Indeed if there was such a $\boldsymbol{P}$ then the sentence $(\mathrm{E}x_0)A(x_0)$ would be absolute w.r.t. M and this is not the case if $\mathscr{X}_1 = \mathrm{Boid}\,M$. Thus the classical Shoenfield method can not be applied here.

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Continuous relations and generalized G_δ sets

by

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Abstract. In the paper some purely topological analogues of the main notions connected with forcing are considered. We compare the properties of the topological notions with the properties of their forcing counterparts.

Introduction. The paper was inspired by studies on forcing.

If we consider the first original notion of forcing, i.e., the set of Cohen forcing conditions $2^{<\omega}$, then $2^{<\omega}$ codes a base of the natural product topology in 2^ω. If $\underset{\sim}{x}$ is a fixed real in the Shoenfield universe $V^{2^{<\omega}}$, then the relation R, $s \Vdash (\check{t} \subseteq \underset{\sim}{x})$, is a relation between finite sequences s and t. If $\mathscr{D}$ is a family of dense subsets of $2^{<\omega}$, then the set X of reals $\mathscr{D}$-generic over $2^{<\omega}$ is a subset of 2^ω. If $\mathscr{D}$ is so large that for $\alpha \in X$, $i_\alpha(\underset{\sim}{x})$ can be defined then the function $f(\alpha) = i_\alpha(\underset{\sim}{x})$ defined for $\alpha \in X$ is a function from X into 2^ω. Since all R, X and f are objects connected with the topological space $\langle 2^\omega, 2^{<\omega}\rangle$, we can ask about their topological characterizations. Moreover, we can study their topological properties. This leads us to the notions of a regular relation, a g. G_δ set, a forcing function, and a continuous relation. These notions are not restricted to the case of the Cantor space $\langle 2^\omega, 2^{<\omega}\rangle$ but the reader should always have in mind this space or the Baire space $\langle \omega^\omega, \omega^{<\omega}\rangle$ as the main illustration. The mentioned notions are inspired respectively by

1) sets of the form $\{x\colon x \text{ is } \boldsymbol{P}\text{-generic over } M\}$ for given $\boldsymbol{P}$, M,

2) relations of the form $\{\langle p, q\rangle\colon p \in \boldsymbol{P}, p \Vdash (\underset{\sim}{x} \in \check{q}), q \in \boldsymbol{Q}\}$ for given $\boldsymbol{P}, \boldsymbol{Q}, M, \underset{\sim}{x} \in M^{\boldsymbol{P}}$,

3) functions of the form $f(\alpha) = i_\alpha(\underset{\sim}{x})$ for given $\boldsymbol{P}, M, \underset{\sim}{x} \in M^{\boldsymbol{P}}$,

4) relations of the form $\{\langle x, y\rangle\colon \langle x, y\rangle \text{ is generic over } \boldsymbol{P}\times\boldsymbol{Q}, M\}$ for given $\boldsymbol{P}$, $\boldsymbol{Q}$, M.

In certain cases our topological notions characterize the appropriate forcing notions, then we indicate it — Fact 2, Corollary 4 but in general the correspondence is not strict. However, it turns out that certain topological theorems about our notions have analogues in the forcing theory. We prove a few such theorems, mainly Fact 1 and Theorem 1. Indeed Fact 1, especially the fact that $\mathrm{Dom} f$ is g. G_δ for f satisfying certain assumptions corresponds to the fact that the function $f(\alpha) = i_\alpha(\underset{\sim}{x})$ is defined for all generic α. Theorem 1 corresponds to the fact that if $\langle x, y\rangle$ is generic over $P\times Q$ then x is generic over P. Also the proof of Theorem 1 uses ideas of the proof of the above fact about forcing.

Note that the topological and the forcing theorems are not strictly equivalent but only analogous. We infer that notions characteristic for forcing and certain theorems about forcing have their purely topological counterparts free of logical notions. Perhaps this can serve to eliminate forcing from certain topological proofs.

Since there are a number of applications of forcing in topology and descriptive set theory, we can ask whether it is possible to characterize forcing as a topological method. Our answer is half positive. There are topological notions and operations that behave like forcing and sometimes can replace forcing, as we show in the present paper. However they are not exactly forcing. Probably there is no quite exact translation between forcing and any topological method. Thus forcing can hardly be called a topological method.

The main exposition of the topological aspects of forcing is Mostowski [5]. I think that the present exposition is in a sense deeper. Mostowski was not interested in eliminating logical notions from forcing. Moreover, Mostowski did not study those relations and functions in which we are interested but certain very general forcing notions, e.g. the relation $p \Vdash \varphi(\underline{x})$.

We have said that our notions and theorems have forcing counterparts. On the other hand they have counterparts in elementary recursion theory and descriptive set theory. For example the name "g.G_δ" comes from the fact that our g.G_δ sets generalize the G_δ sets as regards both their definition and some of their properties. Theorem 1 is a generalization of a theorem about G_δ sets in Polish spaces. At the end of the paper we show that the standard proofs of some theorems about Turing degrees and about degrees of constructibility follow the same pattern. This expresses the connection between forcing and relative recursiveness. The main bridge here is the notion of continuity, which plays a great role in the paper.

Let us consider a topological space $\langle X, \mathcal{O}\rangle$ where $\mathcal{O}$ is a basis of a topology in X. Assume $q, q'\in\mathcal{O}$ and $q\cap q'\neq\emptyset \Rightarrow q\cap q'\in\mathcal{O}$, $q\in\mathcal{O}\Rightarrow q\neq\emptyset$. Let us remind what we mean by a tree in $\mathcal{X}$ and by a branch of a tree — these notions were defined in [1].

Namely let T be called a *tree in* $\langle\mathcal{X}, \mathcal{O}\rangle$ if it is a partially ordered set consisting of a subset of $\mathcal{O}$ with the ordering $\subseteq$. Let us identify the tree T with the subset of $\mathcal{O}$ and write $T\subseteq\mathcal{O}$. Let x be a branch of T iff $(p)_{\mathcal{O}}\big(x\in p\to(\mathrm{E}p')_T\,(x\in p'\subseteq p)\big)$.

We shall say that a subset $A\subseteq\mathcal{X}$ *is generalized* G_δ (write g.G_δ) iff there is a tree $T\subseteq\mathcal{O}$ and a family $\mathcal{D}$ of dense (w.r.t. the ordering) subsets of T such that $\overline{\overline{\mathcal{D}}}<\overline{\overline{\mathcal{X}}}$ and $x\in A\equiv x$ is a $\mathcal{D}$-generic branch of T, i.e. x is a branch of T and $(D)_{\mathcal{D}}\big((\mathrm{E}p)_D(x\in p)\big)$.

The idea to define the notion of a g.G_δ set comes from forcing as we have already said in the introduction. Indeed if M is an inner model, $P\in M$ is a partially ordered set then P determines a topology $\mathcal{O}$ in the space $\mathcal{X}$ of ultrafilters of P (see [2]). Let us identify P with $\mathcal{O}$. Then the set A of M-generic branches of P is a g.G_δ set and $\overline{\overline{\mathcal{P}}}^M(P)<\overline{\overline{\mathcal{P}}}(P)$. Indeed, if $\mathcal{D}=\{D\colon D \text{ dense in } P\,\&\,D\in M\}$, then $x\in A\equiv x$ is a $\mathcal{D}$-generic branch of P. Conversely, if A is g.G_δ and the tree T is in M and the family $\mathcal{D}$ is included in M then A contains the set of M-generic branches of P.

The name "g.G_δ" comes from the fact that every G_δ subset of $\mathcal{X}$ can be represented by the equivalence

$$x\in A\equiv x \text{ is a } \mathcal{D}\text{-generic branch of } T$$

for a tree T and a countable family $\mathcal{D}$. Indeed let $T=\{p\in\mathcal{O}\colon (\mathrm{E}x)(x\in p\cap A)\}$ and let $D_n=\{p\in T\colon p\subseteq A_n\}$ where $A=\bigcap_n A_n$ and A_n are open. Let $\mathcal{D}=(D_n)_{n\in\omega}$.

We shall show certain properties of g.G_δ sets generalizing properties of G_δ sets.

Although the notion of a g.G_δ set comes from forcing, we can show certain descriptive properties of such sets without referring to inner models and forcing. Only in the proofs we shall use forcing ideas. We shall see that certain g.G_δ sets occur in natural topological questions. Especially the notion of a g.G_δ set is connected with the notion of continuity. We shall show for example that the domain of a maximal continuous function from $\langle\mathcal{X}, \mathcal{O}\rangle$ to a 0-dimensional space defined at a dense set is always a g.G_δ set. This will serve us to give a topological characterization of the forcing relation. Contrary to Mostowski [5] this characterization does not mention logical notions.

Let us recall from [1] that a relation $R\subseteq\mathcal{X}\times\mathcal{Y}$ is called *continuous in* $\langle\mathcal{X}, \mathcal{O}\rangle$ w.r.t. $\langle\mathcal{Y}, \mathcal{O}'\rangle$ iff

$$(\langle xy\rangle)_R(q)_{\mathcal{O}'}\big(y\in q\to(\mathrm{E}p)_{\mathcal{O}}(x\in p\,\&\,p\cap\mathrm{Dom}R\subseteq R^{-1}(q))\big).$$

We shall show that if R is continuous and g.G_δ then ΣR (i.e. $\mathrm{Dom}R$) is g.G_δ under some natural assumptions.

Also if R is continuous and g.G_δ, then under certain assumptions ΠR (i.e. $\{x\colon (y)R(x,y)\}$) contains a dense g.G_δ subset. This can serve to study the question whether a projective set A contains a g.G_δ subset and hence whether it contains elements generic over an appropriate inner model.

First let us consider a few natural examples of g.G_δ sets.

EXAMPLE 1. Let $\langle\mathcal{X}, \mathcal{O}\rangle$ be of dim0. For instance $\langle\mathcal{X}, \mathcal{O}\rangle$ can be the space $\langle\omega_1^\omega, \omega_1^{<\omega}\rangle$ where in ω_1 we take the discrete topology and $\omega_1^{<\omega}$ denotes the Tichonow basis. Let $T\subseteq\mathcal{O}$ be a tree such that T is a dense subset of $\mathcal{O}$ w.r.t. the ordering. Then there is a family $\mathcal{D}$ of dense subsets of $\mathcal{O}$ such that x is a branch of $T\equiv x$ is a $\mathcal{D}$-generic branch of $\mathcal{O}$. Hence the set A of branches of T is g.G_δ both with the tree T and with the tree $\mathcal{O}$.

Proof. Let $p\in\mathcal{O}$. Let D_p be defined as $D_p=\{p'\in T\colon p'\leqslant p\vee p'\cap p=\emptyset\}$. By the density of T in $\mathcal{O}$, D_p is dense in T. Let $\mathcal{D}=\{D_p\colon p\in\mathcal{O}\}$. Let us show that x is a $\mathcal{D}$-generic branch of $\mathcal{O}$ iff x is a branch of T. Indeed let x be $\mathcal{D}$-generic. Let $p\in\mathcal{O}$. Let $p'\leqslant p$ be such that $x\in p'\,\&\,p'\in D_p$. Then $p'\cap p\neq\emptyset$ and thus $p'\subseteq p$. Hence x is a branch of T because $p'\in T$.

Conversely let x be a branch of T. Let $p\in\mathcal{O}$. If $x\in p$ then $(\mathrm{E}p')_{\leqslant p}(x\in p'\in T)$. Then $p'\in D_p$. If $x\notin p$ then by the fact that $\langle\mathcal{X}, \mathcal{O}\rangle$ is of dim0, there is a p' such that $p' \cap p = \emptyset \,\&\, x \in p'$. Hence $p' \in D_p$. Thus x intersects D_p. Hence x is $\mathscr{D}$-generic. ■

EXAMPLE 2. Let $\langle \mathscr{X}, \mathcal{O} \rangle = \langle \omega_1^{\omega_1}, \omega_1^{<\omega} \rangle$. Let $A \subseteq \mathscr{X}$ be called $\omega_1 - G_\delta$ iff A is an intersection of ω_1 open sets. Let $T \subseteq \mathcal{O}$ be a tree. Then the set of branches A of T is $\omega_1 - G_\delta$. Indeed let $(K_\xi)_{\xi \in \omega_1}$ be an enumeration of finite subsets of ω_1. Let $x \in \mathscr{X}$. Then x is a branch of T iff $(\xi)_{\omega_1}(\mathrm{E}p)_T(x \in p \subseteq x \upharpoonright K_\xi)$ where we identify a finite sequence with the neighbourhood that it determines.

Let $A_\xi = \bigcup \{p \in T\colon (\mathrm{E}s)_{\omega_1^{<\omega}}(p \leqslant s \,\&\, \operatorname{dom} s = K_\xi)\}$. Let $A = \bigcap_\xi A_\xi$.

Then $x \in A \equiv x$ is a branch of T. Also conversely, if A is $\omega_1 - G_\delta$ then there is a tree T and a family $\mathscr{D}$ of dense subsets of T of power ω_1 such that $x \in A = x$ is a $\mathscr{D}$-generic branch of T. Hence on $\omega_1 - G_\delta$ set is g. G_δ.

EXAMPLE 3. Let $\langle \mathscr{X}, \mathcal{O} \rangle$ be c.c.c. Let $A \subseteq \mathscr{X}$ be $\omega_1 - G_\delta$ and be an intersection of dense open sets. Assume Martin's axiom and $2^\omega > \omega_1$. Then A is dense.

Remark 1. It is reasonable to restrict our notion of a g. G_δ set to the notion of a M-g. G_δ set for an inner model M (e.g. M = L) where by an M-g. G_δ set we mean a g. G_δ set with the tree T and the family $\mathscr{D}$ such that $T \in M$, $\mathscr{D} \subseteq M$. Hence under appropriate assumptions it is easy to find sets that are not M-g. G_δ.

As we shall see by analysing the proofs, our Theorems 1 and 2 hold if we replace everywhere "g. G_δ" by "M-g. G_δ".

Remark 2. If in our definition of a g. G_δ set we drop the restriction $\overline{\overline{\mathscr{D}}} < \overline{\overline{\mathscr{X}}}$ then every set would be g. G_δ (provided that no singleton is open). Indeed let $A \subseteq \mathscr{X}$. Define $D_x = \{p \in \mathcal{O}\colon x \notin p\}$. Then we have

$$x \in A \ \equiv\ x \text{ is a } \mathscr{D}\text{-generic branch of } \mathcal{O}$$

where $\mathscr{D} = \{D_x\colon x \notin A\}$.

Remark 3. If x is a $\mathscr{D}$-generic branch of T then we can assume that

$$(*) \qquad (D)_{\mathscr{D}}(P)_T\big(x \in p \to (\mathrm{E}p')_D(x \in p' \leqslant p)\big).$$

Indeed otherwise take for $D \in \mathscr{D}$,

$$D' = \{p \in T\colon (\exists q)_D(q \leqslant p)\} \quad \text{and} \quad \mathscr{D}' = \{D'\colon D \in \mathscr{D}\}.$$

For the sequel let us mean by a $\mathscr{D}$-generic branch of T a branch with the property $(*)$.

Let us now pay attention to continuous functions.

Let $\langle \mathscr{X}, \mathcal{O} \rangle$, $\langle \mathscr{Y}, \mathcal{O}' \rangle$ be topological spaces, $\overline{\overline{\mathcal{O}'}} < \overline{\overline{\mathscr{X}}}$. Let us recall from [1] that we say that $\langle \mathscr{Y}, \mathcal{O}' \rangle$ has the centralization property iff $(\mathcal{O}'')(\mathcal{O}'' \subseteq \mathcal{O}' \,\&\, \mathcal{O}''$ is centralized $\Rightarrow \bigcap \mathcal{O}'' \neq \emptyset)$.

Let $f\colon \langle \mathscr{X}, \mathcal{O} \rangle \to \langle \mathscr{Y}, \mathcal{O}' \rangle$. Assume that $\operatorname{Dom} f$ is dense in $\langle \mathscr{X}, \mathcal{O} \rangle$. Let us say that f is maximal continuous iff for every f' if f' continuous and $f \subseteq f'$ then $f = f'$.

FACT 1. *Let $f\colon \langle \mathscr{X}, \mathcal{O} \rangle \to \langle \mathscr{Y}, \mathcal{O}' \rangle$ be maximal continuous with a dense domain. Let $\langle \mathscr{Y}, \mathcal{O}' \rangle$ have the centralization property and be T_1. Then f is g. G_δ in $\langle \mathscr{X}, \mathcal{O} \rangle \times \langle \mathscr{Y}, \mathcal{O}' \rangle$ and $\operatorname{Dom} f$ is g. G_δ in $\langle \mathscr{X}, \mathcal{O} \rangle$.*

Proof. Let us define a tree $T \subseteq \mathcal{O} \times \mathcal{O}'$ as follows: let

$$\langle p, q \rangle \in T \ \equiv\ \big(p \cap \operatorname{Dom} f \subseteq f^{-1}(q)\big).$$

Let us show that $\langle x, y \rangle \in f$ if $\langle x, y \rangle$ is a branch of T. Let $\langle x, y \rangle \in f$, $\langle x, y \rangle \in \langle p, q \rangle$. Then by the continuity of f there is a $p' \subseteq p$ such that $x \in p' \,\&\, p' \cap \operatorname{Dom} f \subseteq f^{-1}(q)$. Hence $\langle p', q \rangle \in T \,\&\, \langle p'q \rangle \leqslant \langle pq \rangle$. Thus $\langle x, y \rangle$ is a branch of T. To show the converse let us show that the set of branches of T is a continuous function containing f (and hence it is equal to f). Let $\langle x, y \rangle \in \tilde{f} \equiv \langle x, y \rangle$ is a branch of T. Then $f \subseteq \tilde{f}$ by the previous considerations.

Let us show that $\tilde{f}$ is a function. Indeed suppose that $\langle x, y \rangle$, $\langle x, y' \rangle$ are branches of T, $y \neq y'$. Let $y \in q$, $y' \in q'$, $q \cap q' = \emptyset$ by the centralization property $\langle Y, O' \rangle$ is of dim0. By the fact that $\langle x, y \rangle$, $\langle x, y' \rangle$ are branches of T, there are p, $\bar{q}$, p', $\bar{q}'$ such that $x \in p \cap p'$, $y \in \bar{q} \subseteq q$, $y' \in \bar{q}' \subseteq q' \,\&\, \langle p, \bar{q} \rangle, \langle p', \bar{q}' \rangle \in T$. Hence

$$p \cap \operatorname{Dom} f \subseteq f^{-1}(\bar{q}) \,\&\, p' \cap \operatorname{Dom} f \subseteq f^{-1}(\bar{q}').$$

Let $p'' \subseteq p \cap p'$. Then

$$p'' \cap \operatorname{Dom} f \subseteq f^{-1}(\bar{q}) \,\&\, p'' \cap \operatorname{Dom} f \subseteq f^{-1}(\bar{q}').$$

Let $\bar{x} \in p'' \cap \operatorname{Dom} f$. Such an $\bar{x}$ exists by the density of $\operatorname{dom} f$. Then $f(\bar{x}) \in \bar{q}$ and $f(\bar{x}) \in \bar{q}'$. This is a contradiction, because $\bar{q} \cap \bar{q}' = \emptyset$.

Let us show now that $\tilde{f}$ is continuous. Let $\langle x, y \rangle \in \tilde{f}$, $y \in q$. Let $\langle p, q' \rangle \in T$ be such that $x \in p$, $y \in q' \subseteq q$. Let $x' \in p \cap \operatorname{Dom} \tilde{f}$. Hence $\langle x', y' \rangle \in \tilde{f}$ for a y'. Suppose that $y' \notin q'$. By the fact that $\langle \mathscr{Y}, \mathcal{O}' \rangle$ is of dim0 (this follows easily from the fact that $\langle \mathscr{Y}, \mathcal{O}' \rangle$ has the centralization property and is T_1) there is a q such that $y' \in \bar{q} \,\&\, q' \cap \bar{q} = \emptyset$. Hence there is a $\langle p', \bar{q}' \rangle \in T$ such that $x' \in p' \subseteq p \,\&\, y' \in \bar{q}' \subseteq q$ because $\langle x'y' \rangle$ is a branch of T.

We have $\langle p, q' \rangle \in T$, $\langle p', \bar{q}' \rangle \in T$. Hence $p \cap \operatorname{Dom} f \subseteq f^{-1}(q') \,\&\, p' \cap \operatorname{Dom} f \subseteq f^{-1}(\bar{q}')$. Hence $p' \cap \operatorname{Dom} f \subseteq f^{-1}(q') \,\&\, p' \cap \operatorname{Dom} f \subseteq f^{-1}(\bar{q}') \,\&\, q' \cap \bar{q}' = \emptyset$. As before we obtain a contradiction. Thus $p \cap \operatorname{Dom} \tilde{f} \subseteq \tilde{f}^{-1}(q')$. Hence follows that $\tilde{f}$ is continuous.

Moreover we have proved

$$\langle p, q \rangle \in T \ \equiv\ \big(p \cap \operatorname{Dom} \tilde{f} \subseteq \tilde{f}^{-1}(q)\big).$$

Let us show now that $\operatorname{Dom} f$ is g. G_δ. Let for $q \in \mathcal{O}'$, D_q be defined as

$$p \in D_q \ \equiv\ (\mathrm{E}q')_{\leqslant q}(\langle p, q' \rangle \in T) \vee (\mathrm{E}q')(\langle p, q' \rangle \in T \,\&\, q' \cap q = \emptyset).$$

Let $\mathscr{D} = (D_q)_{q \in \mathcal{O}'}$.

Let us show that

$$x \in \operatorname{Dom} f \ \equiv\ x \text{ is a } \mathscr{D}\text{-generic branch of } \mathcal{O}.$$

Let $x \in \operatorname{Dom} f$. Let q be given. We have $f(x) \in q$ or $f(x) \notin q$. If $f(x) \in q$ then there is a $\langle p, q' \rangle$ such that $x \in p \,\&\, f(x) \in q' \leqslant q \,\&\, \langle p, q' \rangle \in T$. Then $p \in D_q$. Assume $f(x) \notin q$.

Hence let $q' \cap q = \emptyset$, $f(x) \in q'$. Let p', q'' be such that $\langle p', q'' \rangle \in T$, $x \in p'$, $q'' \leqslant q'$, $f(x) \in q''$. Then $p' \in D_q$. Hence x is a $\mathscr{D}$-generic branch of $\mathcal{O}$.

Assume conversely that x is a $\mathscr{D}$-generic branch of $\mathcal{O}$. Define

$$T_x = \{q \in \mathcal{O}' : (\mathrm{E}p)(x \in p \,\&\, \langle p, q\rangle \in T)\}.$$

By the fact that $\check{f}$ is a function, T_x is centralized. Let $y \in \bigcap T_x$. Let us show that $\langle x, y\rangle$ is a branch of T. Indeed let $\langle x, y\rangle \in \langle p, q\rangle$. Let $p' \in D_q$ be such that $x \in p'$. Let q' be such that $\langle p', q'\rangle \in T \,\&\, q' \subseteq q$ or $q' \cap q = \emptyset$. We have: $q' \in T_x$. Thus $y \in q'$ and hence $q' \cap q \neq \emptyset$. Thus $q' \subseteq q$. Let $p'' \subseteq p' \cap p$, $x \in p''$. Then $\langle p'', q'\rangle \in T$ (because T has the property $\langle p, q\rangle \in T \,\&\, \bar{p} \subseteq p \Rightarrow \langle \bar{p}, q\rangle \in T$) and $\langle p'', q'\rangle \leqslant \langle p, q\rangle$. Hence $\langle x, y\rangle$ is a branch of T, and thus $x \in \mathrm{Dom} f$. Thus $\mathrm{Dom} f$ is g. G_δ. ∎

Corollary 1. *From the proof of Fact 1 it follows that if f is maximal continuous with a dense domain and $\langle \mathscr{Y}, \mathcal{O}'\rangle$ is separable then f, $\mathrm{Dom} f$ are G_δ.*

Corollary 2. *If f is maximal continuous with a dense domain then there is a tree T such that*

$$\langle x, y\rangle \in f \equiv \langle x, y\rangle \text{ is a branch of } T$$

(next we shall say in such a case that f is determined by T) and T has the following four properties

(1) $\langle p, q\rangle \in T \,\&\, p' \leqslant p \Rightarrow \langle p', q\rangle \in T$,
$\langle p, q\rangle \in T \,\&\, q \leqslant q' \Rightarrow \langle p, q'\rangle \in T$,

(2) $\langle p, q\rangle \in T \,\&\, \langle p, q'\rangle \in T \Rightarrow q \cap q' \neq \emptyset \,\&\, \langle p, q \cap q'\rangle \in T$,

(3) $(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}(\langle p'', q\rangle \in T) \Rightarrow \langle p, q\rangle \in T$,

(4) $(q)_{\mathcal{O}'}$

$$D_q = \{p : (\mathrm{E}q')((q' \leqslant q \vee q' \cap q = \emptyset) \,\&\, \langle p, q\rangle \in T)\}$$

is dense in $\mathcal{O}$.

Indeed (1), (2) follow directly from Fact 1 that

$$\langle p, q\rangle \in T \equiv (p \cap \mathrm{Dom} f \subseteq f^{-1}(q))$$

and (3), (4) use Fact 1 that $\mathrm{Dom} f$ is dense. Also conversely if T satisfies (1)–(4) and $\mathcal{O}'$ is countable then one can prove that the set of branches of T is a maximal continuous function with a dense domain.

Let us call a tree satisfying (1)–(4) — *regular*.

Assume that $\langle \mathscr{X}, \mathcal{O}\rangle$, $\langle \mathscr{Y}, \mathcal{O}'\rangle$ are definable and identify them with their definitions. Then we can speak about them in boolean extensions of the universe.

In [3], a relation $R \subseteq \mathcal{O} \times \mathcal{O}'$ is called a *forcing relation* if it satisfies (1)–(4). This name is justified. Namely we have

Fact 2. *Let $\mathcal{O}$, $\mathcal{O}'$ be topologies in $\mathscr{X}$, $\mathscr{Y}$ respectively that are absolutely codable, i.e. there are sets $\langle O, \leqslant\rangle$, $\langle O', \leqslant'\rangle$ and isomorphisms φ, φ' between $\langle O, \leqslant\rangle$, $\langle \mathcal{O}, \subseteq\rangle$ and $\langle O', \leqslant'\rangle$, $\langle \mathcal{O}', \subseteq\rangle$ respectively such that the relations $x \in \varphi(p)$ for $p \in O$ and $y \in \varphi(q)$ for $q \in O'$ are absolute w.r.t. boolean extensions of the universe* (see [1]). *Let $\mathcal{O}$, $\mathcal{O}'$ have the centralization property both in the universe and in all its boolean extensions.*

Then $R \subseteq O \times O'$ satisfies (1)–(4) iff there is a $\underset{\sim}{y} \in V^0$ (the Shoenfield universe) such that $R(p, q)$ is the relation $p \Vdash (\underset{\sim}{y} \in \check{q})$.

Remark 4. If $\mathcal{O}$ is an absolutely codable topology in $\mathscr{X}$, $\langle O, \leqslant\rangle$ is the set coding $\mathcal{O}$, then for $p, p' \in O$ the relations

$$p \subseteq p', \quad p \neq \emptyset, \quad p \cap p' \neq \emptyset$$

are absolute under an identification of $\mathcal{O}$ with O.

Indeed $p \subseteq p'$ iff $p \leqslant p'$, $p \neq \emptyset$ simply for all $p \in O$ and $p \cap p' \neq \emptyset$ iff there is an r in O such that $r \leqslant p$, $r \leqslant p'$.

Proof of Fact 2. Let us prove "⇒". Let us identify $\mathcal{O}$ with O, $\mathcal{O}'$ with O'. Work in V^0. Note that (1)–(4) are absolute and so they are true in V^0 for $\check{O}$, $\check{O}'$, $\check{R}$. Let $\underset{\sim}{x}$ be the canonical name of the O-generic element of $\mathscr{X}$. Let $O'_{\underset{\sim}{x}}$ be defined as

$$O'_{\underset{\sim}{x}} = \{q \in \check{O}' : (\mathrm{E}p)_{\check{O}}(\underset{\sim}{x} \in p \,\&\, \check{R}(p, q))\}.$$

Let us show that finite subsets of $O'_{\underset{\sim}{x}}$ have non-empty intersections. So let $q_1 \dots q_n \in O'_{\underset{\sim}{x}}$. Then there are $p_1 \dots p_n \in \check{O}$ such that $\underset{\sim}{x} \in p_1 \cap \dots \cap p_n$ and $\check{R}(p_i, q_i)$. Let $p \in \check{O}$ be such that $\underset{\sim}{x} \in p \subseteq p_1 \cap \dots \cap p_n$. Then by (1), $\check{R}(p, q_i)$ and by (2), $\check{R}(p, q_1 \cap \dots \cap q_n)$. Hence $q_1 \cap \dots \cap q_n \neq \emptyset$. By the centralization property of $\check{O}'$, there is y such that $y \in \bigcap O'_{\underset{\sim}{x}}$. Let $\underset{\sim}{y}$ be a Shoenfield constant satisfying $\underset{\sim}{y} \in \bigcap O'_{\underset{\sim}{x}}$. We shall show that $\underset{\sim}{y}$ is as required.

Let us show first

(**) $$(q)_{\check{O}'}(\underset{\sim}{y} \in q \equiv (\mathrm{E}p)_{\check{O}}(\underset{\sim}{x} \in p \,\&\, \check{R}(p, q))).$$

Let $q \in \check{O}'$. If $(\mathrm{E}p)_{\check{O}}(\underset{\sim}{x} \in p \,\&\, \check{R}(p, q))$, then evidently $\underset{\sim}{y} \in q$ by the definition of $\underset{\sim}{y}$. So assume that $\underset{\sim}{y} \in q$. Consider

$$D_q = \{p \in \check{O} : (\mathrm{E}q')_{\check{O}'}(q' \subseteq q \vee q' \cap q = \emptyset \,\&\, \check{R}(p, q'))\}.$$

By (4), D_q is dense. Hence $(\mathrm{E}p)_{D_q}(\underset{\sim}{x} \in p)$ (note that D_q is absolutely definable with the parameter q which is an element of the standard set $\check{O}'$). Take this p. Let q' be such that $\check{R}(p, q')$ and $q' \subseteq q$ or $q' \cap q = \emptyset$. But $\underset{\sim}{y} \in q' \cap q$ because $q' \in O'_{\underset{\sim}{x}}$. So $q' \cap q \neq \emptyset$ and thus $q' \subseteq q$. Hence by (1), $\check{R}(p, q)$ and thus $(\mathrm{E}p)_{\check{O}}(\underset{\sim}{x} \in p \,\&\, \check{R}(p, q))$. So we have proved (**).

To show that $R(p, q) \equiv p \Vdash (\underset{\sim}{y} \in \check{q})$ it is enough to show that

$$R(p, q) \equiv (V^0 \models (\underset{\sim}{x} \in p \Rightarrow \underset{\sim}{y} \in \check{q})) \quad \text{for} \quad p \in O,\ q \in O'.$$

Assume $R(p, q)$. Then $V^0 \models \check{R}(\check{p}, \check{q})$. Then by (**), in V^0 $\underset{\sim}{x} \in \check{p} \Rightarrow \underset{\sim}{y} \in \check{q}$. Conversely let $p \in O$, $q \in O'$ and assume that $V^0 \models (\underset{\sim}{x} \in \check{p} \Rightarrow \underset{\sim}{y} \in \check{q})$. Suppose $\neg R(p, q)$. If $(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}(R(p'', q))$ then by (3), $R(p, q)$ contradicting our assumption. Thus

$$(\mathrm{E}p')_{\leqslant p}(p'')_{\leqslant p'} \neg R(p'', q).$$

Take this p'. Consider D_q. By (4), D_q is dense. So there is a p'' in D_q such that $p'' \leqslant p'$. Take this p''. Then there is a q' such that $q' \subseteq q$ or $q' \cap q = \emptyset$ and $R(p'', q')$. If $q' \subseteq q$ then by (1), $R(p'', q)$ which contradicts the choice of p'. Hence $q' \cap q = \emptyset$.

Work in V^0. Suppose that $\underline{x} \in \check{p}''$. By the definition of y, $\underline{y} \in \check{q}'$ because $\check{R}(\check{p}'', \check{q}')$. Hence $\underline{y} \in \check{q}$. But $\underline{x} \in \check{p}$ because $\check{p}'' \subseteq \check{p}$ and $\underline{x} \in \check{p} \Rightarrow \underline{y} \in \check{q}$. Hence $\underline{y} \in \check{q}$. Contradiction because $\check{q} \cap \check{q}' = \emptyset$. Thus $\underline{x} \notin \check{p}''$.

We have shown that $V^0 \models \underline{x} \notin \check{p}''$. But $\|\underline{x} \in \check{p}''\| = p'' \neq \emptyset$. Contradiction. Hence $R(p, q)$.

The implication "$\Leftarrow$" is obvious. Thus we have proved Fact 2. ■

COROLLARY 3. *Note that if $\langle \mathscr{X}, \mathscr{O} \rangle, \langle \mathscr{Y}, \mathscr{O}' \rangle$ are the Baire space $\langle \omega^\omega, \omega^{<\omega} \rangle$ or the Cantor space $\langle 2^\omega, 2^{<\omega} \rangle$, then the assumptions of Fact 2 are satisfied. Hence we have for $R \subseteq \omega^{<\omega} \times \omega^{<\omega}$ or $R \subseteq 2^{<\omega} \times 2^{<\omega}$: R satisfies (1)–(4) iff there is a $\underline{y} \in V^{\omega^{<\omega}}$ or $V^{2^{<\omega}}$ such that R is the relation $p \models (\underline{y} \in \check{q})$.*

COROLLARY 4. *Let us call a maximal continuous function with a dense domain "a forcing function". By Fact 1, Corollary 2 and Fact 2, if the topologies in question satisfy the assumptions, then every forcing function restricted to generic x's is a function $f(x) = i_x(\underline{y})$ for a $\underline{y}$, where i_x is the usual contraction of V^0 and x is identified with $\{p \in O:\ x \in p\}$.*

Thus we obtain a characterization of the functions $i_x(\underline{y})$ as restrictions of maximal continuous functions with dense domains to generic x's.

Let us now consider continuous relations. Let us prove a generalization of the following Louveau's lemma:

LEMMA. *Let $\langle \mathscr{X}, \mathscr{O} \rangle, \langle \mathscr{Y}, \mathscr{O}' \rangle$ be Polish spaces. Let $R \subseteq \mathscr{X} \times \mathscr{Y}$ be continuous and G_δ. Then* Dom$R$ *is G_δ in $\langle \mathscr{X}, \mathscr{O} \rangle$.*

THEOREM 1. *Let $\langle \mathscr{X}, \mathscr{O} \rangle, \langle \mathscr{Y}, \mathscr{O}' \rangle$ be given. Assume either*

(1) $\langle \mathscr{Y}, \mathscr{O}' \rangle$ has the centralization property or

(2) $\langle \mathscr{Y}, \mathscr{O}' \rangle$ is a complete metric space.

Let $R \subseteq \mathscr{X} \times \mathscr{Y}$ be g.G_δ in $\mathscr{X} \times \mathscr{Y}$, i.e. there are T, $\mathscr{D}$ such that

$$R(x, y) \equiv \text{"}\langle x, y \rangle \text{ is a } \mathscr{D}\text{-generic branch of } T\text{"}.$$

Fix T, $\mathscr{D}$ and take "..." for the definition of R. Assume that the relation $x \in$ Dom$R$ *under the above definition of R is absolute w.r.t. boolean extensions of the universe, and so are the assumptions (1), (2). Let R be continuous. Then* Dom$R$ *is g.G_δ.*

Proof. First we shall show that we can assume about T the following property:

$$\langle p, q \rangle \in T \mathbin{\&} p' \leqslant p \mathbin{\&} p' \cap \mathrm{Dom}R \neq \emptyset \Rightarrow \langle p', q \rangle \in T.$$

This property is called in [1] "the continuity of T".

First notice that we can assume

$$(\langle p, q \rangle)_T (E\langle x, y \rangle)_R (\langle x, y \rangle \in \langle p, q \rangle).$$

Otherwise take an appropriate subtree. Then define

$$\overline{T} = \{\langle p, q \rangle \in \mathscr{O} \times \mathscr{O}':\ (E\langle p', q' \rangle)_T (\langle p', q' \rangle \leqslant \langle p, q \rangle)\}.$$

Then $R(x, y) \equiv \langle x, y \rangle$ is a $\mathscr{D}$-generic branch of $\overline{T}$. Let now

$$T' = \{\langle p, q \rangle \in \overline{T}:\ p \cap \mathrm{Dom}R \subseteq R^{-1}(q)\}.$$

Let for $D \in \mathscr{D}$,

$$D' = \{\langle p, q \rangle \in T':\ (E\langle \bar{p}, \bar{q} \rangle)_D (\langle p, q \rangle \leqslant \langle \bar{p}, \bar{q} \rangle)\},$$
$$\mathscr{D}' = \{D':\ D \in \mathscr{D}\}.$$

Notice that

$$R(x, y) \equiv \langle x, y \rangle \text{ is a } \mathscr{D}' \text{ generic branch of } T'.$$

Indeed let $R(x, y)$. Then by the continuity of R, $\langle x, y \rangle$ is a branch of T'. By the fact that it is a $\mathscr{D}$ generic branch of $\overline{T}$ it is a $\mathscr{D}'$ generic branch of T'.

Conversely if $\langle x, y \rangle$ is a $\mathscr{D}'$-generic branch of T', then it is a $\mathscr{D}$-generic branch of T and thus $R(x, y)$. Let $P = \{p \in \mathscr{O}:\ (\mathrm{E}q)(\langle p, q \rangle \in T)\}$. Let $q \in \mathscr{O}'$, $D \in \mathscr{D}$, $n \in \omega$. Let

$$D_n^q = \{p \in P:\ (\mathrm{E}q')_{\leqslant q}((\langle p, q' \rangle \in D \mathbin{\&} q' \subseteq q \mathbin{\&} \delta(q') < 1/n \vee (p')_{\leqslant p}(q')_{\leqslant q} (\langle p', q' \rangle \notin T))\}$$

if $\langle \mathscr{Y}, \mathscr{O}' \rangle$ is metric. If $\langle \mathscr{Y}, \mathscr{O}' \rangle$ is not metric we define D^q analogously, dropping only "$\delta(q') < 1/n$".

Then D_n^q is dense in P. Indeed let $p \in P$. Then either $(p')_{\leqslant p}(q')_{\leqslant q} \langle p', q' \rangle \notin T$ and then $p \in D_n^q$ or there is a $q' \leqslant q$, $p' \leqslant p$ such that $\langle p', q' \rangle \in T$. Then there is a $\langle p'', q'' \rangle \leqslant \langle p', q' \rangle$ such that $\langle p'', q'' \rangle \in D \mathbin{\&} \delta(q'') < 1/n$. Hence $p'' \leqslant p \mathbin{\&} p'' \in D_n^q$. Let $\tilde{\mathscr{D}} = \{D_n^q:\ D \in \mathscr{D}, q \in \mathscr{O}', n \in \omega\}$. Let us show that

$$x \in \mathrm{Dom}R \equiv x \text{ is a } \tilde{\mathscr{D}} \text{ generic branch of } P.$$

If $x \in$ Dom$R$ then there is a y such that $R(x, y)$. By the fact that $\langle x, y \rangle$ is a $\mathscr{D}$-generic branch of T we immediately obtain that x is a $\tilde{\mathscr{D}}$-generic branch of P.

Assume now that x is a $\tilde{\mathscr{D}}$-generic branch of P. Define the following tree

$$T_x = \{q \in \mathscr{O}':\ (\mathrm{E}p)(x \in p \mathbin{\&} \langle p, q \rangle \in T)\}.$$

Let $D \in \mathscr{D}$. Let $D_x = \{q \in \mathscr{O}':\ (\mathrm{E}p)(x \in p \mathbin{\&} \langle p, q \rangle \in D)\}$. Let us show that Dx is dense in T_x. Let $q \in T_x$. Then $\langle p, q \rangle \in T \mathbin{\&} x \in p$ for a p. By the fact that x intersects D_1^q, there is a $p' \leqslant p$ such that $x \in p' \mathbin{\&} p' \in D_1^q$.

By the continuity of T, $\langle p', q \rangle \in T$ and hence $p' \in D_1^q$ implies $(\mathrm{E}q')_{\leqslant q} \langle p', q' \rangle \in D$ Hence $q' \in D_x$, $q' \leqslant q$ and thus D_x is dense in T_x.

Let $\mathscr{D}' = \{D_x:\ D \in \mathscr{D}\}$. Let $\boldsymbol{B}$ be a complete boolean algebra such that

$$V^{\boldsymbol{B}} \models (\mathscr{D}' \text{ is countable}).$$

Work in $V^{\boldsymbol{B}}$. Notice that there is a y which is a $\mathscr{D}'$-generic branch of T_x. Indeed let $\mathscr{D}' = \{D_n:\ n \in \omega\}$. If $\langle \mathscr{Y}, \mathscr{O}' \rangle$ is a complete metric space then we can assume that $q \in D_n \to \delta(q) < 1/n$.

Define a sequence $(q_n)_{n \in \omega}$ as follows: let q_0 be an arbitrary element of D_0; assume that q_n has been defined, $q_n \in D_n$,

$$q_0 \supseteq \bar{q}_1 \supseteq \ldots \supseteq \bar{q}_n.$$

We have $q_n \in T_x$. Let $q' \leqslant q_n$ be such that $q' \in D^n$. Then $\bar{q}' \subseteq q_n$. Let $q_{n+1} \in D_{n+1}$ be such that $q_{n+1} \subseteq q'$. If $\langle \mathscr{Y}, \mathcal{O}' \rangle^{q_n}$ is complete metric then $(\mathrm{E}y)(y \in \bigcap_n q_n)$. If $\langle \mathscr{Y}, \mathcal{O}' \rangle$ has the centralization property, then as well such a y exists — in this case we have to be careful to ensure that y is a branch of T_x, but this is not difficult. Let us show that $\langle x, y \rangle$ is a $\mathscr{D}$-generic branch of T.

Indeed let us show first that it is a branch of T. Let

$$\langle p, q \rangle \in \mathcal{O} \times \mathcal{O}', \quad \langle x, y \rangle \in \langle p, q \rangle .$$

We have: y is a branch of T_x, thus there is a $q' \leqslant q$ such that $q' \in T_x$. By the definition of T_x there is a p' such that $x \in p'$ & $\langle p'q' \rangle \in T$. But then x is a branch of P thus there is a p'' such that $x \in p'' \subseteq p \cap p'$.

By the continuity of T, and by the fact that $p'' \in P$ we have $\langle p'', q' \rangle \in T$. Hence $\langle x, y \rangle$ is a branch of T.

Let now $D \in \mathscr{D}$. Let us show that $\langle x, y \rangle$ intersects D. By the fact that y intersects D_x there are q, p such that $x \in p$ & $y \in q$ & $\langle p, q \rangle \in D$. Thus $\langle x, y \rangle$ is a $\mathscr{D}$-generic branch of T. Hence $R(x, y)$. ■

COROLLARY 5. *If R is G_δ in the Baire or Cantor space and continuous then* Dom$R$ *is G_δ.*

COROLLARY 6. *If R is $\omega_1 - G_\delta$, $\overline{\overline{\mathcal{O}}}' < \omega_1$, R continuous, then* Dom$R$ *is $\omega_1 - G_\delta$.*

Both corollaries follow from the proof of the theorem.

In the sequel let us denote Dom$R$ by ΣR and let $\Pi R = \{x\colon (y) R(x, y)\}$.

Up to now we have studied ΣR for a g.$G_\delta R$. Let us now fix our attention on ΠR.

It is very difficult to find a condition on R sufficient for the fact that ΠR is g.G_δ. We shall formulate a condition on R sufficient for the existence of a g.G_δ dense subset of ΠR.

THEOREM 2. *Let $\langle \mathscr{X}, \mathcal{O} \rangle$, $\langle \mathscr{Y}, \mathcal{O}' \rangle$ be T_1 topological spaces with the centralization property. Let $R \subseteq \mathscr{X} \times \mathscr{Y}$ be continuous and g.G_δ. Let T, $\mathscr{D}$ be such that*

$$R(x, y) \equiv \langle x, y \rangle \text{ is a } \mathscr{D}\text{-generic branch of } T .$$

Let $L[T, \mathscr{D}]$ be the smallest inner model M such that $T \in M$, $\mathscr{D} \subseteq M$. Assume that $P(\mathcal{O}) \cap L[T, \mathscr{D}]$ is countable (especially $\mathscr{D}$ is countable). Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a forcing function. Define $R_f(x)$ as $R(x, f(x))$. Assume that for every forcing function f, R_f is dense in $\langle \mathscr{X}, \mathcal{O} \rangle$. Assume that ΠR is absolute. Then ΠR has a dense g.G_δ subset.

Remark 5. If $\langle \mathscr{Y}, \mathcal{O}' \rangle$ has a decomposition $\mathscr{Y} = \bigcup_{i \in I} \mathscr{Y}_i$ into clopen sets such that $\mathscr{Y}_i$ are compact then we can drop the assumption that for every forcing function f, R_f is dense. We obtain as the conclusion:

if ΠR is dense then it contains a dense g.G_δ subset,

if ΠR is non-empty then it contains a non-empty g.G_δ subset. Generally ΠR has a g.G_δ subset A such that $(p)_{\mathcal{O}}(A \cap p \neq \emptyset \equiv \Pi R \cap p \neq \emptyset)$.

We shall not give a proof of this remark here — the proof is rather complicated, and is a slight refinement of the proof of Theorem 4.1 of [4].

Proof of Theorem 2. We shall show that every x which is generic over $\mathcal{O}$, $L[T, \mathscr{D}]$ is in ΠR.

Indeed — let x be generic over $\mathcal{O}$, $L[T, \mathscr{D}]$. Let f be a forcing function. Let us show that R_f is g.G_δ and continuous. Let $D \in \mathscr{D}$. We can assume that D is dense open in T, i.e. if $\langle p, q \rangle \in D$ & $\langle p', q' \rangle \in T$ & $\langle p', q' \rangle \leqslant \langle p, q \rangle$ then $\langle p', q' \rangle \in D$. Let D_f be defined as follows

$$D_f = \{p\colon (\mathrm{E}q)(\langle p, q \rangle \in D \,\&\, p \cap \mathrm{Dom} f \subseteq f^{-1}(q)) \,\&\, (\mathrm{E}x)_p(f(x) \in q)\} .$$

Let us show that D_f is dense in $\mathcal{O}$. Indeed let $p \in \mathcal{O}$. Let $x \in p \cap R_f$. Then $\langle x, f(x) \rangle$ is a $\mathscr{D}$-generic branch of T. Hence there is a $\langle p', q \rangle$ in D such that $p' \subseteq p$ & $\langle x, f(x) \rangle \in \langle p', q' \rangle$. By the continuity of f, there is a $p'' \subseteq p'$ such that $p'' \cap \mathrm{Dom} f \subseteq f^{-1}(q)$.

Notice that by the proof of Theorem 1, we can assume that T is continuous. Hence $\langle p'', q \rangle \in T$ and thus $\langle p'', q \rangle \in D$. Hence $p'' \in D_f$ and thus D_f is dense in $\mathcal{O}$.

Let $\mathscr{D}_f = \{D_f\colon D \in \mathscr{D}\}$. Let us show that

$$x \in R_f \equiv x \text{ is a } \mathscr{D}_f\text{-generic branch of } \mathcal{O} .$$

Indeed let $x \in R_f$. Then immediately x is a $\mathscr{D}_f$-generic branch of $\mathcal{O}$.

Assume conversely that x is a $\mathscr{D}_f$-generic branch of $\mathcal{O}$. Let $D \in \mathscr{D}$. There is a p in D_f such that $x \in p$. Hence there is a q such that $\langle p, q \rangle \in D$. We have $p \cap \mathrm{Dom} f \subseteq f^{-1}(q)$. Hence $f(x) \in q$. Thus $\langle x, f(x) \rangle$ is a $\mathscr{D}$-generic branch of T and hence $x \in R_f$.

Thus for every forcing function f, $x \in R_f$. Notice that if $y \in L[x, T, \mathscr{D}]$, then there is a forcing function f such that $y = f(x)$. Hence

$$L[x, T, \mathscr{D}] \models (y) R(x, y) .$$

By the absoluteness of ΠR, we have $(y) R(x, y)$. Hence $x \in \Pi R$. ■

COROLLARY 7. *If $\mathscr{X}$, $\mathscr{Y}$, R are as before and we assume that for every forcing function f, $\Pi R \cap \mathrm{Dom} f$ is dense, then we can drop the assumption that R_f is dense. Indeed if $x \in \Pi R \cap \mathrm{Dom} f$, then we have $(y) R(x, y)$ and especially $R(x, f(x))$, i.e., $R_f(x)$.*

COROLLARY 8. *If $\mathscr{X}$, $\mathscr{Y}$, R are as before and ΠR intersects every dense in an open set, g.G_δ subset of $\mathscr{X}$, then ΠR has a dense g.G_δ subset. If $\mathcal{O}'$ is countable and ΠR intersects every dense in an open set G_δ set, then ΠR has a dense g G_δ subset.*

COROLLARY 9. *Let $A \subseteq 2^\omega$ be Π^1_1. Then A has a definition*

$$x \in A \equiv (y)_{\omega^\omega}(\mathrm{E}n) R_0(x{\restriction}n, y{\restriction}n, n)$$

where R_0 is recursive. Let

$$R(x, y) \equiv (\mathrm{E}n) R_0(x{\restriction}n, y{\restriction}n, n) .$$

Then R is G_δ and continuous.

Notice that in Theorem 2 we can assume only that R_f is dense only for such functions f that f is determined by a tree in $L[T, \mathscr{D}]$. This follows from the proof of

Theorem 2. For such functions $\mathrm{Dom} f$ is g. G_δ where the appropriate tree is also in $L[T, \mathscr{D}]$ — this follows from the proof of Fact 1. Hence in our case we have: If A intersects every dense in an open set $\Pi^0_2(L)$ set then A contains every Cohen number.

Notice that the converse is also true. Thus if $A \in \Pi^1_1$ then A intersects every dense in an open set $\Pi^0_2(L)$ set iff A contains all Cohen numbers.

Now let us apply our theorems to degrees.

Let us consider a typical form of a theorem about degrees in recursion, give a general version of its proof and then translate it to the case of degrees of constructibility in ZF and give a condition under which the translation holds in ZF and the proof can be translated as well.

Let $\langle \mathscr{X}_0, \mathcal{O}'_0\rangle, \langle \mathscr{X}_1, \mathcal{O}'_1\rangle, \langle \mathscr{Y}, \mathcal{O}'_2\rangle$ be topological spaces, $\mathscr{X}_i, \mathscr{Y} \subseteq 2^\omega$ or $\mathscr{X}_i, \mathscr{Y} \subseteq \omega^\omega$, $\mathcal{O}'_i = 2^{<\omega}$ or $\mathcal{O}'_i = \omega^{<\omega}$. Let $\mathcal{O}_1$, $\mathcal{O}_2$ be other topologies in $\mathscr{X}_1, \mathscr{Y}$ respectively, and let $\mathcal{O}_i$ be finer than $\mathcal{O}'_i$. Let $R(x_0, x_1, y) \subseteq \mathscr{X}_0 \times \mathscr{X}_1 \times \mathscr{Y}$ be closed in $\mathcal{O}'_0 \times \mathcal{O}'_1 \times \mathcal{O}'_2$. Let $\mathrm{Rec} = \{x \in 2^\omega\colon x \text{ is recursive}\}$. For simplicity assume $\mathcal{O}'_0 = \mathcal{O}'_1 = 2^{<\omega}$.

EXAMPLES.

(1) $\langle \mathscr{X}_0, \mathcal{O}_0\rangle = \langle 2^\omega, 2^{<\omega}\rangle$, $\langle \mathscr{X}_1, \mathcal{O}_1\rangle = \langle \mathrm{Rec}, \text{discrete}\rangle$, $\langle \mathscr{Y}, \mathcal{O}_2\rangle = \langle \omega, \text{discrete}\rangle$, $R_1(x_0, x_1, n) \equiv x_0(n) \neq x_1(n)$.

(2) $\langle \mathscr{X}_0, \mathcal{O}'_0\rangle = \langle 2^\omega, 2^{<\omega}\rangle$, $\mathscr{X}_1 = \{x \in 2^\omega\colon (x)_1 \in \mathrm{Rec}\}$, $\mathcal{O}_1 = \{\langle s, (x)_1\rangle\colon x \in \mathscr{X}_1\}$, $\mathscr{Y} = \{y \in 2^\omega\colon (y)_0$ codes a recursive tree $\subseteq 2^{<\omega} \times 2^{<\omega}$ determining a continuous function from 2^ω to 2^ω, $(y)_1 \in \mathrm{Rec}, (y)_2 \in \omega\}$, $\mathcal{O}_2$ discrete. Let

$$R_2(x_0, x_1, y) \equiv [x_0 = (y)_0((x_1)_0) \vee (x_1)_0 = (y)_1] \,\&\, x_0((y)_2) \neq (x_1)_1((y)_2)\,.$$

(3) $\langle \mathscr{X}_0, \mathcal{O}'_0\rangle = \langle 2^\omega, 2^{<\omega}\rangle = \langle \mathscr{X}_1, \mathcal{O}_1\rangle$, $\mathscr{Y} = \{y\colon (y)_0, (y)_1, (y)_2$ code recursive tree $\subseteq 2^{<\omega} \times 2^{<\omega}$ determining continuous functions, $(y)_3 \in \mathrm{Rec}\}$, $\mathcal{O}_2$ discrete. Let

$$\begin{aligned} R_3(x_0, x_1, y) \equiv {} & [x_0 = (y)_0(x_1) \vee x_1 = (y)_0((x_0)_0)] \,\&\\ & \&\, [(x_0)_0 = (y)_1(x_1) \vee x_1 = (y)_0((x_0)_1)] \,\&\\ & \&\, [(x_0)_1 = (y)_1(x_1) \vee x_1 = (y)_3]\,. \end{aligned}$$

(4) Let the spaces $\langle \mathscr{X}_0, \mathcal{O}'_0\rangle$, $\langle \mathscr{Y}, \mathcal{O}_2\rangle$ be as in (3) and $\langle \mathscr{X}_1, \mathcal{O}_1\rangle$ as in (2). Let

$$\begin{aligned} R_4(x_0, x_1, y) \equiv {} & [(x_0)_0 = (y)_0((x_1)_0) \vee ((x_1)_0) = (y)_0((x_0)_1) \vee (x_0)_1\\ & = (y)_1((x_1)_0) \vee (x_1)_0 = (y)_3] \,\&\, (x_0)_1 = (y)_1((x_0)_0) \,\&\\ & \&\, (x_0)_1 \neq (x_1)_1\,. \end{aligned}$$

Let $(T_e)_{e\in\omega}$ enumerate all recursive trees $\subseteq 2^{<\omega} \times 2^{<\omega}$ determining a function from $\mathscr{X}_0$ to $\mathscr{X}_1$ continuous in $2^{<\omega}$ w.r.t. $\mathcal{O}_1$ (it is clear what this means on the basis of Corollary 8). Let $A(x_0)$ be of the forms:

(a) $(e)(\mathrm{E}n)\big(T_e(x_0)(n)$ is not defined or $(\mathrm{E}y)R(x_0, T_e(x_0), y)\big)$,

(b) $(e)(\mathrm{E}n)\big(T_e((x_0)_1)(n)$ is not defined or $(\mathrm{E}y)R((x_0)_0, T_e((x_0)_1), y)\big) \,\&$
$\&\ (e)(\mathrm{E}n)\big(T_e((x_0)_0)(n)$ is not defined or $(\mathrm{E}y)R'((x_0)_1, T_e((x_0)_0), y)\big)$,

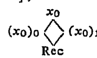
(c) $(e)(\mathrm{E}n)\big(T_e(x_0)(n)$ is not defined or $(\mathrm{E}y)R(x_0, T_e(x_0), y)\big) \,\&$
$\&\ (e)(\mathrm{E}n)\big(T_e((x_0)_1)(n)$ is not defined or $(\mathrm{E}y)R'((x_0)_0, T_e((x_0)_1), y)\big) \,\&$
$\&\ (e)(\mathrm{E}n)\big(T_e((x_0)_0)(n)$ is not defined or $(\mathrm{E}y)R''((x_0)_1, T_e((x_0)_0), y)\big)$,

(d) two first lines of (c).

As far as our examples are considered we have:

If (1) and A is of the form (a) with $R = R_1$, then $A(x_0) \equiv x_0$ is not recursive; if A is of the form (b) and R, $R' = R_1$, then $A(x_0) \equiv (x_0)_0$ is not recursive in $(x_0)_1$ and $(x_0)_1$ is not recursive in $(x_0)_0$.

If (2) and A is of the form (a) with $R = R_2$, then $A(x_0) \equiv x_0$ is of minimal non-recursive Turing degree.

If (3) and A is of the form (c) with $R = R_3$, R', $R'' = R_1$, then $A(x_0) \equiv x_0$ determines an initial segment of Turing degrees of the form 

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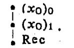
If (4) and A is of the form (d) with $R = R_4$, $R' = R_1$, then $A(x_0) \equiv x_0$ determines an initial segment of the degrees of the form 

.

In all the above cases "$(\mathrm{E}x_0)A(x_0)$" is a theorem of recursion theory.

Let us analyse briefly a typical proof of such a theorem. For simplicity let us consider the case where A is of the form (a), i.e.

$$A(x_0) \equiv (e)(\mathrm{E}n)\big(T_e(x_0)(n) \text{ is not defined or } (\mathrm{E}y)R(x_0, T_e(x_0), y)\big)\,.$$

The cases (b), (c), (d) need a somewhat more subtle treatment.

Let $\langle \mathscr{X}'_1, \mathcal{O}''_1\rangle = \langle\{x \in \omega^\omega\colon x \text{ codes a tree } T_e\}, \text{discrete}\rangle$. Let $\tilde{R}(x_0, x_1, y, n)$ be defined as "$x_1(n)$ is undefined or $R(x_0, x_1(x_0), y)$". Then $\tilde{R}$ is closed in $2^{<\omega} \times \mathcal{O}''_1 \times \mathcal{O}''_2$. We have $A(x_0) \equiv (x_1)_{\mathscr{X}'_1}(\mathrm{E}n)(\mathrm{E}y)\tilde{R}(x_0, x_1, y, n)$.

Let us define a topology $\mathcal{O}_0$ in $\mathscr{X}_0$ with good properties such that R is continuous in $\mathcal{O}_0 \times \mathcal{O}''_1$ w.r.t. $\mathcal{O}_2$. Assume that $\mathcal{O}_2$ is discrete.

Let $p \in \mathcal{O}_0$ if there are $s \in 2^{<\omega}$, e, y, n such that $p = s \cap \{x_0\colon R(x_0, T_e(x_0) y)\}$ or $p = s \cap \{x_0\colon T_e(x_0)(n)$ is not defined$\}$ or if p is a finite intersection of sets of the above form. If R is an alternative as in Example (2), then we can define $p \in \mathcal{O}_0$ if there are e, y's such that

$$p = s \cap \{x_0\colon x_0 = (y)_0(T_e((x_0)_0)) \text{ and } x_0((y)_2) \neq T((x_0)_1)((y)_2)\}$$

or

$$p = s \cap \{x_0\colon T_e(x_0) = (y)_1 \text{ and } x_0((y)_2) \neq T_e((x_0)_1)((y)_2)\}$$

or

$$p = s \cap \{x_0\colon T_e(x_0)((y)_2) \text{ is not defined}\}$$

or p is a finite intersection of sets of the above form.

Then $\mathcal{O}_0$ consists of sets that are closed in the usual topology $2^{<\omega}$, and hence $\mathcal{O}_0$ has the centralization property. Moreover, $\tilde{R}$ is continuous in $\mathcal{O}_0 \times \mathcal{O}''_1$ w.r.t. $\mathcal{O}_2$. Next let us take a subset $\boldsymbol{P}_0 \subseteq \mathcal{O}_0$ such that

(***) $\quad (p^0)_{\boldsymbol{P}_0}(e)(\mathrm{E}x)_{p^0}(x \text{ is a branch of } \boldsymbol{P}_0 \text{ and } (\mathrm{E}y)(\mathrm{E}n)\,\tilde{R}(x, T_e, y, n))$.

We know that if the theorem $(\mathrm{E}x_0)A(x_0)$ is true, then such a $\boldsymbol{P}_0 \subseteq \mathcal{O}_0$ exists. However, it may be difficult to find such a $\boldsymbol{P}_0$ explicitly. Notice that instead of $\mathcal{O}_0$

we could take any topology $\mathcal{O}_0''$ finer than $\mathcal{O}_0$ consisting of sets that are closed in the usual topology in $\mathcal{X}_0$ and then we would preserve the centralization property of $\mathcal{O}_0''$ and the property that R is continuous in $\mathcal{O}_0'' \times \mathcal{O}_1''$ w.r.t. $\mathcal{O}_2$. In practice we take usually the topology of perfect sets with recursive codes included in an element of $\mathcal{O}_0$. Then if we define

$$D_e = \{p\colon (\mathrm{E}y)(\mathrm{E}n)(p \subseteq \{x_0\colon R(x_0, T_e(x), y) \text{ or } T_e(x_0)(n) \text{ is undefined}\})\},$$

then we can usually show that D_e is dense in the topology defined above, and so, if we take $\boldsymbol{P}_0$ to be this topology itself, then $\boldsymbol{P}_0$ has the property (***) (we have to remark that $\boldsymbol{P}_0$ is in fact a topology in the set of branches of $\boldsymbol{P}_0$, not necessarily in the whole $\mathcal{X}_0$, but this does not lead to any difficulties).

Notice that $\Sigma\tilde{R}$ is continuous in $\mathcal{O}_0$ w.r.t. $\mathcal{O}_1''$ by the definition of $\mathcal{O}_0$, and by the continuity of $\tilde{R}$ in $\mathcal{O}_0 \times \mathcal{O}_1''$ w.r.t. $\mathcal{O}_2$ together with Theorem 1, $\Sigma\tilde{R}$ is g.G_δ in $\langle \mathcal{X}_0, \mathcal{O}_0\rangle \times \langle \mathcal{X}_1, \mathcal{O}_1''\rangle$. By the density of the sets D_e, the assumptions of Theorem 2 are satisfied (by $\Sigma\tilde{R}$ in the place of R). Hence $\Pi\Sigma\tilde{R} = A$ is g.G_δ. Analyzing the proof of Theorem 2 we can infer that if $\mathcal{D} = (D_e)_{e\in\omega}$ then any $\mathcal{D}$-generic branch of $\boldsymbol{P}_0$ gives a solution of A.

Let us return to set theory. Let $\langle \mathcal{X}_0, 2^{<\omega}\rangle$, $\langle \mathcal{X}_1, \mathcal{O}_1\rangle$, $\langle \mathcal{Y}, \mathcal{O}_2\rangle$ be spaces such as those considered above, let $R \subseteq \mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{Y}$ be closed in $2^{<\omega} \times 2^{<\omega} \times \mathcal{O}_2'$, let $(T_e)_{e\in\omega}$ be as above and let:

$$A(x_0) \equiv (e)(\mathrm{E}n)(T_e(x_0)(n) \text{ is not defined or } (\mathrm{E}y)R(x_0, T_e(x_0), y)).$$

Assume that $(\mathrm{E}x_0)A(x_0)$ is a theorem of recursion theory. Let us perform the following operation: replace in the definitions of $\langle \mathcal{X}_0, \mathcal{O}_0'\rangle$, $\langle \mathcal{X}_1, \mathcal{O}_1\rangle$, $\langle \mathcal{Y}, \mathcal{O}_2\rangle$, R, $\tilde{R}$, $(T_e)_{e\in\omega}$ the word "recursive" by the word "constructible". We can easily do it in our Examples (1)-(4) to have a better idea of the meaning of this operation. Then (in most of the cases and especially in (1)-(4) we obtain spaces $\langle \mathcal{X}_0, 2^{<\omega}\rangle$, $\langle \mathcal{X}_1, \mathcal{O}_1\rangle$, $\langle \mathcal{Y}, \mathcal{O}_2\rangle$ such that $\mathcal{O}_i$ are L-codable (see [1]), R remains closed in $2^{<\omega} \times \mathcal{O}_1 \times \mathcal{O}_2$ and instead of the enumeration $(T_e)_{e\in\omega}$ we obtain an enumeration $(\tilde{T}_\xi)_{\xi\in\omega_1^L}$ of constructible trees $\subseteq 2^{<\omega} \times 2^{<\omega}$ determining a function continuous in $2^{<\omega}$ w.r.t. $\mathcal{O}_1$. By applying the same operation of translation to $\mathcal{O}_0$ we would be able to prove, under the assumption $\omega_1^L \simeq \omega$, that

$$(\mathrm{E}x_0)(\xi)(x_0 \in \mathrm{Dom}\, \tilde{T}_\xi \to (\mathrm{E}y)R(x_0, \tilde{T}_\xi(x_0), y)),$$

i.e., that there exists an x_0 such that for every x_1 in $\mathcal{X}_1$ which is recursive in x_0 and a constructible parameter (consisting of a tree $\tilde{T}_\xi$) we have $(\mathrm{E}y)R(x_0, x_1, y)$. Usually this is not very interesting. However, we shall show a condition under which the above predicate $A(x_0)$ is equivalent to the statement: for every x_1 in $\mathcal{X}_1$ which is constructible from x_0 we have $(\mathrm{E}y)R(x_0, x_1, y)$. Then $(\mathrm{E}x_0)A(x_0)$ is in our case a theorem about degrees of constructibility. E.g. if (1) and A is of the form (a) with $R = R_1$ (under the appropriate translation), then $A(x_0) \equiv x_0$ is not constructible; if A is of the form (b), then $A(x_0) \equiv (x_0)_0, (x_0)_1$ are mutually non-constructible from each other; if (2) and A is of the form (a) with $R = R_2$, then A means that x_0 is of minimal degree of constructibility; and analogously if (3) (c) or (4) (d), the $A(x_0)$ means that x_0 determines the appropriate initial segment of degrees of constructibility.

As we said before, we can replace the theorem $(\mathrm{E}x_0)A(x_0)$ under the assumption $\omega_1^L \simeq \omega$ by an appropriate consistency theorem. Then in the last two examples, i.e. (3) (c) and (4) (d), we infer that it is consistent that the degrees of constructibility just form the given pattern.

Let $\mathcal{O}_0$ be defined by the appropriate translation of the former definition of $\mathcal{O}_0$, i.e. $p \in \mathcal{O}_0$ iff p is a finite intersection of sets of the form $s \cap \{x_0\colon \tilde{T}_\xi(x_0)(n)$ is not defined$\}$ or $s \cap \{x_0\colon R(x_0, \tilde{T}_\xi(x_0), y)\}$ for given a, n, y. Let $\boldsymbol{P}_0$ be a subset of a refinement of $\mathcal{O}_0$ with the appropriate property (***) and the centralization property. Let us consider the following condition:

(****) for every regular tree $T \subseteq \boldsymbol{P}_0 \times O_1$ and for every $p \in \boldsymbol{P}_0$ there is a $q \in \boldsymbol{P}_0$, $q \leqslant p$ and a tree $\tilde{T} \subseteq 2^{<\omega} \times 2^{<\omega}$ determining a function from $\mathcal{X}_0$ to $\mathcal{X}_1$ continuous in $2^{<\omega}$ w.r.t. $\mathcal{O}_1$ total on q and such that for $x_0 \in q$, $T(x_0) = \tilde{T}(x_0)$.

If $\langle \mathcal{X}_1 \mathcal{O}_1\rangle$ has the Moore property (see [1]), then condition (****) is a consequence of, say, the following condition (f), which can be called "fusion lemma" by an analogy to the Sacks "fusion lemma" from his paper about perfect forcing:

(f) if $(D_n)_{n\in\omega}$ is a constructible family of sets dense in $\boldsymbol{P}_0$ and $p \in \boldsymbol{P}_0$, then there is a $q \leqslant p$ in $\boldsymbol{P}_0$ such that
$(n)(\mathrm{E}m_n)(\mathrm{E}s_1 \ldots s_{m_n})(q \cap s_i \in D_n \text{ and } q \leqslant s_1 \vee \ldots \vee s_{m_n})$.

Indeed, let us show that (f) $\Rightarrow$ (****).

Let T be given and satisfy the assumption of (****), and let $p \in \boldsymbol{P}_0$. Define

$$D_n = \{p' \in \boldsymbol{P}_0\colon (\mathrm{E}p^1)_{\mathcal{O}_1}(\mathrm{E}t)(\mathrm{rank}\, p^1 = n \text{ and } \mathrm{dom}\, t = n \,\&\, p^1 \leqslant t \,\&\, \langle p', p^1\rangle \in T)\}.$$

Let q be such as in the "fusion lemma" for p and the family $(D_n)_{n\in\omega}$ (note that D_n are dense in $\boldsymbol{P}_0$). Let $\tilde{T}$ be defined as

$$\{\langle s, t\rangle\colon s \cap q \in D_n \text{ for an } n \,\&\, (\mathrm{E}p^1)(\mathrm{rank}\, p^1 = n \text{ and } \langle q \cap s, p^1\rangle \in T)\}.$$

Let us show that $\tilde{T}$ determines a function from q to $\mathcal{X}_1$ which is total and continuous on q in $2^{<\omega}$ w.r.t. $\mathcal{O}_1$. Let $x_0 \in q$. Then $\tilde{T}$ is defined at x_0 and

$$\tilde{T}(x_0) = \bigcup \{t\colon (\mathrm{E}s)(\langle s, t\rangle \in \tilde{T})\}.$$

Indeed, for every n there is a t such that $\mathrm{dom}\, t = n$ and there is an s such that $x_0 \in q \cap s$ and $\langle s, t\rangle \in \tilde{T}$. Hence $\tilde{T}(x_0)$ is defined. Let $(p_n^1)_{n\in\omega}$ be such that

$$(\mathrm{E}t)(\mathrm{dom}\, t = n \,\&\, (\mathrm{E}s) \subseteq x_0(\langle q \cap s, p_n^1\rangle \in T) \,\&\, p_n^1 \leqslant t \,\&\, \mathrm{rank}\, p_n^1 = n).$$

Then $\tilde{T}(x_0) \in \bigcap_n p_n^1$ by the centralization property $\langle \mathcal{X}_1, \mathcal{O}_1\rangle$. On the other hand, $T(x_0)$ is then defined and $T(x_0) \in \bigcap_n p_n^1$. Hence $\tilde{T}(x_0) = T(x_0)$. It remains to show the continuity of the function $\tilde{T}$ in $2^{<\omega}$ w.r.t. $\mathcal{O}_1$. Let $\tilde{T}(x_0) \in p^1$. Then there is a $\bar{p}^1 \leqslant p^1$ such that $\tilde{T}(x_0) \in \bar{p}^1$ and there is an s such that $\langle q \cap s, \bar{p}^1 \rangle \in T$ and $x_0 \in s$. Let $x_0' \in q \cap s$. Then $T(x_0') \in p^1$ by the continuity of T and by the fact that T is total on q (we have shown this a few lines above). But as we have just shown, $T(x_0') = \tilde{T}(x_0')$. Hence $\tilde{T}(x_0') \in p^1$, which we wanted to show.

Let us show that $\boldsymbol{P}_0$, ΣR, $\langle \mathcal{X}_0 \boldsymbol{P}_0 \rangle$, $\langle \mathcal{X}_1 \mathcal{O}_1 \rangle$, satisfy the assumptions of Theorem 2.

Notice first that R is closed in $\mathcal{O}_0 \times \mathcal{O}_1 \times \mathcal{O}_2$ because, as we have already remarked, it is closed in $2^{<\omega} \times 2^{<\omega} \times \mathcal{O}_2'$ and $\mathcal{O}_i$ are finer than $\mathcal{O}_i'$. Hence R is g. G_δ. Moreover by Theorem 1, ΣR is g. G_δ. Notice that by the definition of $\mathcal{O}_0$, (ΣR) is continuous in $\langle \mathcal{X}_0, \mathcal{O}_0 \rangle$ w.r.t. $\langle \mathcal{X}_1, \mathcal{O}_1 \rangle$.

Let T be a regular tree included in $\mathcal{O}_0 \times \mathcal{O}_1$.

The crucial point is to show that

$$(\ast) \qquad (p^0)_{\boldsymbol{P}_0} (\mathrm{E}x)_{\mathrm{dom}\, T} \big(x \in p^0 \;\&\; (\mathrm{E}y) R(x, T(x), y)\big),$$

i.e. the density of $(\Sigma R)_f$, where f is determined by T.

Note that for T there is a dense set $D_T = \{p \in \boldsymbol{P}_0 \colon T$ determines a total function on p and there is a tree $T_p \subseteq 2^{<\omega} \times 2^{<\omega}$ determining a total function on p, continuous in $2^{<\omega}$ w.r.t. $\mathcal{O}_1$ such that for $x_0 \in p$, $T(x_0) = T_p(x_0)\}$.

Let us also note that $\boldsymbol{P}_0$ has the following property:

$$(p^0)_{\boldsymbol{P}_0} (\eta) (\mathrm{E}x)_{p0} \big(x \text{ is a branch of } \boldsymbol{P}_0 \text{ and } (\mathrm{E}y)(\mathrm{E}n) \tilde{R}(x, \tilde{T}_\eta, y, n)\big),$$

i.e.,

$$(p^0{}_{\boldsymbol{P}_0} (\eta) (\mathrm{E}x)_{p0} \big(x \text{ is a branch of } \boldsymbol{P}_0 \text{ and either } \tilde{T}_\eta(x)(n) \text{ is not defined or } (\mathrm{E}y) R(x, \tilde{T}_\eta(x), y)\big).$$

Let $p^0 \in \boldsymbol{P}_0$. Let $q \leqslant p^0$ be such that $q \in D_T$. Take T_q. Let x be such that $x \in q$, x is a branch of $\boldsymbol{P}_0$ and there are y, n such that $R(x, T_q, y, n)$. Take y, n. Notice that we cannot have "$T_q(x)(n)$ is not defined" because T_q is total on q. Hence we have $R(x, T_q(x), y)$. Let $r = \{x_0 \colon R(x_0, T_q(x_0) y)\}$. Then $r \in \mathcal{O}_0$. By the fact that $\boldsymbol{P}_0 \subseteq \mathcal{O}_0$ or $\boldsymbol{P}_0$ is finer than $\mathcal{O}_0$, there is an r' such that $r' \subseteq r \cap q$. Let $x_0 \in r'$, x_0 be a branch of $\boldsymbol{P}_0$. Let us show that x_0 is as required in $(\ast)$. We have $x_0 \in \mathrm{dom}\, T$ because $x_0 \in q$. Moreover, we have $R(x_0, T(x_0), y)$ because $R(x_0, T_q(x_0), y)$ and $T(x_0) = T_q(x_0)$.

By Theorem 2 we have:

every filter generic over $\boldsymbol{P}_0$, L determines a real x_0 such that

$$L(x_0) \models (x_1)(\mathrm{E}y) R(x_0, x_1, y),$$

i.e. $L(x_0) \models A(x_0)$. This is what we wanted to show.

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