{j,n} \cap A_{j',n} = \emptyset$ for $j \neq j'$ and each $n$.

Put $B_n = T_1 \cup T_2 \cup A_{j,1} \cup \ldots \cup A_{j,q}$. According to (2) and (6) we may apply the special case (1) to the set $B_n$. Hence $B_n$ separates $M$ between two points $e_j, e_j' \in E$. But $E$ is a finite set, hence there exist $e, e' \in E$ such that $(e_j, e_j') = (e, e')$ for infinitely many $n$. We claim that $B$ separates $M$ between $e$ and $e'$. Just suppose there is an arc $ee' \subset M \setminus B$. Then by (3) there is an index $n$ such that $(e_n, e_n') = (e, e')$ and $ee' \subset M \setminus B_n$. This is a contradiction, which completes the proof.

3.2. COROLLARY. Let $D_1$, $D_2$ be disks in $M$ with disjoint interiors. Let $q = 2(c(M) + 2)$. Let $\{a_1', \ldots, a_q'\} \subset D_1$ and let $\{b_1', \ldots, b_q'\} \subset D_2$ and suppose there is a collection $F_1, \ldots, F_q$ of mutually disjoint subcontinua of $M \setminus (D_1 \cup D_2)$ such that $a_i' \notin F_j$, $b_j' \notin F_j$, $j = 1, \ldots, q$. Then there exist a collection
\[\{A_1, A_1', A_2, \ldots, A_{c(M)+2}\} \subset \{F_j\}
\]
and continua $T_i = D_i$, $i = 1, 2$, such that
\[T_1 \cap \{A_1, A_1', A_2, \ldots, A_{c(M)+2}\} = \emptyset.
\]
and $T_1 \cup T_2 \cup A_1 \cup \ldots \cup A_{c(M)+2}$ separates $M$ between $A'$ and $A''$.

Proof. Let $a_1 < a_2 < \ldots < a_q$ be a cyclic ordering of the set $\{a_1', \ldots, a_q'\}$ on the simple closed curve $D_1$. Let $A_1'$ be the unique element of $\{F_i\}$ such that $a_1' \in A_1'$, where $k = 1, \ldots, c(M) + 2$. Let $b_j'$ be the unique element of $\{b_j'\}$ belonging to $A_j'$. Let $T_1$ be a $q$-od with endpoints $\{a_i\}$ regularly embedded in $D_1$ and let $T_2$ be an analogous $q$-od in $D_2$ corresponding to the set $\{b_j\}$. By applying 3.1 to the set

\[A = \text{Fundamenta Mathematicae} \text{ CXXXIII, } 1.\]
The following two lemmas are variations of a lemma due to Mazurkiewicz (see [4]).

3.3. Lemma. Let \( Y \) be a continuum irreducible between \( a \) and \( b \). Let \( F \) and \( D \) be closed subsets of \( Y \) such that \( a \in F \) and \( b \in D \). Let \( Y \) be an open nonvoid subset of \( Y \) disjoint from \( F \). Then there exists a continuum \( A \subseteq D \) intersecting \( Y \) and irreducible between points \( a', b' \in F \).

**Notation.** We write \( A \in (F, D, V) \) to indicate that \( A \) is a continuum satisfying the conclusion of 3.3.

3.4. Lemma. Let \( Y \) be as in 3.3. For \( F_1, F_2 \) be two closed disjoint subsets of \( Y \) such that \( a \in F_1 \) and \( b \in F_2 \). Let \( D \) be a closed subset of \( Y \) such that \( F_1 \cup F_2 \cup D = Y \). Let \( V \) be an open nonvoid subset of \( Y \) separating \( Y \) between \( F_1 \) and \( F_2 \). Then there exists a continuum \( A \subseteq D \) intersecting \( Y \) and irreducible between \( a' \in F_1 \) and \( b' \in F_2 \).

**Notation.** We write \( A \in (F_1, F_2, D, V) \) to indicate that \( A \) is a continuum satisfying the conclusion of 3.4.

3.5. Lemma. Let \( X \) be an indecomposable continuum, let \( C \) be a component of \( X \), let \( F \) be a closed subset of \( X \) with finitely many components and let \( U \) and \( V \) be two nonvoid open subsets of \( X \). Then there exists a continuum \( C \) disjoint from \( F \) and intersecting both \( U \) and \( V \).

**Proof.** Let \( F \) be a nonvoid open subset of \( X \) such that \( F \subseteq V \) (the set \( X \) is nonvoid because \( F \) is nowhere dense). Let \( F^* \) be the union of components of \( X\setminus F \) each of which meets \( F \). Then \( F^* \) is a closed set with empty interior. So \( U \cap F^* \neq \emptyset \).

Let \( E \) be a component of \( X \setminus F^* \) which intersects \( C \setminus F^* \). Then \( E \) is a subcontinuum of \( C \) intersecting both \( U \) and \( V \) and disjoint with \( F \).

In the following two lemmas we denote by \( X \) an indecomposable continuum lying in \( M \).

3.6. Let \( D \) and \( D' \) be disks in \( M \) with disjoint interiors. Assume \( X \cap D \neq \emptyset \neq X \cap D' \). Let \( R \) be an uncountable collection of components of \( X \). Then there exists a closed set \( F \subseteq R \) (with at most \( c(M)+2 \) components) which separates \( M\setminus D' \) between two points of \( D \cap X \).

**Proof.** Let \( C \in R \). Using a standard trick (see [4] and [2]), one can construct a continuum \( B_2 \subseteq C \setminus D' \) irreducible between \( a_2 \) and \( b_2 \in D' \), which intersects \( D \) (first we construct a continuum \( Y \subseteq C \) irreducible between two points from \( D' \) which intersects \( D \) and then apply 3.3 to the triple \( (Y \setminus D', Y \setminus D, Y \setminus D) \)). Since \( D' \) intersects all components of \( X \), we have \( D' \not\subseteq X \). Hence there is an arc \( L \subseteq D' \) which contains \( D' \cap X \). Thus \( a_2 \), \( b_2 \in L \) for each \( C \in R \). Without loss of generality we may assume that \( a_2 < d < b_2 \) for each \( C \in R \), where \( < \) is one of the natural orders on \( L \). Since \( R \) is uncountable, there is a point \( e \in L \) and an uncountable subcollection \( R' \subseteq R \) such that \( a_2 < d < b_2 \) for each \( C \in R' \). Let \( e \in D' \setminus L \) and let \( e \) be an arc regularly embedded in \( D' \). Then there exist two disks \( D_1, D_2 \) such that \( D' = D_1 \cup D_2 \), \( D_1 \cap D_2 = \{ e \} \), \( a_2 \in D_2 \) and \( b_2 \in D_1 \) for each \( C \in R' \). Applying 3.2 to \( D_1, D_2 \) and \( 2(c(M)+2) \) elements of \( \{ B_2 : C \in R \} \) we obtain a collection \( \{ A', A'', A_1, \ldots, A_{c(M)+2} \} \) and continua \( T \subseteq D_1 \) such that

\[ T \cap (A_1 \cup \ldots \cup A_{c(M)+2}) = \emptyset \]

and \( T \cap (A_1 \cup \ldots \cup A_{c(M)+2}) = \emptyset \). Since \( A' \not\subseteq \emptyset \not\subseteq A'' \), setting \( F = A_1 \cup \ldots \cup A_{c(M)+2} \) we obtain a closed set with the required properties.

The following lemma is of fundamental importance in the next section.

3.7. Lemma. Let \( D \), \( D' \) and \( X \) be as in 3.6. Let \( C \) be a component of \( X \). Then there exists a continuum \( E \subseteq C \) such that \( E \setminus D \) separates \( M \setminus D \) between two points of \( D' \cap X \).

**Proof.** Applying 3.6 to the collection of all components of \( X \) distinct from \( C \), we infer that there exists a closed set \( F' \) disjoint with \( C \) which separates \( M\setminus D' \) between \( x, y \in D' \cap X \). Hence there exist two disks \( D_1, D_2 \) such that

\begin{enumerate}
  \item \( D_1 \cup D_2 = D' \), \( a_2 \in D_1 \) and \( b_2 \in D_2 \).
  \item \( D_1 \cap D_2 = \emptyset \), \( i = 1, 2 \),
  \item \( F' \) separates \( D' \cap D' \) between \( D_1 \) and \( D_2 \).
\end{enumerate}

By a repeated application of 3.5 (using (2)) we construct a sequence \( C_1, C_2, \ldots, C_{c(M)+3} \) of mutually disjoint subcontinua of \( C \) such that \( C_j \) meets both \( D_1 \) and \( D_2 \). By (3) we have \( D_1 \cap D_2 = \emptyset \). Hence by 3.4 we may also assume that

\begin{enumerate}
  \item \( C_j \cap D' = \emptyset \).
  \item \( C_j \cap D' = \emptyset \).
  \item \( C_j \cap D' = \emptyset \).
\end{enumerate}

By 3.2 and (4) there is a collection \( \{ A', A'', A_1, \ldots, A_{c(M)+2} \} \) such that \( F = A_1 \cup \ldots \cup A_{c(M)+2} \) separates \( M \setminus D_1 \cup D_2 \) between \( A' \) and \( A'' \). Hence from (1) and (5) it follows that \( F \) separates \( M \setminus D_1 \cup D_2 \) between two points \( a, b \in (D' \cap X) \cap C \).

There is a continuum \( E \subseteq C \) containing \( F \). Thus \( E \) satisfies the conclusion of our lemma.

4. On indecomposable subcontinua of surfaces.

4.1. Theorem. The union of all external components, i.e., the set \( \bigcup E \), is of the first category in \( X \).
Proof. We follow the ideas used in the proof of the Main Theorem. Most of the argument is the same, and so we shall not repeat it here. An analysis of that proof shows that the property of \( M \) contained in our Lemma 3.7 is all we need to extend the proof to the general case. This completes the proof.

Now we make a comment on the possibility of extending the properties of internal components established in [3] (for indecomposable continua lying in \( S^3 \)) to the general case. An easy inspection shows that the properties listed in Section 3 of [3], except 3.6, follow from property 3.6, from the Main Theorem [2] (4.1 extends it to all surfaces) and from the general results on indecomposable continua.

Now we are going to state a property of indecomposable subcontinua of surfaces which is a generalization of 3.6 from [3] (proved only for the sphere \( S^3 \)). We can prove the generalization proceeding in the same way as in the proof of 3.6 in [3], applying our Lemma 3.7 in place of 2.2 of [3].

4.2. Lemma. Let \( L \subseteq M \) be a continuum intersecting all components of \( X \) and let \( C \) be an arbitrary component of \( X \). If \( X \neq L_x \), then there exists an open neighborhood \( U \) of \( L \) in \( M \) and a continuum \( A \subseteq C \) which separates \( U \) between two points of \( L \).

Having this lemma, we can extend all the properties 3.1–3.9 from [3] to the general case. In particular, we have the following generalization of 3.2 from [3]:

4.3. Theorem. The set \( \emptyset \subseteq X \) is an \( F \)-set in \( X \). Consequently, the union of internal components of \( X \) is a \( G \)-set dense in \( X \).

References


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\[ L_{\omega_1} \]-elementary equivalence of \( \omega_1 \)-like models of PA

by

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Abstract. We show that two recursively saturated, \( \omega_1 \)-like models of PA are \( L_{\omega_1} \)-elementarily equivalent iff they are elementarily equivalent and have the same standard systems. On the other hand, for every countable model \( M \) of PA we construct a continuum of pairwise non-isomorphic, \( \omega_1 \)-like, recursively saturated, elementary end extensions of \( M \).

Any two recursively saturated models of PA (Peano Arithmetic) are \( L_{\omega_1} \)-elementarily equivalent iff they are elementarily equivalent and have the same standard systems. This result was first noted by Craig Smoryński in [8] but its roots go back to some earlier works of George Wilmers and Alex Wilkie (see [8] for historical remarks).

For the purpose of this paper let us call elementary equivalent models with the same standard systems similar.

The result mentioned above is particularly important in the case of countable models of PA since it says that any two recursively saturated countable similar models are isomorphic. The situation is different when we consider uncountable models and, as usual, the first counterexamples can be found among \( \omega_1 \)-like models. Take for instance a "rather classless" model \( M \) of M. Kaufmann [3] (the existence of a rather classless model is provable in ZFC, cf [7]) and a recursively saturated \( \omega_1 \)-like model similar to \( M \) but not "rather classless", hence not isomorphic to \( M \).

In the last section of this paper we show that there is at least a continuum of pairwise non-isomorphic \( \omega_1 \)-like recursively saturated similar models. But our main theorem says that \( \omega_1 \)-like recursively saturated similar models are still very similar, namely they are \( L_{\omega_1} \)-elementarily equivalent.

1. Preliminaries. We assume the readers acquaintance with the basic properties of recursively saturated models of PA (Smoryński [8] is a perfect survey of this subject). With some minor changes we are going to follow the terminology and notation of [3] and [10].

Models of PA are called shorty models. As usual, \( x \in D_n \) is an abbreviation for an arithmetical formula expressing that \( x \) is an element of a set coded by \( y \) and \( (\gamma)_i \) denotes the \( i \)-th element of this set (in increasing order).

If \( b \) is an element of a model \( M \) then

\[ D_b^n = \{ x \in M : M \models x \in D_n \} . \]