Applications of certain \( \mathcal{R} \)-families

by

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Dedicated to Professor J. Aczel on his 60th birthday

Abstract. We give some applications of Morgan's abstract Baire category theory to Cauchy's functional equation and to Hamel bases. Especially we can give a unified representation for all results of Kuczma's report on topologically saturated non measurable sets in \([7]\).

1. Introduction. In the real analysis and in the theory of functional equations there are many results, which remain true, replacing measure theoretical conditions by appropriate topological conditions, though the proofs are sometimes quite different (see [3] and [13]). The aim of this paper is to show, that Morgan's theory of \( \mathcal{R} \)-families (cf. [9]–[11]) is an appropriate theory for a unified representation of many well-known results, concerning measure and category especially in the theory of real-valued additive functions and in the herewith closely related theory of Hamel bases. We remark that all theorems are formulated in \( \mathbb{R} \), though they are sometimes valid in \( \mathbb{R}^n \) or in more general spaces.

2. Preliminaries. We shall use the terminology of [9]–[11] and assume, that the reader is familiar especially with the theory of \( \Psi \)- and \( \Omega \)-families on \( \mathbb{R} \). If \( \mathcal{E} \) is a \( \mathcal{R} \)-family on \( \mathbb{R} \), we say, that \( \mathcal{E} \) is multiplication invariant respectively inversion invariant, if for all \( x \in \mathbb{R} \setminus \{0\} \) we have \( A \cdot x \in \mathcal{E} \) respectively \( -A \in \mathcal{E} \). Now if \( \mathcal{E} \) is multiplication invariant respectively inversion invariant, it is clear, that the families of \( \mathcal{E} \)-singular sets, \( \mathcal{E} \)-sets and sets with the Baire property are also multiplication invariant respectively inversion invariant. Moreover a \( \mathcal{E} \)-family \( \mathcal{E} \) on \( \mathbb{R} \) is said to be an inversion invariant \( \mathcal{E} \)-family \( \mathcal{G} \) on \( \mathbb{R} \) such that \( \mathcal{G} \) contains

\[
\{ x \in \mathbb{R} : |x-y|<1/n \} \quad y \in \mathbb{Q}, \quad n \in \mathbb{N} \}
\]

We define a class of sets, which will play a key role in our considerations:

\[
\mathcal{G}_{\text{inf}} := \{ A \subset \mathbb{R} : \exists B \subset A : B \in \mathcal{B}(\mathbb{R}) \cap \mathcal{G}_{\text{inf}} \}
\]

The next theorem will be used rather often in this note ([16], Theorem 3.4 and [17], Theorem 2):

**Theorem 2.1.** (1) If \( \mathcal{G} \) is a \( \mathcal{E} \)-family on \( \mathbb{R} \) and if \( A \in \mathcal{G}_{\text{inf}} \), then \( A - A \) contains an interval.
3. Main results. The first results in this section were motivated by Kuczma’s report on topologically saturated non-measurable sets during the 17th International Symposium about functional equations in Oberwolfach 1979 [7].
\( \theta(1) = 0 \). Since \( A \in \Psi \) we infer from Theorem 2 in [11], that there is a \( \Theta \)-set \( C \), such that \( B \cap A \in \Psi \) for all \( B \in \Psi \) with \( B \subseteq C \). Now take an \( x \in C \cap A \). From the definition of a \( \Theta \)-family we get a descending sequence \( (A_n) \) of \( \Theta \)-sets, such that \( x \in A_1, A_2 \subseteq C \), and \( \text{diam} A_n \leq 1/n \) for all \( n \in \mathbb{N} \). Choose \( m \in \mathbb{N} \) such that \( 1 + \frac{1}{m} \leq \frac{b-a}{2} \). By our construction we have \( A_{2m} \cap A \in \Psi \) and it follows \( A \in \Psi \) and \( \Psi \). But now \( p(x) \) does not take on values of an interval of positive length for all \( x \in A \): indeed, if \( x \in A \), then by our assumption \( f(x) \) does not take on values of an interval of length \( b-a \).

Since \( f \) is \( \Omega \)-homogeneous, \( f(1) = 0 \) implies \( f(r) = f(1) = 0 \) for all \( r \in \Omega \). It follows that \( f(x) \neq [a, b] \) for all \( x \in M : = A + Q \), where \( c, d \in \Psi \) (see the corollary after Theorem 2 in [3]). So we have \( M \) := \{\( x \in R : f(x) \neq [a, b] \} \subseteq c, d \in \Psi \). Without loss of generality we may suppose that \( b = ra > 0 \), where \( r \in \Omega \}. \) (Observe that \( -f \) is also an additive function). Using that \( f \) is \( \Omega \)-homogeneous we get for all \( n \in \mathbb{N} \), that \( f(x) \neq [a, b, c, d] \) only if \( x \) is an element of the \( \Theta \)-set \( r^m M_0 \). Defining \( B := [a, b, c, d] \) follows \( B \subseteq M \cap \Omega \cup [0, \infty) \), and \( c, d \in \Psi \cap \Omega \). But \( f \) is bounded above on \( c, d \) by \( a \). Again Theorem 4.1 in [16] implies the continuity of \( f \), which is impossible. (We remark that in the proof only “rational multiplication invariance” is used.

**Definition 3.5.** [7] Let \( A \subseteq R \) be an uncountable Borel set. A Hamel basis \( H \) is called a **Burstin basis relative to** \( A \) if \( H \) intersects each uncountable Borel subset of \( A \).

The existence of a Burstin basis relative to \( A \) was proved by Burstin in [2] and by Abian in [1]. In a similar manner the following result can be proven.

**Theorem 3.6.** [7] Every Borel set \( A \subseteq R \), containing a Hamel basis, contains a Burstin basis relative to \( A \).

Now we can prove immediately the following two results.

**Theorem 3.7.** Let \( \Psi \) and \( \sigma \) be nonequivalent \( \Psi \)- and \( \sigma \)-family on \( R \). If \( A \subseteq R \) is a \( \Psi \)-residual Borel set, then each Burstin basis relative to \( A \) is non \( \Psi \)-saturated.

**Proof.** (1) Let \( H \) be a Burstin basis relative to \( A \). If \( H \in \Psi \), then by Theorem 2.1 \( H \subseteq \Omega \) contains an interval \( I \). Then there is an \( r \in \Omega \} \) such that \( r \in I \). Thus there exist \( b, d \in H \) satisfying \( r = b \). Contradicting the linear independence of \( a, b, d \).

(2) Now we assume that \( A \subseteq \Omega \), then by Theorem 12 in [10] \( A \subseteq \Omega \) contains a nonempty perfect set \( P \); but \( P \) is a nonmeasurable Borel set of \( A \) such that \( P \cap H = \Omega \), which is impossible. Since \( \Psi \)-residual, that is \( A \subseteq \Psi \), we get \( \Psi \subseteq \Psi \).

**Theorem 3.8.** Let \( \Psi \) and \( \sigma \) be nonequivalent \( \Psi \)- and \( \sigma \)-family on \( R \). Then there exist Hamel bases \( H \) and \( B \) such that \( H \in \Psi \) and \( B \in \Psi \).

**Proof.** By Theorem 2.1 \( R \) can be decomposed into a \( \Psi \)-set \( C \) and a \( \sigma \)-set \( D \). Observing that \( \Psi \) and \( \sigma \) consist of perfect sets, the proof of Theorem 2.1 in [17] yields that \( C \) and \( D \) are uncountable Borel sets (indeed, \( C \) and \( D \) are either \( \Sigma_p \) or \( \Sigma_p \)-sets). Moreover we have \( C \times \sigma \subseteq \Psi \) and \( D \times \sigma \subseteq \Psi \). From Theorem 2.1 we infer again that \( E(C) = R \) and \( E(D) = R \). But it is known, that this is a necessary and sufficient condition for \( C \) and \( D \) to contain a Hamel basis (cf. [3], p. 42). By Theorem 3.6 \( C \) contains a Burstin basis \( H \) relative to \( C \) and \( D \) contains a Burstin basis \( B \) relative to \( D \). Now Theorem 3.7 yields \( H \subseteq \Sigma_p \subseteq \Psi \) and \( B \subseteq \Sigma_p \subseteq \Psi \).

The next two results are extensions of theorems in [4]. We introduce some notations. If \( H \) is any Hamel basis, then we denote by \( H^* \) the set of all real numbers of the form \( \sum x_i / \vert x_i \vert \) (finite sum) and by \( H^* \) the set of all real numbers of the form \( \sum x_i / \vert x_i \vert \) (finite sum); here \( x_i \in H \) and \( x_i = H^* \). Moreover we define for any \( \Psi \)-family on \( R \):

\[ \Psi(\Psi') := \{ A \subseteq R : |A| > \mathbf{K}_0, \forall C \subseteq \Psi : |C \cap A| < \mathbf{K}_0 \} . \]

If for example \( \Psi \) is the topological example of \( \Psi \), then \( \Psi(\Psi') \) consists of all Lusin sets (see [11], Definition 9).

**Theorem 3.9.** [4] If \( H \) is any Hamel basis and if \( \Psi' \) is a multiplication invariant \( \Psi \)-family on \( R \), then \( H^* \subseteq \Psi(\Psi') \).

**Proof.** It is obvious, that for all \( x \in R \) there is an \( x \in Z \) such that \( x \in H^* \). Thus

\[ R = \bigcup \{ z \subseteq H^* : z \in Z \} \]

Since \( R \subseteq \Psi \), we have \( n^{-1} \in H^* \subseteq \mathbf{K}_0 \) for some \( n \in Z \}. \) Thus \( H^* \subseteq \Psi \). Moreover it is known that \( H^* \) is dense in \( R \). Since \( H^* + H^* = H^* \) (\( H^* \) is an additive group), Theorem 2.1 in [9] yields \( \Psi(\Psi') \).

Now suppose that \( H^* \subseteq \Psi \). By Theorem 2.1 the interior of \( H^* \) is nonempty. Thus \( H^* \) is open and also closed and we get \( \Psi \subseteq \Psi \), which is a contradiction.

**Theorem 3.10.** Let \( \Psi \) and \( \sigma \) be nonequivalent, multiplication invariant \( \Psi \)- and \( \sigma \)-family on \( R \), such that \( \Psi \subseteq \sigma \subseteq \sigma \) and such that \( \sigma \) and \( \Psi \) satisfy c.c. Then \( \Psi \subseteq \sigma \subseteq \sigma \) and \( \sigma \subseteq \sigma \subseteq \sigma \).

**Proof.** (1) By \( \Psi \) we denote the family of all sets, which are complements of members of \( \Psi \). Because of \( \Psi \subseteq \sigma \subseteq \sigma \), we have \( \Psi \subseteq \sigma \subseteq \sigma \). Now let \( \Psi \subseteq \sigma \subseteq \sigma \).

In [4] we can construct a Hamel basis \( H \) such that \( H^* \subseteq \Psi \) and \( \mathbf{K}_0 \subseteq \mathbf{K}_0 \) for all \( \mathbf{K}_0 \). In this stage of the proof we need the multiplication invariance of \( \sigma \).

Now let \( \Psi \subseteq \Psi \). By Theorem 3.1 in [11] \( A \subseteq \Psi \) is contained in a certain set \( \mathbf{K}_0 \), \( \mathbf{K}_0 \), which proves that \( H^* \subseteq \Psi(\Psi') \). Theorem 2.1 yields that \( H^* \) is the disjoint union of
a set $A \in \mathcal{E}$, and a set $B \in \mathcal{D}$. Since $H^+ \in \mathcal{E}(\mathcal{E})$ we get $|A| \leq |B|$. Thus $A \in \mathcal{D}$ and also $H^+ \in \mathcal{D}$.  

The second statement can be proved in exactly the same manner.  

We now prove a result, which can be compared with Theorem 5 in [9].

**Theorem 3.11.** Let $\mathcal{C}$ be a $\mathcal{S}$-family on $R$ such that $|\mathcal{C}| < \aleph_0$ and $\mathcal{C}$ satisfies c.c.c. If $c = a_0$, then each $\mathcal{C}_\mathcal{P}$ set can be decomposed into $c$ disjoint sets, none of which has the Baue property.

**Proof.** Let $A \in \mathcal{C}_\mathcal{P}$. By Theorem 17 in [11] and by Theorem 6 in [10] $A$ contains a set $L \in \mathcal{L}(\mathcal{E})$. If $f : R \times R \to L$ is a bijective function (Observe that $c = a_0$ implies $|R \times R| = |L|$, then

$$\{f((x) \times R) : x \in R\}$$

are $c$ disjoint $\mathcal{C}_\mathcal{P}$-sets, contained in $A$. Now Theorem 19 in [10] yields, that $A$ contains $c$ disjoint sets $B_n$, $a < c a_n$, such that $B_n \notin \mathcal{C}(\mathcal{E})$ for all $a < c a_n$. Consider

$$D := A \cap \bigcup\{B_n : a < c a_n\}.$$ 

If $D \notin \mathcal{C}(\mathcal{E})$, then $\{B_n : a < c a_n\} \cup \{D\}$ is the desired decomposition of $A$. If $D \in \mathcal{C}(\mathcal{E})$, then $\{B_n \cup D\} \cup \{B_n : 0 < c a_n\}$ is a decomposition of $A$ with $B_n \cup D \notin \mathcal{C}(\mathcal{E})$. Indeed, if $B_n \cup D \notin \mathcal{C}(\mathcal{E})$, then $(B_n \cup D) \cap D = B_n \notin \mathcal{C}(\mathcal{E})$, which is impossible.

We close our considerations with two results concerning real-valued additive functions. Smith [18], [19] could give necessary and sufficient conditions for sets $T \subseteq R$ such that every additive function, bounded (respectively bounded above) on $T$, is continuous in $R$. We here replace these conditions by equivalent conditions on $\mathcal{D}$-families on $R$.

**Theorem 3.12.** Let $\mathcal{C}$ be a $\mathcal{S}$-family on $R$ and let $T \subseteq R$. Then every additive function $f : R \to R$ bounded on $T$ is continuous in $R$ if $T \subseteq \mathcal{C}(\mathcal{E})$.

**Proof.** (I) Let $Q(T \setminus T) = \mathcal{C}(\mathcal{E})$ and let $f((x) \times R) \leq M$ for all $x \in T$ and for some $M \in R$. Using that $f$ is $Q$-homogeneous we get that $|f((x) \times R) \leq M$ for all $x \in Q(T \setminus T)$. Now Theorem 4.1 in [16] implies that $f$ is continuous in $R$.

(II) Assume that $Q(T \setminus T) \notin \mathcal{C}(\mathcal{E})$. If $Q(T \setminus T)$ would contain an interval, it would also contain a member $A$ of the family $\{x \in R : |x - y| < 1/n : n \in \mathbb{N}\}$. Since each $\mathcal{S}$-family satisfies $\mathcal{C} \subseteq \mathcal{D}$, we get $A \in \mathcal{C}(\mathcal{E}) \cap \mathcal{E}$, which is impossible. Thus $Q(T \setminus T)$ contains no interval and by Theorem 4 in [18] there is a discontinuous additive function bounded on $T$.

Using again Theorem 4.1 in [16] and the main result in [19], we can prove immediately the following theorem.

**Theorem 3.13.** Let $\mathcal{C}$ be a $\mathcal{S}$-family on $R$ and let $T \subseteq R$. Then every additive function $f : R \to R$ bounded above on $T$ is continuous in $R$ if $Q(T \setminus A) \notin \mathcal{C}(\mathcal{E})$ for all subsets $A$ of $R$, which are $Q$-radial at a point.

Let $\mathcal{C}$ be a $\mathcal{S}$-family on $R$ with $R \notin \mathcal{C}$. We here remark, that the condition "Every additive function $f : R \to R$ upper-bounded on $T \subseteq R$ is continuous in $R$" does not imply that $T \subseteq \mathcal{C}(\mathcal{E})$.

References


Fixed point sets of continuum-valued mappings

by

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Abstract. Let $X$ be a metric continuum and let $C(X)$ denote the hyperspace of subcontinua of $X$. The following question is investigated: When does $X$ have the property that for each non-empty closed subset $A$ of $X$ there exists a continuous function $F: X \to C(X)$ such that $x \in F(x)$ if and only if $x \in A$?

1. Introduction. By a continuum we mean a nonempty compact connected metric space. If $X$ is a continuum, then $2^X$ denotes the hyperspace of closed subsets (subcontinua) of $X$, each with the Hausdorff metric.

A Peano continuum is a locally connected continuum. By a mapping we mean a continuous function. If $X$ is a space and $f: X \to X$ is a mapping, then the fixed point set of $f$ is $\{x \in X: f(x) = x\}$. In [16] L. E. Ward, Jr. defines a space $X$ to have the complete invariance property (CIP) provided that for each nonempty closed subset $A$ of $X$ there exists a mapping $f: X \to X$ such that $A$ is the fixed point set of $f$. Some spaces known to have CIP are one-dimensional Peano continua [9], convex subsets of Banach spaces [16], compact $n$-manifolds [14], locally compact metrizable groups [8], and polyhedra [3]. In [16] Ward asked if every Peano continuum has CIP. This question was answered negatively in [7]. A rather complete bibliography of the literature on fixed point sets and CIP may be found in the survey article by H. Schirmer [14].

Part of the literature on the fixed point property has been concerned with multi-valued (set-valued) mappings. However, the question of which sets can be fixed point sets of multi-valued mappings has not been investigated before. If $X$ is a continuum, $F: X \to 2^X$ is a mapping, and $x \in X$, then $x$ is said to be a fixed point of $F$ provided $x \in F(x)$. The fixed point set of $F$ is $\{x \in X: x \in F(x)\}$. By a continuum-valued mapping we mean a mapping $F: X \to C(X)$.

In this paper we introduce and study the following generalization of CIP to the setting of multi-valued mappings. A continuum $X$ is said to have the complete invariance property for continuum-valued mappings (MCIP) provided that for each nonempty closed subset $A$ of $X$ there exists a mapping $F: X \to C(X)$ such that $A$ is the fixed point set of $F$. 