On indecomposable subcontinua of surfaces *

by

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Abstract. Let X be an indecomposable subcontinuum of a surface. It is proved that the union of all external composants of X is an F_σ-set of the first category in X. This result generalizes an analogous theorem of Krasinkiewicz.

1. By a surface we shall mean a compact connected 2-dimensional manifold (without boundary). Let X be an indecomposable continuum. By a composant of a point $x \in X$ we understand the union of all proper subcontinua of X containing x. Consider X as a subset of a surface M. A composant C of X is called *external* if there exists a continuum $L \subset M$ such that $L \cap C \neq \emptyset$, $L \not\subset C$, and L does not intersect all composants of X. A composant of X which is not external is called *internal* [2].

If $M = S^2$, then by a theorem of Krasinkiewicz we have:

THEOREM ([3], 3.2). *The union of all external composants of X is an F_σ-set of the first category in X.*

In this paper we generalize this result to all surfaces (see Section 4).

2. The homology (cohomology) theory used in this section is the singular one with coefficients in the R-module Z_2, where $R = Z_2$. The Z_2-module $H_q(X, A;\ Z_2)$ will be denoted briefly by $H_q(X, A)$. Similar notation applies to cohomology.

For a surface M we have

$$(*)\qquad H^q(M) = \begin{cases} Z_2 & \text{for } q = 0 \text{ and } q = 2\,, \\ Z_2^{2-\chi(M)} & \text{for } q = 1\,, \\ 0 & \text{for } q > 2\,, \end{cases}$$

where $\chi(M)$ is the Euler characteristic of M (see [1], p. 141). The number $c(M) = 2-\chi(M)$ is called the connectivity of M.

A continuum homeomorphic to a 1-dimensional polyhedron is called a graph. A simply connected graph is called a tree.

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For a graph G we have

$$(**)\qquad H^q(G) = \begin{cases} Z_2 & \text{for } q = 0\,, \\ Z_2^{1-\chi(G)} & \text{for } q = 1\,, \\ 0 & \text{for } q > 1\,, \end{cases}$$

(see [1], p. 141). Put $c(G) = 1-\chi(G)$ and call $c(G)$ the connectivity of G. If G is a tree then $c(G) = 0$.

Now suppose the graph G is a subset of the surface M. Since $G \in \mathrm{ANR}$, from the Lefschetz duality theorem (see [5], p. 290 and p. 297) we get

2.1. Lemma. *$H_q(M \setminus G) = H^{2-q}(M, G)$, for $q \geqslant 0$ (an isomorphism of Z_2-modules). In particular $\dim_{Z_2} H^2(M, G)$ is equal to the number of components of the set $M \setminus G$.*

All results of this paper heavily depend on the following

2.2. Lemma. *Let n be the number of components of the set $M \setminus G$. Then we have*

$$c(G) - c(M) + 1 \leqslant n \leqslant c(G) + 1\,.$$

Proof. Let $i\colon G \subset M$ and $j\colon M \subset (M, G)$ and consider the exact cohomology sequence of (M, G):

$$\ldots \xrightarrow{\delta_{q-1}} H^q(M, G) \xrightarrow{j_q^*} H^q(M) \xrightarrow{i_q^*} H^q(G) \xrightarrow{\delta_q} H^{q+1}(M, G) \xrightarrow{j_{q+1}^*} \ldots$$

(see [5], p. 240). By (*) and (**) a portion of the sequence looks like this:

$$Z_2^{c(M)} \xrightarrow{i_1^*} Z_2^{c(G)} \xrightarrow{\delta_1} H^2(M, G) \xrightarrow{j_2^*} Z_2 \xrightarrow{i_2^*} 0\,.$$

This is a sequence of Z_2-modules and Z_2-morphisms.

For a Z_2-module H set $r(H) = \dim_{Z_2} H$. The exactness of the sequence implies the following relations:

$$r\big(H^2(M, G)\big) = 1 + r(\operatorname{im}\delta_1) \leqslant 1 + c(G)\,,$$
$$r(\operatorname{im}\delta_1) = c(G) - r(\operatorname{im} i_1^*) \geqslant c(G) - c(M)\,.$$

Hence $1 + c(G) - c(M) \leqslant r\big(H^2(M, G)\big) \leqslant 1 + c(G)$. Applying 2.1, we get the conclusion of 2.2.

2.3. Corollary. *If $c(G) > c(M)$, then G separates M.*

2.4. Corollary. *No tree in M separates M.*

3. Separation of surfaces by certain collections of subcontinua. Throughout this section by M we denote a closed connected surface. If D is a disk, then by $\mathring{D}$ we denote its interior and by $\dot{D}$ its boundary. By an n-od we mean a continuum homeomorphic to the cone over an n-point set. We say that an n-od $T \subset D$ is regularly embedded in D provided $T \cap \dot{D}$ is the set of the endpoints of T.

3.1. Lemma. *Let D_1 and D_2 be disks in M with disjoint interiors. Let $q = c(M)+2$ and let T_i be a q-od regularly embedded in D_i, $i = 1, 2$. Let (a_j) be the endpoints of T_1 and let (b_j) be the endpoints of T_2. Let $A_1, \ldots, A_q$ be a collection of mutually disjoint subcontinua of $M \setminus (\mathring{D}_1 \cup \mathring{D}_2)$ such that $a_j, b_j \in A_j$, $j = 1, \ldots, q$. Then $B = T_1 \cup T_2 \cup A_1 \cup \ldots \cup A_q$ separates M between two points of $\mathring{D}_1 \setminus T_1$.*

Proof. There is a finite set $E \subset \mathring{D}_1 \setminus T_1$ which has exactly one point in common with each component of $\mathring{D}_1 \setminus T_1$. Let us observe that the lemma holds true in the special case where

(1) A_j is an arc $a_j b_j$ (or the point a_j in case $a_j = b_j$), $j = 1, \ldots, q$.

In fact, under this assumption the set B is homeomorphic to the suspension over a q-point set. It follows that each point $x \in M \setminus B$ can be joined with $e_x \in E$ by an arc lying in $M \setminus B$. Moreover, B is a graph such that $c(B) = c(M)+1$. By 2.3 there exist $x, y \in M \setminus B$ such that B separates M between x and y. Then $e_x \neq e_y$, which completes the proof.

Let us return to the general case. Since $M \setminus (\mathring{D}_1 \cup \mathring{D}_2)$ is a locally connected continuum, there exist q sequences $\{A_{jn}\}$, $j = 1, \ldots, q$, $n = 1, 2, \ldots$, such that

(2) $A_{jn} \subset M \setminus (\mathring{D}_1 \cup \mathring{D}_2)$ is an arc joining a_j and b_j (and is the point a_j in the case $a_j = b_j$),

(3) for each neighborhood of A_j almost all sets $\{A_{jn}\}_{n=1}^{\infty}$ are contained in this neighborhood,

(4) $A_{j_1 n} \cap A_{j_2 n} = \emptyset$ for $j_1 \neq j_2$ and each n.

Put $B_n = T_1 \cup T_2 \cup A_{1n} \cup A_{2n} \cup \ldots \cup A_{qn}$. According to (2) and (4) we may apply the special case (1) to the set B_n. Hence B_n separates M between two points $e_n, e_n' \in E$. But E is a finite set, hence there exist $e, e' \in E$ such that $(e_n, e_n') = (e, e')$ for infinitely many n. We claim that B separates M between e and e'. Just suppose there is an arc $ee' \subset M \setminus B$. Then by (3) there is an index n such that $(e_n, e_n') = (e, e')$ and $ee' \subset M \setminus B_n$. This is a contradiction, which completes the proof.

3.2. Corollary. *Let D_1, D_2 be disks in M with disjoint interiors. Let $q = 2\big(c(M)+2\big)$. Let $\{a_1', \ldots, a_q'\} \subset \dot{D}_1$ and let $\{b_1', \ldots, b_q'\} \subset \dot{D}_2$ and suppose there is a collection $F_1, \ldots, F_q$ of mutually disjoint subcontinua of $M \setminus (\mathring{D}_1 \cup \mathring{D}_2)$ such that $a_j', b_j' \in F_j$, $j = 1, \ldots, q$. Then there exist a collection*

$$\{A', A'', A_1, A_2, \ldots, A_{c(M)+2}\} \subset \{F_j\}$$

and continua $T_i \subset D_i$, $i = 1, 2$, such that

$$T_i \setminus (A_1 \cup \ldots \cup A_{c(M)+2}) \subset \mathring{D}_i$$

and $T_1 \cup T_2 \cup A_1 \cup \ldots \cup A_{c(M)+2}$ separates M between A' and A''.

Proof. Let $a_1 < a_2 < \ldots < a_q$ be a cyclic ordering of the set $\{a_1', \ldots, a_q'\}$ on the simple closed curve $\dot{D}_1$. Let A_k be the unique element of $\{F_j\}$ such that $a_{2k} \in A_k$, where $k = 1, \ldots, c(M)+2$. Let b_k be the unique element of $\{b_j'\}$ belonging to A_k. Let T_1 be a $q/2$-od with endpoints $\{a_{2k}\}$ regularly embedded in D_1 and let T_2 be an analogous $q/2$-od in D_2 corresponding to the set $\{b_k\}$. By applying 3.1 to the set

$B = T_1 \cup T_2 \cup A_1 \cup \ldots \cup A_{c(M)+2}$, it is easy to determine A' and A'' such that the conclusion of 3.2 is satisfied.

The following two lemmas are variations of a lemma due to Mazurkiewicz (see [4]).

3.3. LEMMA. *Let Y be a continuum irreducible between a and b. Let F and D be closed subsets of Y such that $a, b \in F$ and $F \cup D = Y$. Let V be a nonvoid open subset of Y disjoint with F. Then there exists a continuum $A \subset D$ intersecting V and irreducible between points $a', b' \in F$.*

NOTATION. We write $A \in (F, D, V)$ to indicate that A is a continuum satisfying the conclusion of 3.3.

3.4. LEMMA. *Let Y be as in 3.3. Let F_1, F_2 be two closed disjoint subsets of Y such that $a \in F_1$ and $b \in F_2$. Let D be a closed subset of Y such that $F_1 \cup F_2 \cup D = Y$. Let V be an open nonvoid subset of Y separating Y between F_1 and F_2. Then there exists a continuum $A \subset D$ intersecting V and irreducible between $a' \in F_1$ and $b' \in F_2$.*

NOTATION. We write $A \in (F_1, F_2, D, V)$ to indicate that A is a continuum satisfying the conclusion of 3.4.

3.5. LEMMA. *Let X be an indecomposable continuum, let C be a composant of X, let F be a closed subset of X with finitely many components and let U and V be two nonvoid open subsets of X. Then there exists a continuum in C disjoint with F and intersecting both U and V.*

Proof. Let W be a nonvoid open subset of $V \setminus F$ such that $\overline{W} \subset V$ (the set $V \setminus F$ is nonvoid because F is nowhere dense). Let F^* be the union of components of $X \setminus W$ each of which meets F. Then F^* is a closed set with an empty interior. So $U \setminus F^* \neq \emptyset$. Let E be a component of $X \setminus W$ which intersects $C \setminus F^*$. Then E is a subcontinuum of C intersecting both U and V and disjoint with F.

In the following two lemmas we denote by X an indecomposable continuum lying in M.

3.6. *Let D and D' be disks in M with disjoint interiors. Assume $X \cap \mathring{D} \neq \emptyset \neq X \cap \mathring{D}'$. Let $\boldsymbol{R}$ be an uncountable collection of composants of X. Then there exists a closed set $F \subset \bigcup \boldsymbol{R}$ (with at most $c(M)+2$ components) which separates $M \setminus \mathring{D}'$ between two points of $\mathring{D} \cap X$.*

Proof. Let $C \in \boldsymbol{R}$. Using a standard trick (see [4] and [2]), one can construct a continuum $B_C \subset C \setminus \mathring{D}'$ irreducible between $a_C, b_C \in \dot{D}'$, which intersects $\mathring{D}$ (first we construct a continuum $Y \subset C$ irreducible between two points from $\mathring{D}'$ which intersects $\mathring{D}$ and then apply 3.3 to the triple $(Y \cap D', Y \setminus \mathring{D}', Y \cap \mathring{D})$). Since $\dot{D}'$ intersects all composants of X, we have $\dot{D}' \not\subset X$. Hence there is an arc $L \subset \dot{D}'$ which contains $\dot{D}' \cap X$. Thus $a_C, b_C \in L$ for each $C \in \boldsymbol{R}$. Without loss of generality we may assume that $a_C < b_C$ for each $C \in \boldsymbol{R}$, where $<$ is one of the natural orders on L. Since $\boldsymbol{R}$ is uncountable, there is a point $d \in L$ and an uncountable subcollection $\boldsymbol{R}' \subset \boldsymbol{R}$ such that $a_C < d < b_C$ for each $C \in \boldsymbol{R}'$. Let $e \in \dot{D}' \setminus L$ and let ed be an arc regularly embedded in D'. Then there exist two disks D_1, D_2 such that $D' = D_1 \cup D_2$, $D_1 \cap D_2 = ed$, $a_C \in \dot{D}_1$ and $b_C \in \dot{D}_2$ for each $C \in \boldsymbol{R}'$. Applying 3.2 to D_1, D_2 and $2(c(M)+2)$ elements of $\{B_C\colon C \in \boldsymbol{R}'\}$ we obtain a collection $\{A', A'', A_1, \ldots, A_{c(M)+2}\} \subset \{B_C\colon C \in \boldsymbol{R}'\}$ and continua $T_i \subset D_i$ such that

$$T_i \setminus (A_1 \cup \ldots \cup A_{c(M)+2}) \subset \mathring{D}_i$$

and $T_1 \cup T_2 \cup A_1 \cup \ldots \cup A_{c(M)+2}$ separates M between A' and A''. Since $A' \cap \mathring{D} \neq \emptyset \neq A'' \cap \mathring{D}$, setting $F = A_1 \cup \ldots \cup A_{c(M)+2}$ we obtain a closed set with the required properties.

The following lemma is of fundamental importance in the next section.

3.7. LEMMA. *Let D, D' and X be as in 3.6. Let C be a composant of X. Then there exists a continuum $E \subset C$ such that $E \setminus \mathring{D}$ separates $M \setminus \mathring{D}$ between two points of $\mathring{D}' \cap X$.*

Proof. Applying 3.6 to the collection of all composants of X distinct from C, we infer that there exists a closed set F' disjoint with C which separates $M \setminus \mathring{D}'$ between $x, y \in \mathring{D} \cap X$. Hence there exist two disks D_1, D_2 such that

(1) $D_1 \cup D_2 \subset D \setminus F'$,

(2) $\mathring{D}_i \cap X \neq \emptyset$, $i = 1, 2$,

(3) F' separates $M / \mathring{D}'$ between D_1 and D_2.

By a repeated application of 3.5 (using (2)) we construct a sequence $C_1, C_2, \ldots, C_{2(c(M)+2)}$ of mutually disjoint subcontinua of C such that C_j meets both D_1 and D_2. By (3) we have $D_1 \cap D_2 = \emptyset$. Hence by 3.4 we may also assume that

(4) $C_j \subset C \setminus (\mathring{D}_1 \cup \mathring{D}_2)$ is irreducible between $a_j \in \dot{D}_1$ and $b_j \in \dot{D}_2$.

It follows from (3) that for each j we have

(5) $C_j \cap \mathring{D}' \neq \emptyset$.

By 3.2 and (4) there is a collection $\{A', A'', A_1, \ldots, A_{c(M)+2}\} \subset \{C_j\}$ such that $F = A_1 \cup \ldots \cup A_{c(M)+2}$ separates $M \setminus (\mathring{D}_1 \cup \mathring{D}_2)$ between A' and A''. Hence from (1) and (5) it follows that F separates $M \setminus \mathring{D}$ between two points $a, b \in (\mathring{D}' \cap X) \setminus C$. There is a continuum $E \subset C$ containing F. Thus E satisfies the conclusion of our lemma.

4. On internal composants. Let X be an indecomposable continuum lying in a surface M. Denote by $\boldsymbol{E}$ the collection of all external (relatively to M) composants of X. The notion of an external (internal) composant of X was introduced and studied in [3] in the case where $M = S^2$.

For this case the Main Theorem in [2] states the following theorem (and we shall show below that the same statement is valid for arbitrary surfaces):

4.1. THEOREM. *The union of all external composants, i.e., the set $\bigcup \boldsymbol{E}$, is of the first category in X.*

Proof. We follow the ideas used in the proof of the Main Theorem. Most of the argument is the same, and so we shall not repeat it here. An analysis of that proof shows that the property of M contained in our Lemma 3.7 is all we need to extend the proof to the general case. This completes the proof.

Now we make a comment on the possibility of extending the properties of internal composants established in [3] (for indecomposable continua lying in S^2) to the general case. An easy inspection shows that the properties listed in Section 3 of [3], except 3.6, follow from property 3.6, from the Main Theorem [2] (4.1 extends it to all surfaces) and from the general results on indecomposable continua.

Now we are going to state a property of indecomposable subcontinua of surfaces which is a generalization of 3.6 from [3] (proved only for the sphere S^2). We can prove the generalization proceeding in the same way as in the proof of 3.6 in [3], applying our Lemma 3.7 in place of 2.2 of [3].

4.2. LEMMA. *Let $L \subset M$ be a continuum intersecting all composants of X and let C be an arbitrary composant of X. If $X \not\subset L$, then there exist an open neighborhood U of L in M and a continuum $A \subset C$ which separates U between two points of L.*

Having this lemma, we can extend all the properties 3.1–3.9 from [3] to the general case. In particular, we have the following generalization of 3.2 from [3]:

4.3. THEOREM. *The set $\bigcup E$ is an F_σ-set in X. Consequently, the union of internal composants of X is a G_δ-set dense in X.*

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$L_{\infty\omega_1}$-elementary equivalence of ω_1-like models of PA

by

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Abstract. We show that two recursively saturated, ω_1-like models of PA are $L_{\infty\omega_1}$-elementarily equivalent iff they are elementarily equivalent and have the same standard systems. On the other hand, for every countable model M of PA we construct a continuum of pairwise non-isomorphic, ω_1-like, recursively saturated, elementary end extensions of M.

Any two recursively saturated models of PA (Peano Arithmetic) are $L_{\infty\omega_1}$-elementarily equivalent iff they are elementarily equivalent and have the same standard systems. This result was first noted by Craig Smoryński in [8] but its roots go back to some earlier works of George Wilmers and Alex Wilkie (see [8] for historical remarks).

For the purpose of this paper let us call elementary equivalent models with the same standard systems *similar*.

The result mentioned above is particularly important in the case of countable models of PA since it says that any two recursively saturated countable similar models are isomorphic. The situation is different when we consider uncountable models and, as usual, the first counterexamples can be found among ω_1-like models. Take for instance a "rather classless" model M of M. Kaufmann [3] (the existence of a rather classless model is provable in ZFC, cf [7]) and a recursively saturated ω_1-like model similar to M but not "rather classless", hence not isomorphic to M. In the last section of this paper we show that there is at least a continuum of pairwise non-isomorphic ω_1-like recursively saturated similar models. But our main theorem says that ω_1-like recursively saturated similar models are still very similar, namely they are $L_{\infty\omega_1}$-elementarily equivalent.

1. Preliminaries. We assume the readers' acquaintance with the basic properties of recursively saturated models of PA (Smoryński [8] is a perfect survey of this subject). With some minor changes we are going to follow the terminology and notation of [8] and [10].

Models of PA are called shortly models. As usual, $x \in D_y$ is an abbreviation for an arithmetical formula expressing that x is an element of a set coded by y and $(y)_i$ denotes the ith element of this set (in increasing order).

If b is an element of a model M then

$$D_b^M = \{x \in M\colon\ M \models x \in D_b\}.$$