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TECHNICAL UNIVERSITY OF BRAUNSCHWEIG
Braunschweig

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Fixed point sets of continuum-valued mappings

by

Jack T. Goodykoontz, Jr. and **Sam B. Nadler, Jr.** (Morgantown, WV)

Abstract. Let X be a metric continuum and let $C(X)$ denote the hyperspace of subcontinua of X. The following question is investigated: When does X have the property that for each nonempty closed subset A of X there exists a continuous function $F: X \to C(X)$ such that $x \in F(x)$ if and only if $x \in A$?

1. Introduction. By a *continuum* we mean a nonempty compact connected metric space. If X is a continuum, then $2^X(C(X))$ denotes the hyperspace of closed subsets (subcontinua) of X, each with the Hausdorff metric.

A *Peano continuum* is a locally connected continuum. By a *mapping* we mean a continuous function. If X is a space and $f: X \to X$ is a mapping, then the *fixed point set* of f is $\{x \in X: f(x) = x\}$. In [16] L. E. Ward, Jr. defines a space X to have the *complete invariance property* (CIP) provided that for each nonempty closed subset A of X there exists a mapping $f: X \to X$ such that A is the fixed point set of f. Some spaces known to have CIP are one-dimensional Peano continua [9], convex subsets of Banach spaces [16], compact n-manifolds [14], locally compact metrizable groups [8], and polyhedra [3]. In [16] Ward asked if every Peano continuum has CIP. This question was answered negatively in [7]. A rather complete bibliography of the literature on fixed point sets and CIP may be found in the survey article by H. Schirmer [14].

Part of the literature on the fixed point property has been concerned with multi-valued (set-valued) mappings. However, the question of which sets can be fixed point sets of multi-valued mappings has not been investigated before. If X is a continuum, $F: X \to 2^X$ is a mapping, and $x \in X$, then x is said to be a *fixed point* of F provided $x \in F(x)$. The *fixed point set* of F is $\{x \in X: x \in F(x)\}$. By a *continuum-valued mapping* we mean a mapping $F: X \to C(X)$.

In this paper we introduce and study the following generalization of CIP to the setting of multi-valued mappings. A continuum X is said to have the *complete invariance property for continuum-valued mappings* (MCIP) provided that for each nonempty closed subset A of X there exists a mapping $F: X \to C(X)$ such that A is the fixed point set of F.

We make the following initial observations regarding MCIP. If $A \in 2^X$ and $f\colon X \to X$ is a mapping such that the fixed point set of f is A, then the induced function $F\colon X \to C(X)$ defined by $F(x) = \{f(x)\}$ for each $x \in X$ is a continuum-valued mapping with fixed point set A. It follows that if X has CIP, then X has MCIP. If $A \in 2^X$, then the mapping $F\colon X \to 2^X$ defined by $F(x) = A$ for each $x \in X$ has fixed point set A. This shows why we require F to be continuum-valued in the definition of MCIP. Also, this shows that if X is any continuum and $A \in C(X)$, then A is the fixed point set of a continuum-valued mapping. Next, let X be any continuum, let $A \in 2^X$, and let $p \in A$. Then the function $F\colon X \to C(X)$ defined by

$$F(x) = \begin{cases} \{p\} & \text{if } x \notin A, \\ X & \text{if } x \in A \end{cases}$$

is an upper semi-continuous function and the fixed point set of F is A. This shows why we require F to be continuous rather than upper semi-continuous in the definition of MCIP. Finally, we remark that since we prove in (2.2) that every Peano continuum has MCIP, our generalization of CIP to MCIP provides a setting in which the answer to Ward's question [16] is affirmative.

In Section 2, in addition to showing that Peano continua have MCIP, we also prove an extension result, (2.3), and a general result which implies that any nonempty finite subset of any continuum is the fixed point set of a continuum-valued mapping (see (2.5)). This finite set theorem is not necessarily true for single-valued mappings (see the paragraph following (2.6)). Also, as is shown in (2.8), this result can not be generalized to countably infinite compacta. We also give an example in (2.7) of a continuum X and a closed subset A of X such that A has exactly two components and such that A is not the fixed point set of any mapping from X into $C(X)$.

In Section 3 we prove a number of lemmas which are needed to prove our main results in Section 4. Let L be a connected, locally connected, locally compact, non-compact, separable metric space and $X = L \cup R$ be a metric compactification of L with a compact metric space R as remainder. Our main result in Section 4 is (4.1), which states that if $A \in 2^X$ and $A \cap L \neq \emptyset$, then A is the fixed point set of a mapping $F\colon X \to C(X)$. As is shown in (4.2), this result is a generalization of (2.2). We also prove in Section 4 that if the dimension of L is $\geqslant 2$ or if L contains a simple closed curve, then X has MCIP. We show that X need not have MCIP when L is one-dimensional and does not contain a simple closed curve.

We will assume that the reader is familiar with basic facts and terminology about hyperspaces. Information about hyperspaces may be found in [4] and [12].

If X is a continuum, we will let d denote the metric on X and H denote the Hausdorff metric on 2^X. If $A \subset X$, then int A and $\bar{A}$ will denote the interior of A and the closure of A respectively.

2. Initial results and examples. Our first lemma will be used in the proofs of (2.2) and (4.1). It also provides the basic motivation for much of the material in Section 3.

(2.1) LEMMA. *Let X be a continuum and let $p \in X$. Let $\{\gamma(t)\colon t \in [0,1]\}$ be a path in $C(X)$ such that $\gamma(0) = \{p\}$ and $\bigcup \{\gamma(t)\colon t \in [0,1]\} = X$. Define $\tau\colon X \to [0,1]$ by*

$$\tau(x) = \inf\{t \in [0,1]\colon x \in \gamma(t)\}$$

for each $x \in X$. If τ is continuous and A is a closed subset of X such that $p \in A$, then there is a continuous function $F\colon X \to C(X)$ such that A is the fixed point set of F.

Proof. We may assume that the metric d on X has been normalized so that $d(x, y) \leqslant 1$ for all $x, y \in X$. Define $F\colon X \to C(X)$ by

$$F(x) = \gamma([1 - d(x, A)] \cdot \tau(x))$$

for each $x \in X$. Since γ and τ are continuous, F is continuous. Let $x \in X$. Since γ is continuous, $\{t \in [0,1]\colon x \in \gamma(t)\}$ is a closed subset of $[0,1]$. Hence, (#) $x \in \gamma(\tau(x))$. If $x \in A$, then $F(x) = \gamma(\tau(x))$. Thus, by (#), $x \in F(x)$. If $x \notin A$, then $x \neq p$. Since $\gamma(0) = \{p\}$, we have by (#) that $\tau(x) > 0$. Thus, $[1 - d(x, A)]\tau(x) < \tau(x)$. It follows from the definition of τ that $x \notin F(x)$. Hence A is the fixed point set of F.

We now prove that the class of continua which have MCIP includes the Peano continua.

(2.2) THEOREM. *If X is a Peano continuum, then X has* MCIP.

Proof. Let $A \in 2^X$, let $p \in A$, and let ϱ be a convex metric for X such that for all $x, y \in X$, $\varrho(x, y) \leqslant 1$ ([1] or [11]). For each $t \in [0,1]$, let

$$\gamma(t) = \{x \in X\colon \varrho(p, x) \leqslant t\}.$$

Since ϱ is convex, $\{\gamma(t)\colon t \in [0,1]\}$ is a path in $C(X)$ (in fact, by [10, Lemma 5], $\{\gamma(t)\colon t \in [0,1]\}$ is an order arc in $C(X)$). It is clear that $\gamma(0) = \{p\}$ and $\bigcup \{\gamma(t)\colon t \in [0,1]\} = X$. Define $\tau\colon X \to [0,1]$ as in (2.1). Then, for each $x \in X$, $\tau(x) = \varrho(p, x)$. Hence τ is continuous. It now follows from (2.1) that X has MCIP.

The next lemma provides a method for extending certain continuum-valued mappings and will be applied several times in the paper.

(2.3) LEMMA. *Let Z be a continuum and let Y be a proper subcontinuum of Z. Let $G\colon Y \to C(Y)$ be a mapping with fixed point set A (possibly, $A = \emptyset$) such that $\mathscr{G} = \{G(y)\colon y \in Y\}$ is contained in a locally connected subcontinuum $\mathscr{L}$ of $C(Y)$. Let $\mathscr{K} = \{\varkappa(t)\colon t \in [0,1]\}$ be an order arc in $C(Z)$ from Y to Z and let K be a compact subset of $Z - \bigcup \{\varkappa(t)\colon t < 1\}$. Then, G can be extended to a mapping $F\colon Z \to C(Z)$ such that the fixed point set of F is $A \cup K$, $F(z) = Z$ for all $z \in K$, and*

$$\mathscr{F} = \{F(z)\colon z \in Z\}$$

is contained in a locally connected subcontinuum of $C(Z)$.

Proof. By [17], (1) $C(\mathscr{L})$ is an absolute retract. By [4, 1.1 and 1.2], (2) the union function $\sigma\colon 2^{2^Z} \to 2^Z$ is continuous and σ maps $C(\mathscr{L})$ into $C(Y)$. Let $\mathscr{S} = \sigma[C(\mathscr{L})]$. By (1) and (2) we see that (3) $\mathscr{S}$ is a locally connected subcontinuum of $C(Y)$. Let $\tilde{G}\colon Y \to C(\mathscr{L})$ be the mapping defined by $\tilde{G}(y) = \{G(y)\}$ for each $y \in Y$. First assume that $K = \emptyset$. By (1), $\tilde{G}$ can be extended to a mapping $\psi\colon Z \to C(\mathscr{L})$. Let

$F = \sigma \circ \psi$. Since ψ is an extension of $\tilde{G}$, we see that F is an extension of G. It follows easily using (2) and (3) that F has the properties required in the lemma. Thus, from now on, we assume that $K \neq \emptyset$. Since $\varkappa(0) = Y$, $K \cap Y = \emptyset$. Thus, there exist nonempty open subsets U_n, $n = 0, 1, 2, \ldots$, of Z such that $\overline{U}_0 \cap Y = \emptyset$, $U_n \supset \overline{U}_{n+1}$ for each $n = 0, 1, 2, \ldots$, and $\bigcap_{n=0}^{\infty} U_n = K$. For each $n = 0, 1, 2, \ldots$, we define mappings F_n as follows. By (1), there is a mapping $\Gamma\colon Z - U_0 \to C(\mathscr{L})$ such that $\Gamma(y) = \tilde{G}(y)$ for all $y \in Y$ and $\Gamma(z) = \mathscr{L}$ for all $z \in \overline{U}_0 - U_0$. Let $F_0 = \sigma \circ \Gamma$ Since Γ is continuous and, by (2), σ is continuous, F_0 is continuous. Thus, F_0 is a mapping from $Z - U_0$ into $\mathscr{S}$. Note that F_0 is an extension of G and that $F_0(z) = \sigma(\mathscr{L})$ for all $z \in \overline{U}_0 - U_0$. Since there is an order arc in $C(Y)$ from $\sigma(\mathscr{L})$ to Y (unless $Y = \sigma(\mathscr{L})$), it follows easily that there is an order arc $\mathscr{A} = \{\alpha(t)\colon t \in [0, 1]\}$ in $C(Z)$ from $\sigma(\mathscr{L})$ to Z such that $\mathscr{A} \supset \mathscr{K}$. We assume that (4) α is a homeomorphism from $[0, 1]$ onto $\mathscr{A}$ such that $\alpha(0) = \sigma(\mathscr{L})$ and $\alpha(1) = Z$. Hence, (5) $K \subset Z - \bigcup\{\alpha(t)\colon t < 1\}$. Let $t_0 = 0$. Since each U_n is a nonempty open subset of Z, we see that, for each $n = 1, 2, \ldots$, the following numbers t_n exist:

$$t_n = \inf\{t \in [0, 1]\colon \alpha(t) \cap \overline{U}_{n-1} \neq \emptyset\}.$$

It follows easily from (5), the properties of α, and the properties of the sets U_n, that $t_{n-1} < t_n$ for each $n = 1, 2, \ldots$ and that (6) $t_n \to 1$ as $n \to \infty$. For each $n = 1, 2, \ldots$, let

$$\mathscr{A}_n = \{\alpha(t)\colon t_{n-1} \leqslant t \leqslant t_n\}.$$

For each $n = 1, 2, \ldots$, $\overline{U}_{n-1} - U_{n-1}$ and $\overline{U}_n - U_n$ are disjoint nonempty closed subsets of $\overline{U}_{n-1} - U_n$ and $\mathscr{A}_n$ is an arc. Hence, there exists, for each $n = 1, 2, \ldots$, a mapping $F_n\colon \overline{U}_{n-1} - U_n \to \mathscr{A}_n$ such that $F_n(z) = \alpha(t_{n-1})$ for all $z \in \overline{U}_{n-1} - U_{n-1}$ and $F_n(z) = \alpha(t_n)$ for all $z \in \overline{U}_n - U_n$. Note that for each $n = 0, 1, 2, \ldots$, $F_n(z) = F_{n+1}(z)$ for all $z \in \overline{U}_n - U_n$. Thus, it follows easily using (4) and (6) that the following formula defines a continuous function F from Z into $\mathscr{S} \cup \mathscr{A}$:

$$F(z) = \begin{cases} F_0(z), & \text{if } z \in Z - U_0, \\ F_n(z), & \text{if } z \in \overline{U}_{n-1} - U_n \text{ for some } n = 1, 2, \ldots, \\ Z, & \text{if } z \in K. \end{cases}$$

Since F_0 is an extension of G, F is an extension of G. Thus, since each point of A is a fixed point of G, each point of A is a fixed point of F. Since $F(z) = Z$ for each $z \in K$, each point of K is a fixed point of F. Thus, letting B denote the fixed point set of F, we have that $A \cup K \subset B$. We show that $B = A \cup K$. Since $F_0(z) \in \mathscr{S} \subset C(Y)$ for each $z \in Z - U_0$ and since F_0 is an extension of G, we see that the only possible fixed points of F_0 are points of A. From the properties of the sets U_n and the definition of the numbers t_n, it follows easily that, for each $n = 1, 2, \ldots$, (7) $\alpha(t) \cap U_{n-1} = \emptyset$ if $t_{n-1} \leqslant t \leqslant t_n$ and (8) $\alpha(t_{n-1}) \cap \overline{U}_{n-1} = \emptyset$. By using (7) and (8) we see that F_n has no fixed point for any $n = 1, 2, \ldots$ Therefore, it follows that $B = A \cup K$. Now, to complete the proof of (2.3), it remains to show that $\mathscr{S} \cup \mathscr{A}$ is a locally connected subcontinuum of $C(Z)$. But this follows from (3), the fact that $\mathscr{A}$ is an arc, the fact that $\mathscr{S} \cap \mathscr{A} \neq \emptyset$ [since $\sigma(\mathscr{L}) \in \mathscr{S} \cap \mathscr{A}$], and [5, Thm. 1, p. 230]. Therefore, we have proved (2.3).

If X is a continuum and $B \in C(X)$, then B is said to be *buried* (in X) provided that whenever $Y \in C(X)$ such that $Y \cap B \neq \emptyset$ and $Y \cap (X - B) \neq \emptyset$, then $Y \supset B$. We note that if B is nondegenerate and buried, then $C(X) - \{B\}$ is not arcwise connected ([13, 4.4]). If B is decomposable, then B is buried if and only if $C(X) - \{B\}$ is not arcwise connected ([13, 4.4]).

(2.4) THEOREM. *Let X be a continuum. Let A be a nonempty compact subset of X such that A has only finitely many components $A_1, \ldots, A_n$. If each A_i, except possibly one, is buried, then there is a mapping $F\colon X \to C(X)$ such that the fixed point set of F is A and such that $\mathscr{F} = \{F(x)\colon x \in X\}$ is contained in a locally connected subcontinuum of $C(X)$.*

Proof. If $n = 1$ (i.e., $A = A_1$), then the mapping $F\colon X \to C(X)$, defined by $F(x) = A$ for all $x \in X$, has the properties required in the theorem (since $\mathscr{F} = \{A\}$, $\mathscr{F}$ itself is a locally connected subcontinuum of $C(X)$). Thus, we assume for the purpose of proof that $n > 1$. Also: If one of the components of A is not buried, then we assume without loss of generality that A_1 is the non-buried component of A. Since $n > 1$, there is an order arc $\mathscr{B}_1 = \{\beta^1(t)\colon t \in [0, 1]\}$ in $C(X)$ such that $\beta^1(0) = A_1$, $\beta^1(t) \cap A = A_1$ for all $t < 1$, and $\beta^1(1) \cap A \neq A_1$. Let $X_1 = \beta^1(1)$ and let

$$K_1 = \bigcup\{A_i\colon \beta^1(1) \cap A_i \neq \emptyset \text{ and } i \neq 1\}.$$

Since A_i is buried for each $i \neq 1$, $K_1 \subset X_1 - \bigcup\{\beta^1(t)\colon t < 1\}$. Also note that K_1 is compact. Thus, defining $G\colon A_1 \to C(A_1)$ by $G(y) = A_1$ for all $y \in A_1$, we have by (2.3) that G can be extended to a mapping $F_1\colon X_1 \to C(X_1)$ such that the fixed point set of F_1 is $A_1 \cup K_1$ and such that $\mathscr{F}_1 = \{F_1(x)\colon x \in X_1\}$ is contained in a locally connected subcontinuum of $C(X_1)$. If $X_1 \not\supset A$, then there is an order arc $\mathscr{B}_2 = \{\beta^2(t)\colon t \in [0, 1]\}$ in $C(X)$ such that $\beta^2(0) = X_1$, $\beta^2(t) \cap A = A \cap X_1$ $(= A_1 \cup K_1)$ for all $t < 1$, and $\beta^2(1) \cap A \neq A \cap X_1$. Let $X_2 = \beta^2(1)$ and let

$$K_2 = \bigcup\{A_i\colon \beta^2(1) \cap A_i \neq \emptyset \text{ and } A_i \cap X_1 = \emptyset\}.$$

Since $A_1 \subset X_1$, each A_i which makes up the set K_2 is buried. Thus, we see that $K_2 \subset X_2 - \bigcup\{\beta^2(t)\colon t < 1\}$. Hence, by (2.3), F_1 can be extended to a mapping $F_2\colon X_2 \to C(X_2)$ such that the fixed point set of F_2 is $A_1 \cup K_1 \cup K_2$ and such that $\mathscr{F}_2 = \{F_2(x)\colon x \in X_2\}$ is contained in a locally connected subcontinuum of $C(X_2)$. By continuing the process indicated above, we obtain, after a finite number of steps, a subcontinuum X_k of X and a mapping $F_k\colon X_k \to C(X_k)$ such that the fixed point set of F_k is A and such that $\mathscr{F}_k = \{F_k(x)\colon x \in X_k\}$ is contained in a locally connected subcontinuum of $C(X_k)$. If $X_k = X$, then $F = F_k$ has the properties required in (2.4). Assume that $X_k \neq X$. Then, using the properties of F_k and letting the set K in (2.3) be the empty set, we see from (2.3) that F_k can be extended to a mapping $F\colon X \to C(X)$ such that F satisfies (2.4). This completes the proof of (2.4).

(2.5) Corollary. *Let X be a continuum. If A is a nonempty finite subset of X, then there is a mapping $F\colon X \to C(X)$ such that the fixed point set of F is A and such that $\mathcal{F} = \{F(x)\colon x \in X\}$ is contained in a locally connected subcontinuum of $C(X)$.*

Proof. Since any one-point set is buried, (2.5) is a special case of (2.4).

Let P denote the pseudo-arc. In [8, 5.7] it was shown that any two-point subset of P is a fixed point set of a mapping from P into P and in [8, 5.7] it was asked if P has CIP. We do not know if P has MCIP. However, in this connection, we have the following:

(2.6) Corollary. *Let X be an hereditarily indecomposable continuum. If $A \in 2^X$ such that A has finitely many components, then there is a mapping $F\colon X \to C(X)$ such that the fixed point set of F is A and such that $\mathcal{F} = \{F(x)\colon x \in X\}$ is contained in a locally connected subcontinuum of $C(X)$.*

Proof. Since any subcontinuum of X is buried, (2.6) is a special case of (2.4).

We now give our first example of a continuum X such that X does not have MCIP. We verify that X does not have MCIP by showing that there is a closed subset A of X with exactly two components such that A is not the fixed point set of any mapping from X into $C(X)$. Thus, the condition in (2.4) that all but one component be buried is necessary. We also make the following observations. Schirmer [14] has asked if every chainable continuum has CIP. Let Z be the usual $\sin(1/x)$-continuum ($Z = W_1 \cup J_1$ where W_1 and J_1 are as in (2.7)). Let $p = (0, -1)$ and let $p' = (1, \sin[1])$. An easy argument shows that $\{p, p'\}$ is not the fixed point set of any mapping f from Z into Z (since f would have to map J_1 onto J_1 and have fixed point set $\{p\}$ — see [15, p. 564]). Thus, the answer to Schirmer's question is no. The next two examples show that there are chainable continua which do not even have MCIP. We note that, by (4.6), the $\sin(1/x)$-continuum Z does have MCIP.

(2.7) Example. Let (see Fig. 1)

$$W_1 = \{(x, \sin[1/x])\colon 0 < x \leqslant 1\},$$
$$W_2 = \{(2-x, y)\colon (x, y) \in W_1\},$$
$$W_3 = \{(-x, y-2)\colon (x, y) \in W_1\},$$
$$W_4 = \{(x+2, y-2)\colon (x, y) \in W_1\},$$
$$J_1 = \{(0, y)\colon -1 \leqslant y \leqslant 1\},$$
$$J_2 = \{(2, y)\colon -1 \leqslant y \leqslant 1\},$$
$$J_3 = \{(0, y)\colon -3 \leqslant y \leqslant -1\}, \text{ and}$$
$$J_4 = \{(2, y)\colon -3 \leqslant y \leqslant -1\}.$$

Let $X = \bigcup_{i=1}^{4} (W_i \cup J_i)$. Let $A = W_3 \cup J_3 \cup W_4 \cup J_4$. We show that A is not the fixed point set of any mapping from X into $C(X)$. For this purpose, let $p = (0, -1)$, let $q = (2, -1)$, and, for any nonempty compact subset K of X, let

$$g(K) = \mathrm{glb}\{x\colon (x, y) \in K\}.$$

Suppose that there is a mapping $F\colon X \to C(X)$ such that the fixed point set of F is A. Since $(W_1 \cup W_2) \cap A = \emptyset$, it follows using the continuity of F that

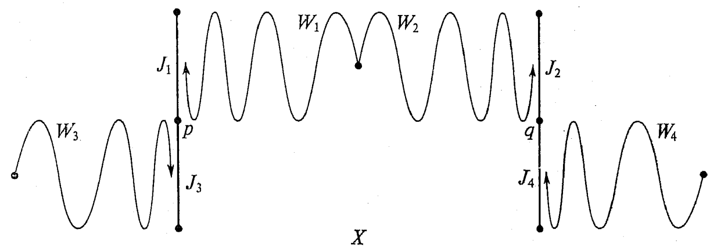

Fig. 1

$x < g(F(x, y))$ for all $(x, y) \in W_1 \cup W_2$ or $x > g(F(x, y))$ for all $(x, y) \in W_1 \cup W_2$. Assume, without loss of generality, that (1) $x < g(F(x, y))$ for all $(x, y) \in W_1 \cup W_2$. Since $p \in F(p)$, we see using (1) and the continuity of F that (2) $F(p) \cap J_3 = \{p\}$. Suppose that $F(p) \neq \{p\}$. Then, using (2) and the continuity of F, it follows that $(x, y) \notin F(x, y)$ for points $(x, y) \in W_3$ sufficiently close to p. This contradicts the fact that $W_3 \subset A$. Hence, $F(p) = \{p\}$. Thus, since F has no fixed point in $J_1 - \{p\}$, it follows easily that $(0, 1) \notin F(0, y)$ for any point $(0, y) \in J_1$. Therefore, by the continuity of F, there is a continuum Z (near J_1) such that $Z \supset J_1$, $Z \cap W_1 \neq \emptyset$, and $(0, 1) \notin F(z)$ for any $z \in Z$. Let $M = \bigcup \{F(z)\colon z \in Z\}$. Since $\{F(z)\colon z \in Z\}$ is a subcontinuum of $C(X)$, we have by [4, 1.2] that M is a continuum. Since $p \in Z \cap A$, $p \in M$. Since $Z \cap W_1 \neq \emptyset$, we have by (1) that there is a point $(x_0, y_0) \in M$ such that $x_0 > 0$. It now follows easily that $M \supset J_1$. Thus, $(0, 1) \in M$. This contradicts the fact that $(0, 1) \notin F(z)$ for any $z \in Z$. Therefore, there is no mapping $F\colon X \to C(X)$ such that the fixed point set of F is A. This completes (2.7).

We now give another example of a continuum X such that X does not have MCIP. In this example we show that a certain countably infinite subset of X is not the fixed point set of any mapping from X into $C(X)$. Thus, the condition that A in (2.5) be finite is necessary.

(2.8) Example. Let W_1, J_1, W_3, J_3, and p be as in (2.7) and let $X = W_1 \cup J_1 \cup W_3 \cup J_3$. For each $n = 0, 1, 2, \ldots$, let $p_n \in W_1$ and $p'_n \in W_3$ be the points defined by $p_n = (2/[3\pi + 4n\pi], -1)$ and $p'_n = (2/[-3\pi - 4n\pi], -1)$. Let

$$A = \{p_n\colon n = 0, 1, 2, \ldots\} \cup \{p'_n\colon n = 0, 1, 2, \ldots\} \cup \{p\}.$$

Suppose that there is a mapping $F\colon X \to C(X)$ such that the fixed point set of F is A. Note that $p \in F(p) \in C(X)$. Thus: If $F(p) \supset J_1 \cup J_3$, then, using the continuity of F, it follows that points in $(J_1 \cup J_3) - \{p\}$ sufficiently close to p would be fixed points of F. Hence, since F has no fixed points in $(J_1 \cup J_3) - \{p\}$, we have that $F(p) \not\supset J_1 \cup J_3$. Therefore, $(0, -3) \notin F(p)$ or $(0, 1) \notin F(p)$. Without loss of generality, assume that $(0, 1) \notin F(p)$. Then, since F has no fixed point in $J_1 - \{p\}$, it follows easily that $(0, 1) \notin F(0, y)$ for any point $(0, y) \in J_1$. Hence, by the continuity of F, there is a subcontinuum Z (near J_1) of $J_1 \cup W_1$ such that $Z \supset J_1$, $Z \cap W_1 \neq \emptyset$, and $(0,1)\notin F(z)$ for any $z\in Z$. Let $M=\bigcup\{F(z)\colon z\in Z\}$. Since $\{F(z)\colon z\in Z\}$ is a subcontinuum of $C(X)$, we have by [4, 1.2] that M is a continuum. Since $p\in Z\cap A$, $p\in M$. Since Z is a subcontinuum of X such that $Z\supset J_1$ and $Z\cap W_1\neq\emptyset$, we see that $p_i\in Z$ for some i (large enough). Thus, since $p_i\in F(p_i)$, $p_i\in M$. Hence, M is a subcontinuum of X such that $p, p_i\in M$. Therefore, we see that $M\supset J_1$. Thus, $(0,1)\in M$. This contradicts the fact that $(0,1)\notin F(z)$ for any $z\in Z$. Therefore, there is no mapping $F\colon X\to C(X)$ such that the fixed point set of F is A. This completes (2.8).

(2.9) Remark. The example in (2.8) can be modified to show that there exist continua of all dimensions which fail to have MCIP. Observe that each of $W_1\cup J_1$ and $W_3\cup J_3$ is a compactification of a half-line with an arc as remainder. If we replace each of W_1 and W_3 by the product of a half-line and an n-cell (or the product of a half-line and a Hilbert cube) and compactify on J_1 and J_3 in the analogous manner, then the resulting continuum will be $(n+1)$-dimensional (or infinite-dimensional) at every point. Using essentially the same argument as in (2.8), it can be shown that these continua fail to have MCIP.

3. Existence of certain order arcs. In the proof of (2.2) we showed that if X is a Peano continuum and $A\in 2^X$, then there exists a mapping $F\colon X\to C(X)$ such that A is the fixed point set of F and such that $\{F(x)\colon x\in X\}$ is an order arc in $C(X)$. If $\{\gamma(t)\colon t\in[0,1]\}$ is an order arc in $C(X)$, then there is an associated function $\tau\colon X\to[0,1]$ defined by

$$\tau(x)=\inf\{t\in[0,1]\colon x\in\gamma(t)\}.$$

In view of (2.1), it is of interest to know when τ is continuous. In this section we will prove a number of technical lemmas ((3.1) through (3.6)) concerning the existence, when X is a Peano continuum, of a special class of order arcs in $C(X)$ for which the associated function τ is continuous. Our main lemma (3.7) will be used in Section 4 to prove (4.1), which is a generalization of (2.2), and to prove (4.3), which shows that a large class of continua have MCIP.

(3.1) Lemma. *Let X be a Peano continuum, let $Y\in 2^X$, and let $B\in C(X)$. Let μ be a Whitney map for $C(X)$. For each $(y,t)\in Y\times[0,+\infty)$, let*

$$g(y,t)=\bigcup\{A\in\mu^{-1}([0,t])\colon y\in A\}$$

and, for each $t\in[0,+\infty)$, let

$$f(B,t)=\bigcup\{A\in\mu^{-1}([0,t])\colon A\cap B\neq\emptyset\}.$$

Then, g is a uniformly continuous function from $Y\times[0,+\infty)$ into $C(X)$ and $f(B,\cdot)$ is a uniformly continuous function from $[0,+\infty)$ into $C(X)$.

Proof. Let $(y,t)\in Y\times[0,+\infty)$ and let $\mathscr{A}(y,t)=\{A\in\mu^{-1}([0,t])\colon y\in A\}$. Note that $\mathscr{A}(y,t)$ is a collection of connected sets and, since $\{y\}\in\mathscr{A}(y,t)$, $\bigcap\mathscr{A}(y,t)\neq\emptyset$. Hence, $\bigcup\mathscr{A}(y,t)$ is connected. Since $\mathscr{A}(y,t)$ is a compact subset of $C(X)$, $\bigcup\mathscr{A}(y,t)$ is compact [4, p. 23]. Thus, since $g(y,t)=\bigcup\mathscr{A}(y,t)$, we have proved that $g(y,t)\in C(X)$. In the proof of [4, 4.1] it is observed that any Peano

continuum satisfies [4, 3.2]. Hence, it follows from the proof in [4, 3.3] that g, which is G in the proof of [4, 3.3], is continuous. Therefore, g is uniformly continuous since Y is compact and since, for some t_0, $g(y,t)=X$ for $t\geqslant t_0$ and for all $y\in Y$. To verify the properties of $f(B,\cdot)$, let $Y=B$ and, for each $t\in[0,+\infty)$, let

$$\Gamma(t)=\{g(b,t)\colon b\in B\}.$$

Note that $f(B,t)=\bigcup\Gamma(t)$ for each $t\in[0,+\infty)$. Fix $t=t_1\in[0,+\infty)$. Since $g(\cdot,t_1)$ is continuous on B, since $B\in C(X)$, and since $g(b,t_1)\in C(X)$ for each $b\in B$, we see that $\Gamma(t_1)\in C(C(X))$. Hence [4, 1.2], $\bigcup\Gamma(t_1)\in C(X)$. Thus, we have proved that $f(B,t)\in C(X)$ for each $t\in[0,+\infty)$. Since g is uniformly continuous on $B\times[0,+\infty)$, it follows easily that Γ is continuous on $[0,+\infty)$. Hence, since union is continuous [4, p. 23] and since $f(B,t)=\bigcup\Gamma(t)$ for each $t\in[0,+\infty)$, $f(B,\cdot)$ is continuous on $[0,+\infty)$. Therefore, since $f(B,t)=X$ for sufficiently large t, $f(B,\cdot)$ is uniformly continuous on $[0,+\infty)$. This completes the proof of (3.1).

(3.2) Lemma. *Let X be a continuum and let $\alpha\colon[0,1]\to C(X)$ be an order arc. Assume that whenever $t_1,t_2\in[0,1]$ with $t_1<t_2$, then $\alpha(t_1)\subset\operatorname{int}[\alpha(t_2)]$. Then the function $\tau\colon\alpha(1)\to[0,1]$, defined by*

$$\tau(x)=\inf\{t\in[0,1]\colon x\in\alpha(t)\}$$

for each $x\in\alpha(1)$, is continuous.

Proof. Let $x_0\in\alpha(1)$. We show that τ is continuous at x_0. Let $t_0=\tau(x_0)$. First assume that $0<t_0<1$. Let $\varepsilon>0$ such that $0<t_0-\varepsilon$ and $t_0+\varepsilon<1$. Let

$$V=\operatorname{int}[\alpha(t_0+\varepsilon)]-\alpha(t_0-\varepsilon).$$

Since V is an open subset of X and $V\subset\alpha(1)$, V is an open subset of $\alpha(1)$. Since $t_0=\tau(x_0)$, $x_0\notin\alpha(t_0-\varepsilon)$ and, by the continuity of α, $x_0\in\alpha(t_0)$. Thus, since $\alpha(t_0)\subset\operatorname{int}[\alpha(t_0+\varepsilon)]$, we have that $x_0\in V$. It follows easily that for any $y\in V$,

$$t_0-\varepsilon<\tau(y)<t_0+\varepsilon.$$

Hence, we have proved that τ is continuous at x_0 in the case when $t_0\neq 0,1$. Next assume that $t_0=0$. Let $\varepsilon>0$ such that $\varepsilon<1$. Then $x_0\in\operatorname{int}[\alpha(\varepsilon)]$, $\operatorname{int}[\alpha(\varepsilon)]$ is an open subset of $\alpha(1)$, and $\tau(y)<\varepsilon$ for any $y\in\operatorname{int}[\alpha(\varepsilon)]$. Finally assume that $t_0=1$. Let $\varepsilon>0$ such that $\varepsilon<1$. Let

$$W=\alpha(1)-\alpha(1-\varepsilon).$$

Then $x_0\in W$ (since $\tau(x_0)=1$), W is an open subset of $\alpha(1)$, and $\tau(y)>1-\varepsilon$ for any $y\in W$. This completes the proof of (3.2).

(3.3) Lemma. *Let X be a Peano continuum and let B be a proper subcontinuum of X. Let μ be a Whitney map for $C(X)$. For each $t\geqslant 0$, let*

$$f(B,t)=\bigcup\{A\in\mu^{-1}([0,t])\colon A\cap B\neq\emptyset\}.$$

Let $T = \inf\{t \geqslant 0: f(B,t) = X\}$. *Then*:

(1) $\{f(B,t): t \in [0,T]\}$ *is an order arc in* $C(X)$ *from* B *to* X;

(2) *if* $t_1, t_2 \in [0,T]$ *and* $t_1 < t_2$, *then* $f(B,t_1) \subset \mathrm{int}[f(B,t_2)]$;

(3) *the function* τ *is continuous where* $\tau: X \to [0,T]$ *is defined by*

$$\tau(x) = \inf\{t \in [0,T]: x \in f(B,t)\}$$

for each $x \in X$.

Proof. By (3.1), $f(B,\cdot)$ is a continuous function from $[0,+\infty)$ into $C(X)$. Thus, letting

$$\Gamma = \{f(B,t): t \in [0,T]\},$$

we have that Γ is a subcontinuum of $C(X)$. Furthermore, it is clear that $f(B,t_1) \subset f(B,t_2)$ whenever $t_1, t_2 \in [0,T]$ and $t_1 \leqslant t_2$. Hence [10, Lemma 5], Γ is an order arc in $C(X)$ from B to X. This proves (1). To prove (2), let $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$. Let $x \in f(B,t_1)$. Then there exists $A \in \mu^{-1}([0,t_1])$ such that $A \cap B \neq \emptyset$ and $x \in A$. Since $\mu(A) < t_2$ and X is a Peano continuum, there is a closed connected neighborhood V in X of x such that $\mu(A \cup V) \leqslant t_2$. Thus, since $(A \cup V) \cap B \neq \emptyset$, $A \cup V \subset f(B,t_2)$. Hence, since $x \in \mathrm{int}[V]$, we have that $x \in \mathrm{int}[f(B,t_2)]$. Therefore, we have proved (2). It follows from (1) and (2) that we can use (3.2) to see that (3) holds. This completes the proof of (3.3).

(3.4) Lemma. *Let M be a Peano continuum and let $Y, Z \in 2^M$. Let μ be a Whitney map for $C(M)$. For each $(y,t) \in Y \times [0,+\infty)$, let*

$$g(y,t) = \bigcup \{A \in \mu^{-1}([0,t]): y \in A\}.$$

Define $k: Y \to [0,+\infty)$ *by*

$$k(y) = \inf\{t \in [0,+\infty): g(y,t) \cap Z \neq \emptyset\}$$

for each $y \in Y$. *Then* k *is continuous.*

Proof. Let $y_0 \in Y$ and let $\varepsilon > 0$. Since M is a Peano continuum, there are arbitrarily small closed connected neighborhoods in M of y_0. Using these neighborhoods, compactness of $C(M)$, and a sequence argument, it follows that there is a closed connected neighborhood V in M of y_0 such that

(#) if $A \in C(M)$ and $A \cap V \neq \emptyset$, then $\mu(A \cup V) \leqslant \mu(A) + \varepsilon$.

We show that $k(y_0) - \varepsilon \leqslant k(y) \leqslant k(y_0) + \varepsilon$ for each $y \in V \cap Y$. Let $y \in V \cap Y$. Each $g(y_0,t)$, being the union of a compact subset of $C(M)$, is compact [4, p. 23]. Hence, since Z is compact, it follows easily that $g(y_0, k(y_0)) \cap Z \neq \emptyset$. Thus, there exists $A \in \mu^{-1}([0,k(y_0)])$ such that $y_0 \in A$ and $A \cap Z \neq \emptyset$. Since $y_0 \in A \cap V$, $A \cap V \neq \emptyset$. Hence, by (#), $\mu(A \cup V) \leqslant k(y_0) + \varepsilon$. Also, since $y \in V$, $y \in A \cup V$. Thus, $A \cup V \subset g(y, k(y_0)+\varepsilon)$. Hence, since $A \cap Z \neq \emptyset$, $g(y, k(y_0)+\varepsilon) \cap Z \neq \emptyset$. Therefore, $k(y) \leqslant k(y_0) + \varepsilon$. Now suppose that $k(y) < k(y_0) - \varepsilon$. Then there exists $t_1 < k(y_0) - \varepsilon$ such that $g(y,t_1) \cap Z \neq \emptyset$. Hence, there exists $A \in \mu^{-1}([0,t_1])$ such that $y \in A$ and $A \cap Z \neq \emptyset$. Since $y \in A \cap V$, $A \cap V \neq \emptyset$. Hence, by (#), $\mu(A \cup V) \leqslant t_1 + \varepsilon$. Also, since $y_0 \in V$, $y_0 \in A \cup V$. Thus, $A \cup V \subset g(y_0, t_1+\varepsilon)$. Hence, since $A \cap Z \neq \emptyset$, $g(y_0, t_1+\varepsilon) \cap Z \neq \emptyset$. Hence, $k(y_0) \leqslant t_1 + \varepsilon$. This contradicts the fact that $t_1 < k(y_0) - \varepsilon$. Therefore, $k(y) \geqslant k(y_0) - \varepsilon$. This completes the proof of (3.4).

(3.5) Lemma. *Let M be a Peano continuum, let q be a noncut point of M, and let B be a subcontinuum of M such that $q \notin B$. Let U be an open subset of M such that $q \in U$ and $\overline{U} \cap B = \emptyset$. Then there exists an order arc $\{\beta(t): t \in [0,T]\}$ in $C(M)$ such that*

(1) $\beta(0) = B$;

(2) $\beta(T) \supset M - U$;

(3) $q \notin \beta(T)$;

(4) *the function* τ *is continuous where* $\tau: \beta(T) \to [0,T]$ *is defined by*

$$\tau(x) = \inf\{t \in [0,T]: x \in \beta(t)\}$$

for each $x \in \beta(T)$.

Proof. By [5, Thm. 3, p. 257], M is the union of finitely many locally connected continua $M_1, M_2, \ldots, M_n$ such that $\bigcup\{M_i: q \in M_i\} \subset U$. Assume (by changing the indexing if necessary) that $q \in M_i$ if and only if $i \geqslant i_0$ for some i_0. Since $M - \{q\}$ is a connected open subset of M, it follows using [5, Thm. 1, p. 254] that, for each $i < i_0$, there is an arc $A_i \subset M - \{q\}$ from a point of M_1 to a point of M_i. Then [5, Thm. 1, p. 230], letting

$$X = \bigcup\{M_i \cup A_i: i < i_0\},$$

X is a Peano continuum. Note that $q \notin X$ and $X \supset M - U$. Let μ_1 be a Whitney map for $C(M)$ and let $\mu_2 = \mu_1 | C(X)$. Note that μ_2 is a Whitney map for $C(X)$. For each $t \geqslant 0$, let

$$f(B,t) = \bigcup\{A \in \mu_2^{-1}([0,t]): A \cap B \neq \emptyset\}.$$

Let $T = \inf\{t \geqslant 0: f(B,t) = X\}$. Since $\overline{U} \cap B = \emptyset$ and $X \supset M - U$, B is a proper subcontinuum of X. Hence, by (1) of (3.3), $\{f(B,t): t \in [0,T]\}$ is an order arc in $C(X)$ from B to X. Let $Y = (\overline{M - X}) \cap X$ and define $j: Y \to [0,T]$ by

$$j(y) = \inf\{t \in [0,T]: y \in f(B,t)\}$$

for each $y \in Y$. Since j is the restriction to Y of the mapping in (3) of (3.3), we have that j is continuous. For each $(y,t) \in Y \times [0,+\infty)$, let

$$g(y,t) = \bigcup\{A \in \mu_1^{-1}([0,t]): y \in A\}$$

and define $k: Y \to [0,+\infty)$ by

$$k(y) = \inf\{t \in [0,+\infty): q \in g(y,t)\}.$$

By (3.4), with $Z = \{q\}$, we have that k is continuous. Let us note for future use that g and $f(B, \cdot)$ are uniformly continuous by (3.1). We now define β. Note that if $y \in Y \cap f(B, t)$, then $t - j(y) \geqslant 0$. Hence, since $T > 0$, we see that the following formula for β "makes sense": For each $t \in [0, T]$, let

$$\beta(t) = f(B, t) \cup \left[\bigcup \left\{ g\left(y, \frac{t-j(y)}{T} \cdot k(y)\right) \colon y \in Y \cap f(B, t) \right\}\right].$$

We first show that $\beta(t) \in C(M)$ for each $t \in [0, T]$. Let $t_0 \in [0, T]$. Let $S(t_0)$ denote the part of $\beta(t_0)$ in the square parentheses — thus, $\beta(t_0) = f(B, t_0) \cup S(t_0)$. Since g maps into $C(M)$ by (3.1), $S(t_0)$ is the union over a collection (possibly empty) of subcontinua of M each of which intersects $f(B, t_0)$. Thus, since $f(B, t_0)$ is a connected subset of $\beta(t_0)$, it follows that $\beta(t_0)$ is connected. Since $Y \cap f(B, t_0)$ is compact and since j, k, and g are continuous, we see that $S(t_0)$ is the union over a compact collection. Hence [4, p. 23], $S(t_0)$ is compact. Thus, $\beta(t_0)$ is compact. Therefore, we have proved that $\beta(t_0) \in C(M)$. Next we show that β is uniformly continuous. Since $f(B, \cdot)$ is uniformly continuous on $[0, T]$, there exists $\delta_1 > 0$ such that if $t_1, t_2 \in [0, T]$ and $|t_1 - t_2| < \delta_1$, then

(a) $$H(f(B, t_1), f(B, t_2)) < \tfrac{1}{2}\varepsilon.$$

Note that, since k is continuous on the compact set Y, k is bounded. Also note that $g(y, 0) = \{y\}$ for each $y \in Y$. Hence, by the uniform continuity of g, there exists $\delta_2 > 0$ such that if $t_1, t_2 \in [0, T]$ and $|t_1 - t_2| < \delta_2$, then

(b) $$H\left(\{y\}, g\left(y, \frac{|t_1 - t_2|}{T} \cdot k(y)\right)\right) < \tfrac{1}{2}\varepsilon \quad \text{for each } y \in Y.$$

Let $s_0 = \inf\{t \in [0, T] \colon Y \cap f(B, t) \neq \emptyset\}$. Using the uniform continuity of j, k, and g, we see that there exists $\delta_3 > 0$ such that if $|t_1 - t_2| < \delta_3$ and $s_0 \leqslant t_1 \leqslant t_2 \leqslant T$, then

(c) $$H\left(g\left(y, \frac{t_2 - j(y)}{T} \cdot k(y)\right), g\left(y, \frac{t_1 - j(y)}{T} \cdot k(y)\right)\right) < \varepsilon \text{ for each } y \in Y \cap f(B, t_1).$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Let $t_1, t_2 \in [0, T]$ such that $|t_1 - t_2| < \delta$ and assume that $t_1 \leqslant t_2$. Let $z \in \beta(t_2)$. If $z \in f(B, t_2)$, then by (a) there exists $x \in f(B, t_1)$ such that $d(z, x) < \frac{1}{2}\varepsilon$. Hence, $x \in \beta(t_1)$ and $d(z, x) < \varepsilon$. So, assume that $z \notin f(B, t_2)$. Then, since $z \in \beta(t_2)$, there exists $y \in Y \cap f(B, t_2)$ such that

$$z \in g\left(y, \frac{t_2 - j(y)}{T} \cdot k(y)\right).$$

If $y \notin Y \cap f(B, t_1)$, then $t_1 < j(y)$. Since $y \in Y \cap f(B, t_2)$, $j(y) \leqslant t_2$. Hence, $t_2 - j(y) < t_2 - t_1 < \delta \leqslant \delta_2$. Thus, we can apply (b) to conclude that $d(z, y) < \frac{1}{2}\varepsilon$. Since $y \in f(B, t_2)$, there exists by (a) a point $x \in f(B, t_1)$ such that $d(y, x) < \frac{1}{2}\varepsilon$.

Thus, $x \in \beta(t_1)$ and $d(z, x) < \varepsilon$. If $y \in Y \cap f(B, t_1)$, then $s_0 \leqslant t_1$. Hence, we can apply (c) to obtain a point

$$x \in g\left(y, \frac{t_1 - j(y)}{T} \cdot k(y)\right)$$

such that $d(z, x) < \varepsilon$. Hence, $x \in \beta(t_1)$ and $d(z, x) < \varepsilon$. Therefore, we have proved that each point $z \in \beta(t_2)$ is within ε of some point $x \in \beta(t_1)$. It is easy to see that $\beta(t_1) \subset \beta(t_2)$ since $t_1 \leqslant t_2$. Hence, we have proved that

$$H(\beta(t_1), \beta(t_2)) < \varepsilon$$

and, therefore, we have proved that β is (uniformly) continuous. Since β is a continuous function from $[0, T]$ into $C(M)$, $\{\beta(t) \colon t \in [0, T]\}$ is a subcontinuum of $C(M)$. Hence, since $\beta(t_1) \subset \beta(t_2)$ whenever $t_1, t_2 \in [0, T]$ and $t_1 \leqslant t_2$, $\{\beta(t) \colon t \in [0, T]\}$ is an order arc in $C(M)$ by [10, Lemma 5]. Now we verify (1) through (4) of the lemma. By hypothesis, $\overline{U} \cap B = \emptyset$. Hence, since $M - X \subset U$, $Y \cap B = \emptyset$. Hence, since $f(B, 0) = B$, $Y \cap f(B, 0) = \emptyset$. Thus, $\beta(0) = f(B, 0) = B$. This proves (1). Since $f(B, T) = X$ and since $X \supset M - U$, we see that $\beta(T) \supset M - U$. This proves (2). To prove (3), suppose that $q \in \beta(T)$. Then, since $f(B, T) = X$ and $q \notin X$, there exists $y \in Y \cap f(B, T)$ such that

$$q \in g\left(y, \frac{T - j(y)}{T} \cdot k(y)\right).$$

Since $Y \cap B = \emptyset$, $j(y) > 0$. Since $q \notin X$, $q \notin Y$ and thus, since $g(y, 0) = \{y\}$, $k(y) > 0$. Hence,

$$\frac{T - j(y)}{T} \cdot k(y) < k(y)$$

and therefore, by definition of k,

$$q \notin g\left(y, \frac{T - j(y)}{T} \cdot k(y)\right)$$

which is a contradiction. Therefore, $q \notin \beta(T)$ and we have proved (3). To prove (4), we show that β satisfies the following:

(*) if $t_1, t_2 \in [0, T]$ and $t_1 < t_2$, then $\beta(t_1) \subset \operatorname{int}[\beta(t_2)]$.

To prove (*), let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and let $x \in \beta(t_1)$. First assume that there exists $y \in Y \cap f(B, t_1)$ such that

(#) $$x \in g\left(y, \frac{t_1 - j(y)}{T} \cdot k(y)\right).$$

Be replacing the mapping $f(B, t)$ in (3.3) with the mapping $g(y, t)$, we see that, by (2) of (3.3), $g(y, s_1) \subset \operatorname{int}[g(y, s_2)]$ whenever $s_1 < s_2$. Thus, by (#),

$$x \in \operatorname{int}\left[g\left(y, \frac{t_2 - j(y)}{T} \cdot k(y)\right)\right]$$

and therefore, since $y \in Y \cap f(B, t_2)$, we see that $x \in \operatorname{int}[\beta(t_2)]$. Next assume that (#) does not hold for any $y \in Y \cap f(B, t_1)$. Then, since $x \in \beta(t_1)$, it follows easily that $x \in f(B, t_1) - Y$. Hence, $x \in X - Y$. Let W denote the interior, in X, of $f(B, t_2)$. Since $x \in f(B, t_1)$, we have by (2) of (3.3) that $x \in W$. Thus, since $x \in X - Y$ and since $X - Y$ is an open subset of M, it follows that $x \in \operatorname{int}[W]$. Hence, $x \in \operatorname{int}[f(B, t_2)]$ which implies that $x \in \operatorname{int}[\beta(t_2)]$. This completes the proof of (*). By (*) we can apply (3.2) to conclude that (4) holds. This completes the proof of (3.5).

(3.6) Lemma. *Let M be a Peano continuum, let q be a noncut point of M, and let B be a subcontinuum of M such that $q \notin B$. Then there exists an order arc $\{\alpha(t)\colon t \in [0, 1]\}$ in $C(M)$ such that*

(1) $\alpha(0) = B$;

(2) $\alpha(1) = M$;

(3) *$q \notin \alpha(t)$ for any $t < 1$;*

(4) *the function τ is continuous where $\tau\colon M \to [0, 1]$ is defined by*

$$\tau(x) = \inf\{t \in [0, 1]\colon x \in \alpha(t)\}$$

for each $x \in M$.

Proof. For each $i = 1, 2, \ldots$, let $s_i = 1 - 2^{1-i}$, let $I_i = [s_i, s_{i+1}]$, and let U_i be an open subset of M of diameter $\leqslant 2^{-i}$ such that $q \in U_i$ and $\overline{U}_i \cap B = \emptyset$. By (3.5) there is an order arc $\{\alpha_1(t)\colon t \in I_1\}$ in $C(M)$ such that $\alpha_1(s_1) = B$, $\alpha_1(s_2) \supset M - U_1$, $q \notin \alpha_1(s_2)$, and $\alpha_1(t_1) \subset \operatorname{int}[\alpha_1(t_2)]$ whenever $s_1 \leqslant t_1 < t_2 \leqslant s_2$. Assume inductively that we have defined an order arc $\{\alpha_n(t)\colon t \in I_n\}$ in $C(M)$ such that $\alpha_n(s_{n+1}) \supset M - U_n$, $q \notin \alpha_n(s_{n+1})$, and $\alpha_n(t_1) \subset \operatorname{int}[\alpha_n(t_2)]$ whenever $s_n \leqslant t_1 < t_2 \leqslant s_{n+1}$. Since $q \notin \alpha_n(s_{n+1})$, there exists $j(n) \geqslant n+1$ such that $\overline{U}_{j(n)} \cap \alpha_n(s_{n+1}) = \emptyset$. Thus, by (3.5), there exists an order arc $\{\alpha_{n+1}(t)\colon t \in I_{n+1}\}$ in $C(M)$ such that $\alpha_{n+1}(s_{n+1}) = \alpha_n(s_{n+1})$, $\alpha_{n+1}(s_{n+2}) \supset M - U_{j(n)}$, $q \notin \alpha_{n+1}(s_{n+2})$, and $\alpha_{n+1}(t_1) \subset \operatorname{int}[\alpha_{n+1}(t_2)]$ whenever $s_{n+1} \leqslant t_1 < t_2 \leqslant s_{n+2}$. Thus, by induction, we have defined α_i for each $i = 1, 2, \ldots$ such that $\alpha_{i+1}(s_{i+1}) = \alpha_i(s_{i+1})$. Hence, by letting $\alpha'(t) = \alpha_i(t)$ if $t \in I_i$ for each $t \in [0, 1)$, we see that α' is a continuous function from $[0, 1)$ into $C(M)$. From the construction we see that if $t \in I_i$ for some $i \geqslant 2$, then $H(\alpha'(t), M) \leqslant 2^{1-i}$. Hence, the function $\alpha\colon [0, 1] \to C(M)$ defined by

$$\alpha(t) = \begin{cases} \alpha'(t), & \text{if } 0 \leqslant t < 1, \\ M, & \text{if } t = 1 \end{cases}$$

is continuous. It is evident from the construction that α satisfies (1), (2), and (3). It is also clear from the construction that

(#) $\qquad \alpha(t_1) \subset \operatorname{int}[\alpha(t_2)] \quad$ whenever $\quad 0 \leqslant t_1 < t_2 \leqslant 1$.

By (#), α is one-to-one. Hence, $\{\alpha(t)\colon t \in [0, 1]\}$ is an arc which, by (#), is an order arc (in $C(M)$). Hence, again using (#), we can apply (3.2) to conclude that α satisfies (4). Therefore, we have proved (3.6).

(3.7) Main lemma. *Let L be a connected, locally connected, locally compact, noncompact separable metric space. Let $X = L \cup R$ be a metric compactification of L with a compact metric space R as remainder. Let B be a subcontinuum of X such that $B \subset L$. Then there exists an order arc $\{\alpha(t)\colon t \in [0, 1]\}$ in $C(X)$ such that*

(1) $\alpha(0) = B$;

(2) $\alpha(1) = X$;

(3) *$R \cap \gamma(t) = \emptyset$ for any $t < 1$;*

(4) *the function τ is continuous where $\tau\colon X \to [0, 1]$ is defined by*

$$\tau(x) = \inf\{t \in [0, 1]\colon x \in \gamma(t)\}$$

for each $x \in X$.

Proof. Let M denote the quotient space obtained from X by identifying all points of R to one point (denoted by) q. For each $x \in X$ we let $[x]$ denote the member of M containing x; thus, $[x] = \{x\}$ if $x \notin R$ and $[x] = q$ if $x \in R$. Let $\nu\colon X \to M$ denote the quotient map, i.e., $\nu(x) = [x]$ for each $x \in X$. Since L is connected, X is a continuum. Hence, since ν is continuous, M is a continuum (M is metric by [5, Thm. 1, p. 64] and [5, Thm. 3, p. 21]). Let us note for future use that

(*) $\quad \nu|L$ is a homeomorphism from L onto $\nu(L) = M - \{q\}$.

It follows from (*) that M is locally connected at each point other than q and, hence [5, Thm. 3, p. 247], at q. Thus, M is a Peano continuum. Since L is connected, q is a noncut point of M. Since B is a subcontinuum of L, $\nu(B)$ is a subcontinuum of M such that $q \notin \nu(B)$. Therefore, we can apply (3.6) to M, q, and $\nu(B)$ to obtain an order arc $\{\alpha(t)\colon t \in [0, 1]\}$ in $C(M)$ such that $\alpha(0) = \nu(B)$, $\alpha(1) = M$, $q \notin \alpha(t)$ for any $t < 1$, and $\tau'\colon M \to [0, 1]$, defined by

$$\tau'([x]) = \inf\{t \in [0, 1]\colon [x] \in \alpha(t)\}$$

for each $[x] \in M$, is continuous. For each $t \in [0, 1]$, let

$$\gamma(t) = \nu^{-1}(\alpha(t)).$$

Since $\alpha(t)$ is a subcontinuum of $\nu(L)$ for each $t < 1$, it follows from (*) that, whenever $0 \leqslant t < 1$, $\gamma(t) \in C(X)$ and γ is continuous at t. Since $\alpha(1) = M$, $\gamma(1) = X \in C(X)$. To prove continuity of γ at $t = 1$, let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0, 1]$ converging to $t = 1$. Assume without loss of generality that $t_n \leqslant t_{n+1} < 1$ for each n. Since α is continuous at $t = 1$ and since $\alpha(1) = M$, $D = \bigcup\{\alpha(t_n)\colon n = 1, 2, \ldots\}$ is a dense subset of M. Thus, since $q \notin \alpha(t_n)$ for any n, D is a dense subset of $M - \{q\} = \nu(L)$. Hence, by (*), $\nu^{-1}(D)$ is a dense subset of L. Thus, since $\overline{L} = X$, $\nu^{-1}(D)$ is a dense subset of X. Thus, since $\nu^{-1}(\alpha(t_n)) \subset \nu^{-1}(\alpha(t_{n+1}))$ for each n, $\{v^{-1}(\alpha(t_n))\}_{n=1}^{\infty}$ converges to X as $n \to \infty$. Therefore, since $\gamma(1) = X$, we have proved that γ is continuous at $t = 1$. This completes the proof that γ is a continuous function from $[0, 1]$ into $C(X)$. Since $\{\alpha(t): t \in [0, 1]\}$ is an order arc in $C(M)$ and since γ is a continuous function from $[0, 1]$ into $C(X)$, it follows easily using the formula for γ that $\{\gamma(t): t \in [0, 1]\}$ is an order arc in $C(X)$. Since $\alpha(0) = v(B)$ and $B \subset L$, $\gamma(0) = B$. Since $q \notin \alpha(t)$ for any $t<1$, $R \cap \gamma(t) = \emptyset$ for any $t<1$. It remains to prove (4). Let τ be as defined in (4). A simple computation using the formulas for τ, γ, and τ' shows that $\tau(x) = (\tau' \circ v)(x)$ for each $x \in X$. Therefore, the continuity of τ follows from the continuity of τ' and v. This completes the proof of (4) and, therefore, we have proved (3.7).

4. Main results. Throughout this section L will denote a connected, locally connected, locally compact, noncompact, separable metric space, and $X = L \cup R$ will denote a metric compactification of L with a compact metric space R as remainder.

(4.1) THEOREM. *If A is a closed subset of X such that $A \cap L \neq \emptyset$, then there is a continuous function F: $X \to C(X)$ such that A is the fixed point set of F.*

Proof. Let $p \in A \cap L$. Let γ and τ be as in (3.7) for the case when $B = \{p\}$. Since τ is continuous, it follows from (2.1) that there is a continuous function F: $X \to C(X)$ such that A is the fixed point set of F.

(4.2) Remark. Let us note that (4.1) is actually a generalization of (2.2). Let M be a Peano continuum, let $A \in 2^M$, and let q be a noncut point of M [5, Thm. 5, p. 177]. Then $L = M - \{q\}$, $R = \{q\}$, and $X = M$ satisfy the assumptions at the beginning of this section. If $A \cap L \neq \emptyset$, then, by (4.1), there is a continuous function from X into $C(X)$ with fixed point set A. If $A \cap L = \emptyset$, then $A = \{q\}$ and, thus, A is the fixed point set of the mapping F: $X \to C(X)$ defined by $F(x) = \{q\}$ for each $x \in X$. Therefore, M has MCIP.

(4.3) THEOREM. *If L contains a simple closed curve, then X has* MCIP.

Proof. Let $A \in 2^X$. If $A \cap L \neq \emptyset$, then, by (4.1), A is the fixed point set of a mapping from X into $C(X)$. If $A \cap L = \emptyset$, then $A \subset R$. Let S be a simple closed curve such that $S \subset L$. Let γ be as in (3.7) for the case when $B = S$. Let g: $S \to S$ be a fixed point free mapping and let G: $S \to C(S)$ be the induced mapping defined by $G(x) = \{g(x)\}$ for each $x \in S$. Then the fixed point set of G is empty and $\mathcal{G} = \{G(x): x \in S\}$ is a locally connected subcontinuum of $C(S)$. Since $\{\gamma(t): t \in [0, 1]\}$ is an order arc in $C(X)$ from S to X and since A is a compact subset of $X - \bigcup \{\gamma(t): t<1\}$, it follows from (2.3) that G can be extended to a mapping F: $X \to C(X)$ such that the fixed point set of F is A. This completes the proof of (4.3).

(4.4) COROLLARY. *If the dimension of L is $\geqslant 2$, then X has* MCIP.

Proof. From the assumptions about L made at the beginning of this section, and by using [5, Thm. 1, p. 254] and [5, Thm. 5, p. 253], we see that L is the union of countably many nondegenerate locally connected continua $L_1, L_2, \ldots, L_n, \ldots$ Suppose that L does not contain a simple closed curve. Then [5, p. 300], each L_n is a dendrite and, thus, is one-dimensional. Hence, by the Sum Theorem [2, III, p. 30], L is one-dimensional, a contradiction. Thus, L contains a simple closed curve. Therefore, (4.4) follows from (4.3).

(4.5) THEOREM. *If R is a continuum and R has the property that for each nonempty closed subset A of R there exists a mapping G: $R \to C(R)$ with fixed point set A such that $\mathcal{G} = \{G(y): y \in R\}$ is contained in a locally connected subcontinuum $\mathcal{L}$ of $C(R)$, then X has* MCIP.

Proof. Use (4.1) and (2.3) (with $K = \emptyset$).

(4.6) COROLLARY. *If R is a Peano continuum, then X has* MCIP.

Proof. It follows from (2.2) that R satisfies the hypotheses of (4.5).

We remark that the examples in (2.8) and (2.9) show that the connectedness of L is necessary in (4.3), (4.4), (4.5), and (4.6).

The continuum $X = L \cup R$ need not have MCIP when L is one-dimensional and does not contain a simple closed curve. We conclude by giving several examples which illustrate the situation when L is one-dimensional and does not contain a simple closed curve.

(4.7) EXAMPLE. Let $L = [0, \infty)$, let R be the continuum X in (2.8), and let $X = L \cup R$ be a metric compactification of L with remainder R. Let A be as in (2.8) and suppose that F: $X \to C(X)$ is a mapping with fixed point set A. By the result in (2.8), F can not map R into $C(R)$. Let $r_0 \in R$ such that $F(r_0) \cap L \neq \emptyset$ and let $t_0 = \inf\{t \in L: t \in F(r_0)\}$. Let U be a connected open subset of L such that $t_0 \in U$ and $\overline{U} \cap R = \emptyset$. Let $t_1 = \sup(U)$. Since F is continuous at r_0, there exists a point $t_2 \in L$ such that $F(t_2) \cap U \neq \emptyset$ and $t_2 > t_1$. Since $F(t_2)$ is compact and connected and since $t_2 \notin F(t_2)$, $t_2 > \sup(F(t_2))$. It now follows easily that F has a fixed point p such that $0 \leqslant p \leqslant t_2$, which is a contradiction. Hence X does not have MCIP.

(4.8) EXAMPLE. Let X be as in (4.7). Let $L_1 = (-\infty, 0]$, R_1 be an arc, and $X_1 = L_1 \cup R_1$ be a metric compactification of L_1 with remainder R_1. Let Y be

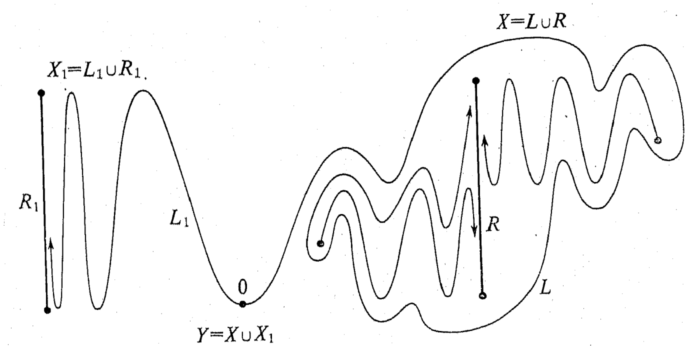

Fig. 2

the space obtained by taking the disjoint union of X and X_1 and then identifying the point 0 in X and the point 0 in X_1 (see Figure 2). Let A be as in (2.8) and suppose that $F\colon Y \to C(Y)$ is a mapping with fixed point set A. Since F can not map R into $C(R)$, an argument similar to the argument in (4.7) shows that there exist $t_1, t_2 \in L$ such that $t_1 < t_2$ and such that $F(t_2) \subset R_1 \cup L_1 \cup [0, t_1]$. Since F has no fixed points in $L_1 \cup L$, it follows easily that for each $t \in L_1 \cup [0, t_2]$ there exists $t' < t$ such that $F(t) \subset R_1 \cup (-\infty, t']$. Consequently, F must map R_1 into $C(R_1)$. Since R_1 has the fixed point property for continuum-valued mappings [15], it follows that F has a fixed point in R_1, which is a contradiction. Hence Y does not have MCIP.

We now show that the example in (4.8) can be modified slightly so that the resulting continuum will have MCIP.

(4.9) EXAMPLE. Let X and L_1 be as in (4.8). Let R_2 be any Peano continuum which is not a dendrite and let $X_2 = L_1 \cup R_2$ be a metric compactification of L_1 with remainder R_2. Let Z be the space obtained by taking the disjoint union of X and X_2 and then identifying the point 0 in X and the point 0 in X_2. We will show that Z has MCIP. Let $A \in 2^Z$. By (4.1), we may assume that $A \subset R_2 \cup R$. If $A \cap R_2 \neq \emptyset$, then, by (2.2), there exists a mapping $G\colon R_2 \to C(R_2)$ with fixed point set $A \cap R_2$. If $A \cap R_2 = \emptyset$, then, by [6], there exists a fixed point free mapping $G'\colon R_2 \to C(R_2)$. It is clear that there exists an order arc $\{\varkappa(t)\colon t \in [0,1]\}$ in $C(Z)$ from R_2 to Z such that $A \cap R \subset Z - \bigcup \{\varkappa(t)\colon t < 1\}$. Since $C(R_2)$ is locally connected, it now follows from (2.3) that G (or G') can be extended to a mapping $F\colon Z \to C(Z)$ such that the fixed point set of F is A.

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DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
Morgantown, West Virginia 26506

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