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A note on Robinson's non-negativity criterion

by

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Abstract. Let V be a real algebraic variety in $\boldsymbol{R}^n$ and $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ a system of polynomial inequalities over V, such that its solution set is Zariski dense on V. Then we prove that Robinson's criterion gives necessary and sufficient conditions for $f \geqslant 0$ to be a consequence of the given system if and only if $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ is *locally slack* on V in the sense of Stengle [S]. As a corollary we quickly obtain several recent results on positive function over the set V_c of central points of V, the image of V_c for finite birational morphisms and Efroymson's [E1] characterization of central points.

1. Introduction. Robinson's non-negativity criterion states that if a polynomial inequality $f \geqslant 0$ is a consequence of a finite system of polynomial inequalities $f_1 \geqslant 0, \ldots, f_r \geqslant 0$, then $f = \sum \varrho_\lambda^2 h_\lambda$, where the ϱ_λ are rational functions and the h_λ are products of some f_i (cf. [Ro]). Stengle [S] extended the result by showing that on a real variety V, if $f \geqslant 0$ is a consequence of $f_1 \geqslant 0, \ldots, f_r \geqslant 0$ then $a^2 f \equiv \sum \varrho_\lambda^2 h_\lambda$ $(\operatorname{mod} \mathscr{I}(V))$, where ϱ_λ and h_λ are as above and a is outside $\mathscr{I}(V)$. Motzkin [M] suggested that Robinson's criterion could be interpreted as the counter part for inequalities of the Hilbert Nullstellensatz; but as Stengle remarked, this comparison is inappropriate in that there exists in some cases a polynomial f which is expressible in the form $\sum \varrho_\lambda^2 h_\lambda$ but which is not positive definite over the closed semi-algebraic set $f_1 \geqslant 0, \ldots, f_r \geqslant 0$. The need for a study of this situation is noted in several recent papers, e.g. Stengle (loc. cit.), Gondard [G], Lorenz [L], where some counter-examples are also given to the sufficiency of the criterion. It is the purpose of this note to clarify the problem and then to apply our results to very recent theorems of Schwartz [Sch], Bochnak–Efroymson [B–E] and Adkins [A]. Our treatment is related to work of Dubois [D] and Brumfiel [B].

2. Locally slack systems on a semi-algebraic set. Locally slack systems of inequalities $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ were first introduced by Stengle [loc. cit.] as a type of system of use in obtaining sufficiency for Robinson's criterion. However he was not able (see p. 96 [S]) to give a direct description of these systems in terms of the $f_1, \ldots, f_r$ (see 2.3 below). We are going to generalize his definition.

2.1. Definition. Given a semi-algebraic set $S \subseteq \boldsymbol{R}^n$, we say that a system of inequalities $\{f_1 \geq 0, \ldots, f_r \geq 0\}$ ($f_i \in \boldsymbol{R}[X]$) is *locally slack on* S, if for every $f \in \boldsymbol{R}[X]$ the ideal $\mathscr{I}(H)$, where H is defined by

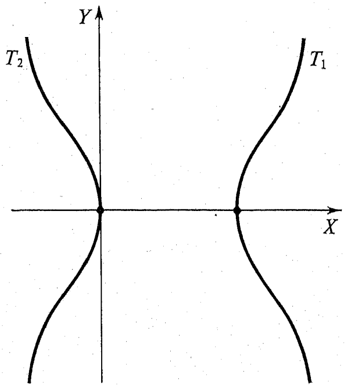
$$H = \{z \in S;\ f_1(z) \geq 0, \ldots, f_r(z) \geq 0, f(z) > 0\},$$

is either $\boldsymbol{R}[X]$ or $\mathscr{I}(S)$. In case a trivial inequality such as $1 \geq 0$ is locally slack on S then we say that S is *locally slack*.

Note that if $\{f_1 \geq 0, \ldots, f_r \geq 0\}$ is locally slack on S then the solution set of the system on S, i.e. $S \cap \{f_1 \geq 0, \ldots, f_r \geq 0\}$, is a locally slack semi-algebraic set; but the converse is false. For local slackness depends on the *ambient* set as well as the geometry of the intersection. For a variety V in $\boldsymbol{R}^n$, local slackness is equivalent to the condition of being locally real at every point ([D–E], [E$_1$]). The study of positive definite functions over $\{f_1 \geq 0, \ldots, f_r \geq 0\} \cap S$, where $\{f_1 \geq 0, \ldots, f_r \geq 0\}$ is locally slack on the semi-algebraic set S, can be reduced to the case of a locally slack, closed, semi-algebraic set T (take $T = \overline{\{f_1 \geq 0, \ldots, f_r \geq 0\} \cap S}$, bar denoting strong closure). Now T is a finite union of sets of the form $\{g_1 \geq 0, \ldots, g_m \geq 0\}$ (Recio [R$_1$], Bochnak–Efroymson [B–E], Delzell [De], Coste–Coste-Roy [C–C]). Let W denote the smallest algebraic set containing T, which is defined by $\mathscr{V}(\mathscr{I}(T))$. It is irreducible because T is locally slack. Now suppose that we have a proper decomposition $T = \bigcup_{i=1}^{s} T_i$, where each T_i is the solution set of a system $\{g_{i_1} \geq 0, \ldots, g_{i_m} \geq 0\}$ on $\boldsymbol{R}^n$. Then W is also the smallest algebraic set containing T_i, but in general $\{g_{i_1} \geq 0, \ldots, g_{i_m} \geq 0\}$ is not locally slack on W, as shown in the following example: take

$$T_1 = \{x^3 - x^2 - y^2 \geq 0\},$$
$$T_2 = \{(x-1)^2 + y^2 + (x-1)^3 \leq 0\}.$$

Then $T = T_1 \cup T_2$ is locally slack on $\boldsymbol{R}^2 = W$, but neither of the T_i is locally slack on W.



But also $T = T_1' \cup T_2'$ where

$$T_1' = \{x^3 - x^2 - y^2 \geq 0,\ x \geq 1\},$$
$$T_2' = \{(x-1)^2 + y^2 + (x-1)^3 \leq 0,\ -x \geq 0\}$$

and T_1', T_2' are now locally slack on $\boldsymbol{R}^2$. For locally slack closed semi-algebraic sets T admitting a decomposition $T = \bigcup_i T_i$, with $T_i = \{g_{i_1} \geq 0, \ldots, g_{i_m} \geq 0\}$ locally slack on W, the study of positive definite functions over T is reduced to the simpler case of a locally slack system over an algebraic variety.

Let $\{f_1 \geq 0, \ldots, f_r \geq 0\}$ be a locally slack system on a real algebraic variety V, and let H be the solution set of such system on $\boldsymbol{R}^n$. A close following of Stengle's proof of Theorems 7 and 8 in [S] yields:

2.2. Proposition. *For all* $f \in \boldsymbol{R}[X]$, $f \geq 0$ *holds over* $V \cap H$ *if and only if* $f \equiv \sum \varrho_\lambda^2 h_\lambda \pmod{\mathscr{I}(V)}$, *where the* ϱ_λ *belong to* $\boldsymbol{R}(V)$, *each* h_λ *is a product of some* f_i.

Thus Robinson's criterion provides a necessary and sufficient condition in the case of a locally slack system. It turns out that it is thus *only* for locally slack systems.

2.3. Proposition. *Assume that for the* H *above*, $\mathscr{I}(V \cap H) = \mathscr{I}(V)$. *Then the assertion in Proposition* 2.2 *is valid if and only if the system* $\{f_1 \geq 0, \ldots, f_r \geq 0\}$ *defining* H *is locally slack on* V.

Proof. The 'if' clause is treated in Proposition 2.2. For the converse, assume that the system is not locally slack on V; i.e. there exists f such that $\mathscr{I}[V \cap H \cap \{f > 0\}]$ is neither (1) nor $\mathscr{I}(V)$; according to Recio [R$_2$] this means there exists c not in $\mathscr{I}(V)$ such that

$$(fc)^{2m} + f \cdot A + B \in \mathscr{I}(V),$$

where $m > 0$, each of A and B being a sum of products of squares and some f_i. Now if A were in $\mathscr{I}(V)$ then, by Stengle's Nullstellensatz fc would be in $\mathscr{I}(V)$, which is absurd. Hence we have

$$-f \equiv \frac{B}{A} + \frac{(f \cdot c)^{2m}}{A} \pmod{\mathscr{I}(V)}.$$

Thus the condition in 2.2 fails and the converse is proved.

Now let M be a subset of the irreducible variety V. Let $\mathscr{S}(\boldsymbol{R}(V))$ be the set of all sums of squares of elements of $\boldsymbol{R}(V)$ and let $\mathscr{P}(M)$ be the set of polynomials f in $\boldsymbol{R}[V]$ which are non negative over M.

2.4. Proposition ([B], [E$_2$]). *Let* V_c *denote the strong closure of the set of simple points of* V(= *set of all central points* = *closure of the set of all points which are regular in the highest dimension*). *Then*

$$\mathscr{P}(V_c) = \boldsymbol{R}[V] \cap \mathscr{S}(\boldsymbol{R}(V)).$$

Proof. If f is positive definite over V_c then $\mathscr{I}[V \cap \{-f > 0\}]$ properly contains $\mathscr{I}(V)$; by means of the argument above we conclude that $f = -(-f)$ belongs to

$\mathscr{S}(\boldsymbol{R}(V))$. Conversely, if f is a sum of squares it can be negative only at points where some denominators vanish, therefore never over V_c (cf. [D–E]).

2.5. COROLLARY ([E₂]). *A point p of V belongs to V_c if and only if for all $f \in \boldsymbol{R}[V] \cap \mathscr{S}(\boldsymbol{R}(V))$, $f(p) \geqslant 0$.*

Proof. Suppose $p \notin V_c$; then there exists $f \in \boldsymbol{R}[V]$ such that $f(B_p) < 0$ (for a convenient ball B_p centered at p) and $f(V_c) \geqslant 0$. According to the above proposition we have then $f \in \boldsymbol{R}[V] \cap \mathscr{S}(\boldsymbol{R}(V))$. The converse follows immediately from the same proposition.

2.6. COROLLARY. *Notation as above, we have*

$$\mathscr{P}(V) = \boldsymbol{R}[V] \cap \mathscr{S}(\boldsymbol{R}(V))$$

if and only if $V = V_c$, i.e. V is locally slack.

2.7. COROLLARY (Schwartz [Sch.], Bochnak–Efroymson [B–E]). *Let M be any subset of the irreducible variety V. Then*

(a) *$\mathscr{P}(M) \subset \boldsymbol{R}[V] \cap \mathscr{S}(\boldsymbol{R}(V))$ if and only if $M \cap V_c$ is dense in V_c.*

(b) *$\mathscr{P}(M) \supset \boldsymbol{R}[V] \cap \mathscr{S}(\boldsymbol{R}(V))$ if and only if $M \cap V_c$ is dense in M.*

Proof. (a) If $M \cap V_c$ is dense in V_c then $f(M) \geqslant 0$ implies $f(V_c) \geqslant 0$ and Proposition 2.4 applies to give the desired inclusion. Assume now that $M \cap V_c$ is not dense in V_c. Take an f with the properties that $-f(M) \geqslant 0$ while $f(B_x) > 0$ for a ball B_x centered at some x in V_c. Then $-f$ can not be a sum of squares for the same reason as used in the last part of the proposition.

(b) Assume $M \cap V_c$ is dense in M. Then any sum of squares, being positive over V_c, must be positive over M. Conversely assume $M \cap V_c$ is not dense in M. As above there exists $x \in M$ and $f \in \boldsymbol{R}[V]$ such that $f(V_c) \geqslant 0$ but $f(x) < 0$ (x being in M but outside the closure of $M \cap V_c$); this f is a sum of squares outside $\mathscr{P}(M)$.

2.8. Remark. The statement of 2.3 (and hence of 2.6) can be couched solely in algebraic terms — see the paragraph at the beginning of Section 2. In fact, $\mathscr{I}(V)$ and $\mathscr{I}[V \cap \{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}]$ are algebraically described by means of the nullstellensatz of Dubois [D₀] and Stengle [S]. The local slackness of $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ on V can be formulated algebraically as shown in the proof of Proposition 2.3 (cf. Recio [R₂], Alonso [Al].

2.9. Remark. We conjecture that any locally slack closed semi-algebraic set T has a decomposition $T = \bigcup T_i$, where each T_i is defined by a locally slack system $\{g_{i_1} \geqslant 0, \ldots, g_{i_m} \geqslant 0\}$. Given any real variety V, V_c is locally slack and closed semi-algebraic ([R₁], [B]), therefore the conjecture applies to this important case. For V_c we even conjecture that the above decomposition can be obtained with $r = 1$, i.e. $V_c = \{g_1 \geqslant 0, \ldots, g_m \geqslant 0\}$.

3. Some applications of the theory.

3.1. PROPOSITION (Adkins [A]). *Let $\varrho\colon V \to W$ be a finite birational morphism of real irreducible varieties (i.e. $\varrho^*\colon \boldsymbol{R}[W] \to \boldsymbol{R}[V]$ is an integral extension and $\boldsymbol{R}(W) = \boldsymbol{R}(V)$). Then $\varrho(V_c) = W_c$.*

Proof. It is clear that $\mathscr{I}(\varrho(V_c)) = \mathscr{I}(W)$ (because $\mathscr{I}(\varrho(V)) = \mathscr{I}(W)$ and $\varrho(V \setminus V_c)$ has strictly smaller dimension than W). Now by an argument of Brumfiel in 8.9.4 of [B], using "integral" in place of "semi-integral", $\varrho(V_c)$ is defined as the closed subset of W where positive functions are just the restriction to $\boldsymbol{R}[W]$ of the positives on V_c. According to 2.4 and using birationality, these are precisely the sums of squares of $\boldsymbol{R}(V)$. Now applying 2.7 we conclude that $\varrho(V_c) = W_c$, completing the proof.

Note. For a deeper study of the above situation cf. Dubois–Recio [D–R].

3.2. COROLLARY ([E₂]). *A point x on the real variety V is locally real if and only if in a desingularization $\pi\colon \tilde{V} \to V$ there is a real point in $\tilde{V}$ over x.*

Proof. The morphism π verifies the hypotheses of 3.1 so, since $\tilde{V} = \tilde{V}_c$, we have $\pi(\tilde{V}) = V_c$.

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Added in proof. Any locally slack semialgebraic set T has in fact — as conjectured — a locally slack decomposition $T = \bigcup T_i$, $T_i = \{g_{i_1} \geqslant 0, \ldots, g_{i_m} \geqslant 0\}$ (cf. Dubois–Recio, unpublished). Also, if V is an algebraic surface then $V_c = \{g_1 \geqslant 0, g_2 \geqslant 0\} \cap V$ (cf. Ruiz "Geometric and arithmetic aspects of the 17th Hilbert Problem for real analytic germs", unpublished dissertation, U. Madrid).

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Applications of certain $\mathfrak{K}$-families

by

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Dedicated to Professor J. Aczél on his 60th birthday

Abstract. We give some applications of Morgan's abstract Baire category theory to Cauchy's functional equation and to Hamel bases. Especially we can give a unified representation for all results of Kuczma's report on topologically saturated non measurable sets in [7].

1. Introduction. In the real analysis and in the theory of functional equations there are many results, which remain true, replacing measure theoretical conditions by appropriate topological conditions, though the proofs are sometimes quite different (see [3] and [13]). The aim of this paper is to show, that Morgan's theory of $\mathfrak{K}$-families (cf. [9]–[11]) is an appropriate theory for a unified representation of many well-known results, concerning measure and category especially in the theory of real-valued additive functions and in the herewith closely related theory of Hamel bases. We remark that all theorems are formulated in $\boldsymbol{R}$, though they are sometimes valid in $\boldsymbol{R}^n$ or in more general spaces.

2. Preliminaries. We shall use the terminology of [9]–[11] and assume, that the reader is familiar especially with the theory of $\mathfrak{P}$- and $\mathfrak{S}$-families on $\boldsymbol{R}$. If $\mathscr{C}$ is a $\mathfrak{K}$-family on $\boldsymbol{R}$, we say, that $\mathscr{C}$ is *multiplication invariant* respectively *inversion invariant*, if for all $x \in \boldsymbol{R} \setminus \{0\}$ we have $A \cdot x \in \mathscr{C}$ respectively $-A \in \mathscr{C}$. Now if $\mathscr{C}$ is multiplication invariant respectively inversion invariant, it is clear, that the families of $\mathscr{C}$-singular sets, $\mathscr{C}_{\mathrm{I}}$-sets and sets with the Baire property are also multiplication invariant respectively inversion invariant. Moreover a $\mathfrak{S}^*$-family $\mathscr{C}$ on $\boldsymbol{R}$ is said to be an *inversion invariant* $\mathfrak{S}$-family $\mathscr{C}$ on $\boldsymbol{R}$ such that $\mathscr{C}$ contains

$$\{\{x \in \boldsymbol{R}: |x-y| < 1/n\}: y \in \boldsymbol{Q}, n \in \boldsymbol{N}\}.$$

We define a class of sets, which will play a key role in our considerations:

$$\mathscr{C}_{\mathrm{III}} := \{A \subset \boldsymbol{R}: \exists B \subset A: B \in \mathfrak{B}(\mathscr{C}) \cap \mathscr{C}_{\mathrm{II}}\}.$$

The next theorem will be used rather often in this note ([16], Theorem 3.4 and [17], Theorem 2):

THEOREM 2.1. (1) *If $\mathscr{C}$ is a $\mathfrak{S}$-family on $\boldsymbol{R}$ and if $A \in \mathscr{C}_{\mathrm{III}}$, then $A-A$ contains an interval.*