Combinatorics on σ-algebras and a problem of Banach

by

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Abstract. A σ-algebra S on the reals is called *measurable* if there exists a probability measure on S vanishing on atoms of S. Banach [0] asked if the union of two countably generated measurable σ-algebras can generate a non-measurable σ-algebra. This problem was solved positively by Grzegorek [4]. Under the assumption of Martin's axiom we show a large family of σ-algebras with the property that all its small subfamilies generate measurable σ-algebras and all large subfamilies — non measurable σ-algebras. We also consider a group-invariant version of Banach's problem and various questions concerning the structure of σ-algebras and their measurability (1).

0. The key-notion of this paper is a σ-algebra of subsets of a set X. We also often use the abbreviation "σ-algebra on X" meaning a family of subsets of X containing X as an element and closed under complements and countable unions. A non-void set a is called an *atom of the σ-algebra* S if $a \in S$ and for every $b \subset a$ if $b \in S$ then $b = \emptyset$ or $b = a$. We say that a σ-algebra S is $\varkappa$-*generated* if there exists a family $T \subset S$, $|T| = \varkappa$ such that S is the smallest σ-algebra containing T. The family T is then called a *generating family* and the σ-algebra generated by T is denoted $\sigma(T)$.

By a measure on a σ-algebra S on X we mean a function $m\colon S \to [0, 1]$ with the following properties:

1° $m(a) = 0$ if a is an atom of S,

2° $m(X) = 1$,

3° $m(\bigcup_{n\in\omega} A_n) = \sum_{n\in\omega} m(A_n)$ if A_n are pairwise disjoint elements of S.

A measure m on S is complete if subsets of measure zero sets are elements of S and is uniform if sets of cardinality $<|X|$ are subsets of measure zero sets.

A σ-algebra S on X is called *measurable* if there exists a measure on S and uniformly measurable if there exists a uniform measure on S.

A family I of subsets of X is called a σ-*ideal* on X if I contains singletons, $X \notin I$ and I is closed under countable unions and under the operation of taking subsets. A σ-ideal I on X is called σ-*saturated* if every pairwise disjoint family of subsets of X outside of I is countable.

(1) The results contained in this paper formed a part of the author's Ph. D. thesis.

If $\mathscr{A} = \{A_n: n \in \omega\}$ is a countable family of subsets of X then A_f denotes the set $\bigcap_{n\in\omega} A_n^{f(n)}$, where $A^0 = A$, $A^1 = X \setminus A$, $f \in 2^\omega$. Every A_f is clearly an atom of $\sigma(\mathscr{A})$. The function f is called the index of this atom. The sets A_n are independent if every $f \in 2^\omega$ is the index of an atom of $\sigma(\{A_n: n \in \omega\})$.

A sequence $\{f_\alpha: \alpha < \varkappa\} \subset \omega^\omega$ is called a *$\varkappa$-scale* if for all $\alpha < \beta < \varkappa$, $f_\alpha \prec f_\beta$ and for every $g \in \omega^\omega$ there exists $\alpha < \varkappa$ such that $g \prec f_\alpha$ ($f \prec g$ means $f(n) < g(n)$ for all but finitely many n).

The last group of notions concerns the properties of invariance. Let G be a group of bijections of a set X. We say that a σ-algebra S on X is G-invariant if for every $A \in S, f \in G$ the image $f*(A)$ is an element of S. A measure m on a G-invariant σ-algebra is G-invariant if for every $A \in S, f \in G$ we have $m(f*(A)) = m(A)$. A set $A \subset X$ is G-almost invariant if for every $f \in G$ $|f*(A) \triangle A| < |X|$.

1. The results in the present section were inspired by the following problem of Banach:

Do there exist two countably generated measurable σ-algebras with the union generating a non-measurable σ-algebra?

This question was positively answered by Grzegorek [4]. In connection with it F. Galvin asked (personal communication) for which cardinal parameters the following sentence is true:

There exist $\varkappa$ λ-generated σ-algebras on a cardinal α such that the union of any β of them generates a measurable σ-algebra but the union of any γ of them generates a non-measurable σ-algebra.

Our first theorem solves this problem for small values of cardinal parameters involved, under the assumption e.g. of Martin's axiom.

THEOREM 1.1. *Assume that a 2^ω-scale exists and that Lebesgue measure is uniform. Let $0 < \varkappa \leqslant \omega$. Then there exists a countable family of sets of reals and its subfamilies A_ξ: $\xi < 2^\omega$ such that the σ-algebras $S_\xi = \sigma(A_\xi)$ have the following property: any union of $< \varkappa$ of them generates a uniformly measurable σ-algebra but any union of $\geqslant \varkappa$ of them generates a σ-algebra non-measurable uniformly.*

We split the proof into several lemmas.

LEMMA 1.2. *Assume that a 2^ω-scale exists. There exists a set $T \subset \omega^\omega$, $|T| = 2^\omega$ with the following properties:*

1° *For any sequence $\{W_n: n \in \omega\}$ of infinite sets of natural numbers, any sequence $\{a_n: n \in \omega\}$ of natural numbers such that $a_n \notin W_n$ and any function $h \in 2^\omega$ not eventually equal 1, there exists an $f \in T$ such that $f(n) \in b_n^{h(n)}$, where $b_n^0 = W_n$, $b_n^1 = \{a_n\}$.*

2° *For any function $f \in \omega^\omega$*

$$\big|\{g \in T: \forall n \in \omega\, g(n) < f(n)\}\big| < 2^\omega.$$

Proof. Let $\{\tau_\alpha: \alpha < 2^\omega\}$ be a one-to-one enumeration of sequences $\{G_n: n \in \omega\}$ such that G_n is an infinite or one-element set of natural numbers and moreover it is infinite for infinitely many n. For $\alpha < 2^\omega$, $\tau_\alpha(n)$ denotes the nth element of the sequence τ_α and k_α — the infinite sequence of those n for which $\tau_\alpha(n)$ is infinite.

Let $\{f_\alpha: \alpha < 2^\omega\}$ be a 2^ω-scale and g_α, for $\alpha < 2^\omega$, be such a function from ω^ω that:

1. $g_\alpha(n) \in \tau_\alpha(n)$ for all $n \in \omega$,
2. $g_\alpha(n) > f_\alpha(n)$ for $n \in \operatorname{Rg}(k_\alpha)$,
3. $g_\alpha \neq g_\beta$ for $\beta < \alpha$.

The set $T = \{g_\alpha: \alpha < 2^\omega\}$ is as required.

LEMMA 1.3. *Assume that a set T from Lemma 1.2 exists and that Lebesgue measure is uniform. There exists a matrix $\{B_{nk}: n, k \in \omega\}$ of sets of reals which the following property:*

$\sigma(\{B_{nk}: \langle n, k\rangle \in Z\})$ is a uniformly measurable σ-algebra iff the set $\{n \in \omega: \{k \in \omega: \langle n, k\rangle \notin Z\}$ is an infinite proper subset of $\omega\}$ is infinite.

Proof. It suffices to construct an appropriate matrix on the set T from the previous lemma. Let $B_{nk} = \{f \in T: f(n) = k\}$. For the proof of left-to-right implication assume that a set $Z \subset \omega \times \omega$ does not satisfy the above condition. Let for $n > n_0$ one of the following possibilities hold: either $\forall k \in \omega\, \langle n, k\rangle \notin Z$ or $\exists k_n \forall k > k_n\, \langle n, k\rangle \in Z$.

Consider only these numbers $n > n_0$ for which the first possibility holds. If there are finitely many of them, the σ-algebra in question has only countably many atoms hence it is non-measurable. If not (call the set of those numbers C) we can apply a generalization of the reasoning from [1].

Assume that $\sigma(\{B_{nk}: \langle n, k\rangle \in Z\})$ is a uniformly measurable σ-algebra. Then for $n \in C$ there exists $l_n \geqslant k_n$ such that the set $\bigcup_{k \leqslant l_n} B_{nk}$ has measure at least $1 - 2^{-n+1}$. Hence the set $B = \bigcap_{n \in C} \bigcup_{k \leqslant l_n} B_{nk}$ has measure at least 1/2.

Consider these functions from the set B which for $n \notin C$ have value 0.

By the previous lemma there are only $< 2^\omega$ of them. On the other hand, by the definition of the set C, belonging to a given atom of the σ-algebra in question does not depend on the values of the function for arguments outside of C. Hence the set B is a union of $< 2^\omega$ atoms and as an element of the matrix it should have measure 0, contradiction.

For the proof of right-to-left implication assume that a set Z satisfies the condition in the lemma. Let $\{m_n: n \in \omega\}$ be the increasing sequence of natural numbers for which the set $Z_n = \{k \in \omega: \langle m_n, k\rangle \in Z\}$ is non-void. By a_n we denote the least element of Z_n and put $Z_n' = \omega \setminus Z_n$. Let

$$\Omega = \bigcap_{n \in \omega} \Big(B_{m_n a_n} \cup \bigcup_{k \in Z_n'} B_{m_n k}\Big)$$

and

$$X_n = B_{m_n a_n} \cap \Omega, \qquad Y_n = \Big(\bigcup_{k \in Z_n'} B_{m_n k}\Big) \cap \Omega.$$

Clearly $X_n = \Omega \setminus Y_n$.

In the space Ω the sets X_n are almost independent (in the sense that only for countably many functions $f \in 2^\omega$, eventually equal to 1, the sets $\bigcap_{n \in \omega} X_n^{f(n)}$ might be empty). Since the Lebesgue measure on Borel subsets of the Cantor set is uniform, we can define a uniform measure m on $\sigma(\{X_n: n\in\omega\})$ in the space Ω. Next, putting $m_1(B_{m_n k}) = 0$ for $k\in Z_n\setminus\{a_n\}$ we can extend m to a uniform measure m_1 on the σ-algebra $\sigma(\{B_{nk}: \langle n,k\rangle\in Z\})$ which finishes the proof of the lemma.

The next lemma is due to F. Galvin (personal communication).

LEMMA 1.4. *Let $m\in\omega$. There exist 2^ω sets of natural numbers, such that the intersection of every m-element family is infinite and not equal to ω but the intersection of every $(m+1)$-element family is finite.*

Proof. We construct this family on the set of m-element sets of finite 0-1 sequences rather than on ω. For any function $f\in 2^\omega$ we define the set X_f as the family of all sets of the form $\{f\restriction k, t_1^k, \ldots, t_{m-1}^k\}$, where k is a natural number and t_i^k is a 0-1 sequence of length k. It is easy to check that the family $\{X_f: f\in 2^\omega\}$ satisfies the required conditions.

In the next lemma K will denote the family of all subsets A of $\omega\times\omega$ for which $\{n\in\omega: \{k\in\omega: \langle n,k\rangle\in A\}$ is infinite$\}$ is finite.

LEMMA 1.5. *There exist 2^ω subsets of $\omega\times\omega$ such that the intersection of every finite subfamily is not an element of K but the intersection of every infinite subfamily is an element of K.*

Proof. Let $\{\{X_\alpha^n: \alpha<2^\omega\}: n\in\omega\}$ be a sequence of families of sets constructed in the previous lemma for every natural n. We define the family $\{X_\alpha: \alpha<2^\omega\}$ as follows:

$$X_\alpha = \bigcup_{n\in\omega}\{n\}\times X_\alpha^n, \qquad \alpha<2^\omega.$$

By the properties of X_α^n and the definition of K we get that these sets are as required.

Now we are already able to finish the proof of Theorem 1.1. We consider two cases.

Case 1. κ is a positive natural number. Let $\{X_\alpha: \alpha<2^\omega\}$ be the family of subsets of ω from Lemma 1.4 constructed for $m=\kappa$. We put $Y_\alpha = \omega\setminus X_\alpha$. Let $\{B_{nk}: n,k\in\omega\}$ be the matrix from Lemma 2.3. The required family A_α: $\alpha<2^\omega$ of its subsets is defined by the formula

$$A_\alpha = \{B_{nk}: n\in\omega, k\in Y_\alpha\} \quad \text{for} \quad \alpha<2^\omega.$$

It follows from Lemmas 1.3 and 1.4 that this family has all properties required in the theorem.

Case 2. $\kappa=\omega$. Let $\{X_\alpha: \alpha<2^\omega\}$ be the family of subsets of $\omega\times\omega$ constructed in Lemma 1.5. Put $Y_\alpha = \omega\times\omega\setminus X_\alpha$. For the matrix $\{B_{nk}: n,k\in\omega\}$ from Lemma 2.3 we define its subsets A_α: $\alpha<2^\omega$ as follows:

$$A_\alpha = \{B_{nk}: \langle n,k\rangle\in Y_\alpha\} \quad \text{for} \quad \alpha<2^\omega.$$

The properties follow from Lemmas 1.3 and 1.5. This finishes the proof of Theorem 1.1.

In Pelc, Prikry [10] it was proved that if the continuum hypothesis is assumed, there exist ω_1 countably generated σ-algebras with countable unions generating measurable σ-algebras and uncountable unions generating non-measurable σ-algebras. Together with Theorem 1.1 this gives the following

COROLLARY 1.6. *Assume the continuum hypothesis. Let $\kappa>0$ be a cardinal $\leqslant\omega_1$. There exist countably generated σ-algebras S_α: $\alpha<\omega_1$ on the reals with the following property:*

the union of every subfamily of cardinality $<\kappa$ generates a measurable σ-algebra but the union of every subfamily of cardinality κ generates a non-measurable σ-algebra.

Remark 1.7. J. Cichoń [2] has recently strengthened Corollary 1.6 proving it in ZFC.

Remark 1.8. It is well known that the assumptions of Theorem 1.1 are strictly weaker than Martin's Axiom. We do not know however if all extra assumptions could be removed.

2. In this section we consider a refinement of the problem of Banach for two σ-algebras. The examples of σ-algebras given in section 1 as well as those from Grzegorek [4] do not have any "good" properties enjoyed e.g. by Borel or Lebesgue measurable sets, in particular they are not translation invariant.

The following theorem, proved in Pelc, Prikry [10] provides a solution to the translation-invariant version of Banach's problem.

THEOREM 2.1. *Assume the continuum hypothesis. There exist countably generated σ-algebras S_1, S_2 on the interval $[0,1]$ and measures m_1, m_2 on S_1, S_2 respectively with the following properties:*

1° both σ-algebras S_1 and S_2 contain all Borel sets and are translation invariant,

2° both measures m_1 and m_2 are extensions of the Lebesgue measure on Borel sets and are translation invariant,

3° the σ-algebra $\sigma(S_1\cup S_2)$ is non-measurable.

We consider a more general situation namely the case of G-invariant σ-algebras on the reals, where G is an arbitrary group of bijections of the reals.

THEOREM 2.2. *Assume that the Lebesgue measure λ is uniform. Let G be any group of bijections of the reals, $|G|\leqslant 2^\omega$. There exist countably generated σ-algebras S_1, S_2 on the reals and uniform measures m_1, m_2 on S_1, S_2 respectively with the following properties:*

1° the σ-algebras of sets measurable with respect to measure completions of m_i $(i=1,2)$ are G-invariant and the measure completions are G-invariant,

2° the σ-algebra $\sigma(S_1\cup S_2)$ is non-measurable.

Proof. Let $G=\{T_\alpha: \alpha<2^\omega\}$. Let x_ξ: $\xi<2^\omega$ be a one-to-one enumeration of the reals. We define by induction elements y_α: $\alpha<2^\omega$ and sets V_α: $\alpha<2^\omega$ such that $|V_\alpha|\leqslant|\alpha|\cdot\omega$. Suppose that y_β: $\beta<\alpha$ and V_β: $\beta<\alpha$ are already defined. Let y_α be the first (in the x_ξ-enumeration) element outside of $\bigcup_{\beta<\alpha} V_\beta$. We denote $T^{+1} = T$ and put

$$V_\alpha = \{T_{\beta_1}^{\pm 1} \circ \ldots \circ T_{\beta_n}^{\pm 1}(y_\beta)\colon \beta, \beta_1, \ldots, \beta_n \leqslant \alpha,\ n \in \omega\};$$

next we define pairwise disjoint sets W_α: $\alpha < 2^\omega$ by the formula $W_\alpha = V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$. Any union of these sets is G-almost translation invariant. To show it, take $W = \bigcup_{\alpha\in A} W_\alpha$ for $A \subset 2^\omega$ and an arbitrary $T_\alpha \in G$.

Let $x \in W$, then $x \in W_\xi$ for some $\xi \in A$. If $\alpha < \xi$ it is clear that $T_\alpha^{-1}(x) \in W_\xi$. If $\xi \leqslant \alpha$ then the family of all such possible elements has cardinality $< 2^\omega$. Hence we have proved that $|W \setminus T_\alpha * (W)| < 2^\omega$. Similarly taking T_α^{-1} instead of T_α we show that $|T_\alpha * (W) \triangle W| < 2^\omega$.

Let now X and Y be a disjoint partition of 2^ω into sets of cardinality 2^ω.

We define $A = \bigcup_{\alpha\in X} W_\alpha$, $B = \bigcup_{\alpha\in Y} W_\alpha$. Let moreover $F_1\colon 2^\omega \xrightarrow[\text{onto}]{1-1} X$, $F_2\colon 2^\omega \xrightarrow[\text{onto}]{1-1} Y$. By the uniformity of Lebesgue measure there exists a non-measurable σ-algebra $\mathfrak{N}$ on 2^ω which contains singletons (cf. [3]). Let $\mathfrak{M}$ denote the Borel σ-algebra on 2^ω. We define four σ-algebras:

$$\mathfrak{M}_1 = \{\bigcup_{\alpha\in F_1*(M)} W_\alpha\colon M \in \mathfrak{M}\}, \quad \text{on } A,$$

$$\mathfrak{M}_2 = \{\bigcup_{\alpha\in F_2*(M)} W_\alpha\colon M \in \mathfrak{M}\}, \quad \text{on } B,$$

$$\mathfrak{N}_1 = \{\bigcup_{\alpha\in F_1*(N)} W_\alpha\colon N \in \mathfrak{N}\}, \quad \text{on } A,$$

$$\mathfrak{N}_2 = \{\bigcup_{\alpha\in F_2*(N)} W_\alpha\colon N \in \mathfrak{N}\}, \quad \text{on } B.$$

Now we define as in [3]:

$$S_1 = \{P \cup Q\colon P \in \mathfrak{M}_1,\ Q \in \mathfrak{N}_2\},$$

$$S_2 = \{P \cup Q\colon P \in \mathfrak{N}_1,\ Q \in \mathfrak{M}_2\}$$

and put

$$m_1(P \cup Q) = \lambda(M), \quad \text{where} \quad P = \bigcup_{\alpha\in F_1*(M)} W_\alpha,$$

$$m_2(P \cup Q) = \lambda(M), \quad \text{where} \quad Q = \bigcup_{\alpha\in F_2*(M)} W_\alpha.$$

Clearly m_1 and m_2 are uniform measures on S_1, S_2 respectively. Property 2° can be proved exactly as in [3]. In order to show property 1° notice that sets of cardinality $< 2^\omega$ (as subsets of unions of $< 2^\omega$ atoms) are $\overline{m}_i$-measurable ($i = 1, 2$) if $\overline{m}_i$ denotes the measure completion of m_i. Let $\overline{S}_1$ denote the σ-algebra of $\overline{m}_1$-measurable sets and consider an arbitrary set $E \in \overline{S}_1$. It is of the form $E = E_1 \triangle E_2$, where $E_1 \in \mathfrak{M}_1$ and $E_2 \subset E' \in S_1$, $m_1(E') = 0$. Let T_α be an arbitrary element of the group G. Since $|T_\alpha * (E_1) \triangle E_1| < 2^\omega$ and $|T_\alpha * (E') \triangle E'| < 2^\omega$ (both sets are unions of atoms), we get:

$$T_\alpha * (E_2) \subset T_\alpha * (E') \quad \text{where} \quad T_\alpha * (E') \in \overline{S}_1 \quad \text{and} \quad \overline{m}_1(T_\alpha * (E')) = 0.$$

Since

$$T_\alpha * (E) = T_\alpha * (E_1) \triangle T_\alpha * (E_2) = E_1 \triangle C \triangle T_\alpha * (E_2),$$

where $|C| < 2^\omega$, we get:

$$T_\alpha * (E) \in \overline{S}_1 \quad \text{and} \quad \overline{m}_1(T_\alpha * (E)) = \overline{m}_1(E).$$

A similar reasoning applies to the σ-algebra S_2 which proves the property 1° and also the whole theorem.

The following corollary of the above proof sheds some light on the existence of translation invariant measures on groups.

Corollary 2.3. *Assume that the Lebesgue measure is uniform. On every group G of cardinality 2^ω there exists a countably generated σ-algebra S and a measure m on S such that the measure completion $\overline{S}$ of the σ-algebra S and the measure completion $\overline{m}$ of the measure m are G-invariant.*

3. The results of this section show what impact have the atoms of a σ-algebra on its measurability. It might seem e.g. that measurability is a property of small σ-algebras at least in the sense that a subalgebra of a measurable σ-algebra is measurable. This is certainly true if they have the same atoms. In general however it turns out to be false.

Proposition 3.1. *There exists a countably generated measurable σ-algebra S on the reals which contains a countably generated non-measurable σ-algebra.*

Proof. Let C be a universally null subset of the reals and $\mathscr{B}$ the σ-algebra of Borel subsets of the reals. Then the σ-algebra $S = \sigma(\mathscr{B} \cup \{C\})$ is measurable but it contains a countably generated non-measurable σ-algebra

$$S' = \{C \cap B\colon B \in \mathscr{B}\} \cup \{(C \cap B) \cup (R \setminus C)\colon B \in \mathscr{B}\}.$$

Reversing Banach's problem one can ask whether a measurable σ-algebra can split into non-measurable parts, i.e. if there exist countably generated non-measurable σ-algebras with a measurable union. Here the answer turns out to be negative.

Proposition 3.2. *For any non-measurable countably generated σ-algebras S_1, S_2 on X, the σ-algebra $\sigma(S_1 \cup S_2)$ is also non-measurable.*

Proof. Let $S_1 = \sigma(\{A_n\colon n \in \omega\})$, $S_2 = \sigma(\{B_n\colon n \in \omega\})$ be non-measurable σ-algebras. Assume to the contrary that $S = \sigma(S_1 \cup S_2)$ is measurable and let m be a measure on S. Since S_1 is non-measurable, one of its atoms A_f must have positive measure m. Let $\mathscr{B} = \{X \in S\colon X \subset A_f\}$. If $X \in \mathscr{B}$ then $X = A_f \cap Y$ for some $Y \in S_2$. For $Y \in S_2$ define

$$\tilde{m}(Y) = \frac{m(A_f \cap Y)}{m(A_f)}.$$

For any atom a of S_2 the set $a \cap A_f$ is an atom of S, hence $m(a) = 0$. It follows that m is a measure on S_2, contradiction.

Proposition 3.1 shows that large atoms may cause pathological situations from the point of view of measurability. Hence the interesting case is when σ-algebras in question contain all singletons (which are then their atoms). The next proposition shows that every countably generated measurable σ-algebra can be extended to such a σ-algebra without loosing measurability.

PROPOSITION 3.3. *Let S be a countably generated σ-algebra on the reals carrying a measure μ. There exists a countably generated σ-algebra $S_1 \supset S$ containing singletons and carrying a measure μ_1 which extends μ.*

Proof. We may assume that S is a σ-algebra on a set $X \subset 2^\omega \times 2^\omega$ such that:

1. $\langle x, x\rangle \in X$ for all $x \in pr_1(X)$.
2. The sets $A_n = \{\langle x, y\rangle : x(n) = 1\} \cap X$ are generators of S.

Let S_1 be the σ-algebra of Borel subsets of X (in the subspace topology). Clearly $S_1 \supset S$, S_1 is countably generated and contains singletons. For $Y \in S_1$ let $Y^* = \{\langle x, y\rangle : \langle x, x\rangle \in Y\} \cap X$. It is easy to see that $Y^* \in S$ whenever $Y \in S_1$. The measure μ_1 on S_1 defined by the formula $\mu_1(Y) = \mu(Y^*)$ extends the measure μ.

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