LEMMA 1. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be recursively saturated structures for a countable language,* $\mathfrak{M} \subseteq \mathfrak{N}$. *Then* $\mathfrak{M} \prec_{\infty\omega} \mathfrak{N}$ *iff* $\mathrm{HYP}_{\mathfrak{M}} \prec_{\infty\omega} \mathrm{HYP}_{\mathfrak{N}}$.

Proof. ($\Leftarrow$) is clear. For ($\Rightarrow$), first suppose $\mathfrak{M}$ and $\mathfrak{N}$ are countable. Then for all finite $\vec{m}$ from M, $(\mathfrak{M}, \vec{m}) \equiv_{\infty\omega} (\mathfrak{N}, \vec{m})$, so $(\mathfrak{M}, \vec{m}) \cong (\mathfrak{N}, \vec{m})$; therefore $(\mathrm{HYP}_{\mathfrak{M}}, \vec{m}) \cong (\mathrm{HYP}_{\mathfrak{N}}, \vec{m})$. Since every element of $\mathrm{HYP}_{\mathfrak{M}}$ is definable with parameters in $\mathfrak{M}$, this shows $\mathrm{HYP}_{\mathfrak{M}} \prec_{\infty\omega} \mathrm{HYP}_{\mathfrak{N}}$. The countability assumption is eliminated by Lévy absoluteness: see for example Barwise [B, II9.2]. ■

LEMMA 2. *Let* (Thm. 2)′ *denote Theorem* 2 *with* (v) *deleted and* (i) *replaced by*

(i′) $\mathfrak{M} \prec \mathfrak{N}$.

Then ZFC $\vdash$ (*Theorem* 2) $\leftrightarrow$ (*Thm.* 2)′.

Proof. ($\rightarrow$) is clear. For ($\leftarrow$), first notice that (v) is superfluous, by Lemma 1. Since recursively saturated models are homogeneous (Schlipf [Sch, II1.8 (i)]), it suffices to show that $\mathfrak{M}$ and $\mathfrak{N}$ realize the same types. But this is well known; we thank Jim Schmerl for pointing this out, as it simplifies the proof of Theorem 2. To prove this, first choose $m \in M$ greater than all elements definable in $\mathfrak{M}$. Then given $\vec{n}$ from $\mathfrak{N}$, we find $\vec{a}$ in M such that $\vec{a}$ and $\vec{n}$ realize the same type. Just choose $\vec{a}$ in $\mathfrak{N}$ to realize the type $\{\varphi(\vec{x}) \leftrightarrow \varphi(\vec{n})\colon \varphi \in L\} \cup \{x_i < m\colon x_i \text{ occurs in } \vec{x}\}$. ■

Proof of Theorem 2. Actually, (Thm. 2)′ is just the theorem of [Ka]. Now $\diamondsuit_{\omega_1}$ is assumed for that result, but Shelah [She] has shown how to eliminate this added hypothesis. So by Lemma 2, we are done. ■

References

[B] J. Barwise, *Admissible Sets and Structures*, Springer-Verlag, 1975.

[Ka] M. Kaufmann, *A rather classless model*, Proc. Amer. Math. Soc. 62 (1977), pp. 330–333.

[MM] W. Marek and A. Mostowski, *On extendability of models of* ZF *set theory to models of Kelley–Morse theory of classes*, in: *Logic Conference Kiel 1974*, Lecture Notes in Mathematics, 499, 1975, pp. 460–542.

[MS] — and M. Srebrny, *Urelements and extendability*, in: *Set Theory and Hierarchy Theory — A Memorial Tribute to Andrzej Mostowski*, Lecture Notes in Mathematics, 537, 1976, pp. 203–220.

[Sch] J. Schlipf, *A guide to the identification of admissible sets above structures*, Ann. Math. Logic 12 (1977), pp. 151–192.

[She] S. Shelah, *Models with second order properties* II: *Trees with no undefined branches*, Ann. Math. Logic 14 (1978), pp. 73–87.

PURDUE UNIVERSITY
W. Lafayette, IN 47907

Received 22 September 1980

Smooth dendroids without ordinary points

by

J. J. Charatonik (Wrocław)

Abstract. Smooth dendroids are constructed which are composed of end points and of ramification points only.

Let X be a metric continuum. If for every two points a and b of X there exists in X an arc (i.e. a continuous and one-to-one image of the closed unit interval $[0,1]$ of reals) with end points a and b, then X is said to be *arcwise connected*. For an arcwise connected continuum X we accept the following three definitions. A point p of X is called an *end point* of X if p is an end point of every arc containing p and contained in X. A point p of X is called an *ordinary point* of X if there are in X exactly two arcs with p as the common end point and which are disjoint out of p. A point p of X is called a *ramification point* of X if there are in X three (or more) arcs with p as the common end point and which are disjoint out of p. In other words end points, ordinary points and ramification points of X are exactly points of order 1, 2 and $n \geqslant 3$ in the classical sense respectively (see [17], pp. 219–221; [9], Chapter IV, I, pp. 63–64; compare [12], pp. 301–302 and [3], pp. 229–230). Thus, given an arcwise connected continuum X, we can distinguish three disjoint sets of its points: the set $E(X)$ of end points of X, the set $O(X)$ of ordinary points of X, and the set $R(X)$ of ramification points of X, and we have

$$X = E(X) \cup O(X) \cup R(X).$$

It is easy to construct some particular examples of arcwise connected continua X with the property that some of these sets are empty.

A continuum is called *hereditarily unicoherent* if the intersection of any two its subcontinua is connected. A *dendroid* means an arcwise connected and hereditarily unicoherent metric continuum. A *dendrite* means a locally connected metric continuum that contains no simple closed curve. The concept of a dendroid is a generalization of one of a dendrite: every dendrite is a dendroid, and every locally connected dendroid is a dendrite (see [14], X, 2, Theorems 1 and 2, p. 306).

It is easy to observe, using the Menger n-spoke theorem ([14], VI, 1, pp. 213–214; [2], Theorem 13.20, p. 478; cf. [11], § 51, I, p. 277) that for locally connected continua the concept of a finite order of a point in the sense of ord introduced by Menger [13], § 2, p. 279; cf. [14], III, 1, p. 97) or by Urysohn ([15], p. 481; cf. [16]) (compare [11], § 51, I, p. 274) coincides with the concept of a finite order of a point in the classical sense. In particular, for locally connected continua, points of order two coincide with ordinary points in the sense defined above. Thus, for every dendrite X, the set $O(X)$ of all ordinary points of X is dense in X (see [14], X, 3, The first theorem, p. 309; cf. [11], § 51, VI, Theorem 8, p. 302). It is easy to show that for locally connected continua which do contain a simple closed curve it is not true. Moreover, there are locally connected continua, even one-dimensional, with a finite number of ordinary points (e.g. the triangular Sierpiński curve, see [11], § 51, I, 6, p. 276) or even containing no ordinary point (see e.g. [2], p. 476–478).

A problem has been asked in [4] as to whether the local connectedness is also an essential assumption is this statement: does there exist a dendroid X for which the set $O(X)$ is not dense in X? We give here an affirmative answer to this question constructing an example of a dendroid X for which $O(X)$ is empty. Moreover, the dendroid X has some additional properties, e.g. it is smooth. The construction is rather geometrical: it resembles one of brush continua ([3], p. 234; see also [7], § 9, p. 318) together with its application to obtain a dendroid Λ such that $R(\Lambda)$ is homeomorphic to Λ and that all points of Λ are of order at most 4 in the classical sense (see [3], pp. 245–251). However, some inverse limit techniques are used in the proof of properties of the present example. These techniques are similar to ones used by Anderson and Choquet in [1] and exploited by many authors in various constructions, in particular by the author in [5].

A dendroid X is said to be *smooth* if there exists a point $p \in X$ (called an *initial point* of X) such that for every convergent sequence of points a_n of X the condition $\lim_{n\to\infty} a_n = a$ implies that the sequence of arcs pa_n is convergent to the arc pa. The set of all points p which can be taken as initial points of a smooth dendroid X is called the *initial set* of X and is denoted by $I(X)$. A more detailed information on smooth dendroids is contained in [7].

Now we are able to formulate and prove the main result of the paper.

THEOREM 1. *There exists a dendroid X such that*

(1) *X has no ordinary point, i.e., $O(X) = \emptyset$;*

(2) *each ramification point of X is of order continuum;*

(3) *the set $E(X)$ of end points of X is an uncountable dense set;*

(4) *the set $R(X)$ of ramification points of X is an uncountable dense set;*

(5) *X is a subset of the Hilbert cube; each arc $A \subset X$ is the union of at most countably many straight line segments, and lengths $\lambda(A)$ of arcs A (with respect to the induced metric) are finite and bounded in common;*

(6) *X is smooth;*

(7) *the set $I(X)$ of initial points of X is a straight line segment;*

(8) *the set $I(X)$ is exactly the set of points at which X is locally connected.*

Proof. We begin with an auxiliary construction. Let $\mathscr{C}$ denote the Cantor ternary set of numbers, i.e.,

$$\mathscr{C} = \{x \in [0,1]:\ x = \sum_{k=0}^{\infty} 2c_k/3^k,\ \text{where } c_k \in \{0,1\}\}, \tag{9}$$

and let $\varphi\colon \mathscr{C} \to [0,1]$ be the well known step-function of $\mathscr{C}$ onto $[0,1]$, i.e., the function defined by

$$\varphi(x) = \sum_{k=0}^{\infty} c_k/2^k \tag{10}$$

for all x having the form mentioned in (9) (see [10], § 16, II, p. 150, footnote (1)).

The aim of the auxiliary construction is to define an operation Γ which, performed on a closed straight line segment S, gives as a result a dendroid $\Gamma(S)$ containing S. It will be done in two steps.

Step 1. Let a straight line segment S with end points a and b be given, and let P be a plane containing S. To make the construction easier, let us choose in P a cartesian rectangular coordinate system (x_1, x_2) in such a manner that the two end points of S have coordinates $(0,0)$ for a and $(1,0)$ for b. Consider a rectangle R with vertices $(0,0)$, $(1,0)$, $(1,\frac{1}{2})$, $(0,\frac{1}{2})$ for which S serves as a side. Note that $1:\frac{1}{2}$ is the ratio of lengths of the sides of R. We will call S the base of R.

Denote by C the Cantor ternary set of points lying in the side of R which is opposite to S:

$$C = \{(x_1, \tfrac{1}{2}):\ x_1 \in \mathscr{C}\},$$

and let $f\colon C \to S = \{(x_1, 0):\ 0 \leqslant x_1 \leqslant 1\}$ be a mapping defined by

$$f\big((x_1, \tfrac{1}{2})\big) = \big(\varphi(x_1), 0\big) \quad \text{for} \quad x_1 \in \mathscr{C}, \tag{11}$$

where φ is given by (10). Thus f is continuous and onto. Now join each point $p \in C$ with its image $f(p) \in S$ by a straight line segment A_p (lying in R and having $p \in C$ and $f(p) \in S$ as its end points), and observe that the union M of all these segments:

$$M = \bigcup \{A_p:\ p \in C\}$$

is a dendroid such that

$$E(M) = C \quad \text{and} \quad R(M) = S \setminus \{a, b\}$$

(cf. [3], Example E4, p. 240).

Let $T \subset R$ be the set of all interior and all boundary points of the triangle with vertices $a = (0,0)$, $b = (1,0)$ and $(\frac{1}{2}, \frac{1}{2})$, and put $D = M \cap T$. Thus D is a dendroid composed of straight line segments $G_p = A_p \cap T$ (called generators of D), where $p \in C$, the end points $f(p)$ of which form the segment S. We will call S the base of the dendroid D. Note that $E(D)$ is homeomorphic to C, in particular a and b are end points of D, and that $R(D) = S \setminus \{a, b\}$. Observe further that for every generator G_p of D its length is less than half of length of the base S of D:

(12) $$\lambda(G_p) < \tfrac{1}{2}\lambda(S).$$

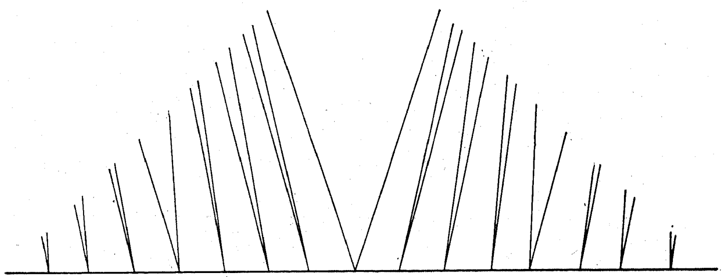
The above construction leads to the formula

(13) $$D = \bigcup \{G_p\colon p \in C\}.$$

It is evident directly from the construction of D and from the definition of smoothness that

(14) D is a smooth dendroid, and $I(D) = S$

(see the figure).



Step 2. Now take a straight line perpendicular to the plane P at the point $a = (0, 0)$ as the third axis, and let the unit 3-dimensional cube I^3 be given in the Euclidean 3-space equipped with the coordinate system (x_1, x_2, x_3) just defined:

$$I^3 = \{(x_1, x_2, x_3)\colon 0 \leqslant x_i \leqslant 1 \text{ for } i = 1, 2, 3\}.$$

Consider the "Cantor book" of rectangles lying in I^3, with sides of length 1 and $\frac{1}{2}$, and having the segment $S = \{(x_1, 0, 0)\colon 0 \leqslant x_1 \leqslant 1\}$ as the common side. More precisely, for each real number $q \in \mathscr{C}$ we take the plane P_q defined by the equation $x_3 = qx_2$ and, in the common part $P_q \cap I^3$ we distinguish a rectangle R_q with the unit segment S of the x_1-axis as its base and with the two other sides (of length $\frac{1}{2}$) lying in the faces $x_1 = 0$ and $x_1 = 1$ of I^3 respectively. Thus the plane P_q in which the rectangle R_q lies and the plane $x_3 = 0$ in which the base of I^3 is located form an angle of measure q. The union

(15) $$B = \bigcup \{R_q\colon q \in \mathscr{C}\}$$

is just the "Cantor book".

In each rectangle R_q we put an isometric copy D_q of the dendroid D in such a way that S is the common base for all D_q's, and define

(16) $$\Gamma(S) = \bigcup \{D_q\colon q \in \mathscr{C}\}.$$

For a fixed point $q \in \mathscr{C}$ let $G_{p,q}$ (where $p \in C$) denote a generator of D_q. Thus by (13) we have

(17) $$D_q = \bigcup \{G_{p,q}\colon p \in C\} \quad \text{for all } q \in \mathscr{C},$$

whence it follows from (16) that

(18) $$\Gamma(S) = \bigcup \{G_{p,q}\colon (p, q) \in C \times \mathscr{C}\}.$$

The following properties of $\Gamma(S)$ are immediate consequences of the above construction.

(19) $\Gamma(S)$ is a dendroid.

(20) The set $E(\Gamma(S))$ of end points of $\Gamma(S)$ is composed of end points a and b of S and of all end points of the generators $G_{p,q}$ which are not in S. Thus $E(\Gamma(S))$ is homeomorphic to the Cantor set.

(21) The set $R(\Gamma(S))$ of ramification points of $\Gamma(S)$ is equal to $S \setminus \{a, b\}$.

(22) Each ramification point r of $\Gamma(S)$ is of order continuum. Namely there exists a Cantor fan $\bigcup \{G_{p,q}\colon q \in \mathscr{C}\}$, (where $p \in f^{-1}(r)$ and $f\colon C \to S$ is defined by (11)) having the point r as its top.

It follows from (12) that

(23) $$\lambda(G_{p,q}) < \tfrac{1}{2}\lambda(S) \quad \text{for each } (p, q) \in C \times \mathscr{C}.$$

Further, as a consequence of (14), since the segment S is common for all dendroids D_q, we conclude by (16) that

(24) the dendroid $\Gamma(S)$ is smooth, and $I(\Gamma(S)) = S$,

and that

(25) $\Gamma(S)$ is locally connected exactly at points of S.

The auxiliary construction is finished.

Now we apply it to define an inductive procedure which will lead to the required dendroid. To do this, we need a new concept. An arc A contained in a dendroid X will be called *free* if (i) all its points except the end points are of order 2 in X, and (ii) it is a maximal one in the sense that it is not a proper subset of another arc satisfying (i).

Let a dendroid Y be given which lies in a k-dimensional cube I^k and which is represented as the union of some family of straight line segments. On each free straight line segment S in Y we perform the operation Γ in such a way that the generators $G_{p,q}$ of $\Gamma(S)$ (see formula (18)) are located in the cube I^{k+2} for which the cube I^k serves as its k-dimensional face. Thus each generator $G_{p,q}$ has exactly one of its end points in common with Y, and since any two distinct generators either are disjoint or have a point in common only, the resulting space, which is denoted by $\Delta(Y)$, is a dendroid. By definition we have

(26) $$\Delta(Y) = Y \cup \bigcup \{\Gamma(S)\colon S \text{ is a free straight line segment in } Y\}.$$

Let S be a fixed straight line segment of length 1. Put $X_1 = S$, $X_{i+1} = \Delta(X_i)$ for $i = 1, 2, \ldots$ and define

(27) $$X = \overline{\bigcup_{i=1}^{\infty} X_i}\,.$$

In details, we have

$$X_1 = S\,, \quad X_2 = \Gamma(S) = \bigcup \{G_{p,q}\colon (p, q) \in C \times \mathscr{C}\}\,.$$

Since $C \times \mathscr{C}$ is homeomorphic to $\mathscr{C}$, we can replace the double indexes by single ones; putting G_α^1 for $G_{p,q}$ we can write

$$X_2 = \bigcup \{G_\alpha^1\colon \alpha \in \mathscr{C}\}$$

and by construction (see (22)) we have

$$\lambda(G_\alpha^1) < \tfrac{1}{2} \quad \text{for each } \alpha \in \mathscr{C}\,.$$

Further we put

$$X_3 = \bigcup \{\Gamma(G_\alpha^1)\colon \alpha \in \mathscr{C}\} = \bigcup \{\bigcup \{G^1_{\alpha,(r,s)}\colon (r, s) \in C \times \mathscr{C}\}\colon \alpha \in \mathscr{C}\}\,.$$

Replacing the triple indexes α, (r, s) again by single ones and denoting $G^1_{\alpha,(r,s)}$ by G_β^2 for some $\beta \in \mathscr{C}$ we can write

$$X_3 = \bigcup \{G_\alpha^2\colon \alpha \in \mathscr{C}\}$$

and by construction we have $\lambda(G_\alpha^2) < (\frac{1}{2})^2$ for each $\alpha \in \mathscr{C}$.

In general we have

(28) $$X_{i+1} = \bigcup \{G_\alpha^i\colon \alpha \in \mathscr{C}\}$$

and

(29) $$\lambda(G_\alpha^i) < (\tfrac{1}{2})^i\,,$$

where $i = 1, 2, \ldots$ The construction is made in such a way that

$$X_1 \subset X_2 \subset X_3 \subset \ldots \subset X_i \subset \ldots$$

and $X_1 \subset I$, $X_2 \subset I^3$, $X_3 \subset I^5, \ldots, X_{i+1} \subset I^{2i+1}, \ldots$ and finally

$$X = \overline{\bigcup_{i=1}^{\infty} X_i} \subset I^{\aleph_0}\,.$$

The following properties of X_i are evident consequences of the construction above and of properties (19)–(25) of $\Gamma(S)$.

(30) X_{i+1} is a dendroid for each $i = 1, 2, \ldots$

Since end points of X_i remain end points of X_{i+1} after performing the operation Γ on each generator of X_i, we see that

(31) the set $E(X_{i+1})$ of end points of X_{i+1} is composed of all end points of X_i and of all end points of the generators G_α^i (see (28)) which are not in X_i.

The inclusion $X_i \subset X_{i+1}$ implies that each ramification point of X_i is a ramification point of X_{i+1}; observe also that each ordinary point of X_i, i.e., a point which lies in a generator G_α^{i-1} without the two end points of G_α^{i-1}, becomes the top of a Cantor fan of generators G_β^i in the next step of the construction. Thus

(32) the set $R(X_{i+1})$ of ramification points of X_{i+1} is equal to $X_i \setminus E(X_i)$,

and

(33) each ramification point of X_{i+1} is of order continuum.

Since X_{i+1} is created from X_i performing the operation Δ on it, i.e., performing Γ on each free straight line segment of X_i, one can easily verify by (24) and (25) using an inductive procedure that

(34) X_{i+1} is smooth for each $i = 1, 2, \ldots$, and $I(X_{i+1}) = S$,

and

(35) if X_i is not locally connected at a point, then X_{i+1} also is not locally connected at this point.

To see that the limit space X defined by (27) is a smooth dendroid we need another description of X, namely as the inverse limit of an inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$ of smooth dendroids X_i with monotone bonding mappings f_i.

To this end we define $f_i\colon X_{i+1} \to X_i$ as monotone retraction which shrinks each generator G_α^i of X_{i+1} (see (28)) to its end point in X_i. In other words, for each point $x \in X_{i+1}$ we take a generator G_α^i of X_{i+1} to which x belongs and we define $f_i(x)$ as the only point of the intersection $G_\alpha^i \cap X_i$. Thus for every point $y \in X_i$ the inverse image $f_i^{-1}(y)$ is just the Cantor fan composed of generators G_α^i having the point y in common, i.e., such that y is the top of this Cantor fan. So f_i is continuous and monotone retraction of X_{i+1} onto X_i. It is evident from the definitions that $\{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence. Observe that condition (29) implies

$$\operatorname{diam} f_i^{-1}(y) < 2 \cdot (\tfrac{1}{2})^i$$

for each point $y \in X_i$, where $i = 1, 2, \ldots$ Hence Theorem 1 of [1], p. 348 can be applied, and thereby

$$\underleftarrow{\lim}\{X_i, f_i\}_{i=1}^{\infty} = \overline{\bigcup_{i=1}^{\infty} X_i}\,,$$

whence

(36) $$X = \underleftarrow{\lim}\, \{X_i, f_i\}_{i=1}^{\infty}$$

by (27). Since the inverse limit of an inverse sequence of smooth dendroids with monotone bonding mappings is a smooth dendroid provided there exists a thread composed of initial points (see [6], Corollary 2), and since the mentioned condition is satisfied by (34), we conclude that X is a smooth dendroid. The equality

(37) $$I(X) = S$$

is evident from the construction. Since the limit space is a dendroid, it can also be easily seen from (27) that each point of the remainder $X\setminus\bigcup_{i=1}^{\infty} X_i$ is an end point of X. Thus the set of end points is composed of all points of the remainder and of end points of all X_i's for $i = 1, 2, \ldots$, i.e.,

(38) $$E(X) = \bigcup_{i=1}^{\infty} E(X_i) \cup (X\setminus\bigcup_{i=1}^{\infty} X_i).$$

The other points of X are points of $X_i\setminus E(X_i)$ for some $i = 1, 2, \ldots$ which are ramification points of X_{i+1} by (32), whence we conclude that

(39) $$R(X) = \bigcup_{i=1}^{\infty} R(X_{i+1}) = \bigcup_{i=1}^{\infty} [X_i\setminus E(X_i)],$$

and that no point of X is an ordinary point. Thus (1) is true. (2) is a consequence of (39) and (33). Conditions (3) and (4) are readily seen from the construction. To verify (5) it is enough to note that for every arc $A\subset X$ its length $\lambda(A)$ does not exceed the sum $\lambda(S)+\sum_{i=1}^{\infty}\lambda(G^i_{\alpha_i})<1+\sum_{i=1}^{\infty}(\frac{1}{2})^i = 2$ (see (29)). Conditions (6) and (7) have been already proved (see (37)). And finally (8) is an immediate consequence of the inductive procedure, see (35). Thus the proof of the theorem is complete.

It is known that there exists a universal smooth dendroid, i.e., a smooth dendroid U such that every smooth dendroid can be homeomorphically embedded into U (see [8]). We have the following

PROBLEM 1. Is the dendroid X constructed in the theorem a universal one?

PROBLEM 2. Is it true that if the dendroid X is composed of end points and of ramification points only, and if each ramification point of X is of order continuum, then X is universal?

If we replace — in the construction of X described in the proof of Theorem 1 — the "Cantor book" (see (15)) by a book composed of $n = 1, 2, \ldots, \aleph_0$ sheets, we get a dendroid $X^{(n)}$ (instead of X) having all properties (1)–(8) of Theorem 1 except property (2) which is changed into

(40) each ramification point of $X^{(n)}$ is of order a) either $n+2$ or $2n+2$ if n is finite; b) $\aleph_0$ if $n = \aleph_0$.

Really, let $\mathscr{C}^{(n)}$ be a closed subset of $\mathscr{C}$ of cardinality n. Taking

$$B^{(n)} = \bigcup\{R_q: q\in\mathscr{C}^{(n)}\}$$

instead of (15) and defining (see (16))

$$\Gamma^{(n)}(S) = \bigcup\{D_q: q\in\mathscr{C}^{(n)}\}$$

we get a dendroid (see (19)) $\Gamma^{(n)}(S)$ which can also be written in the form (see (18))

$$\Gamma^{(n)}(S) = \bigcup\{G_{p,q}: (p,q)\in C\times\mathscr{C}^{(n)}\},$$

and which has properties (20), (21), (23), (24), (25) with $\Gamma^{(n)}$ and $\mathscr{C}^{(n)}$ in place of Γ and $\mathscr{C}$ respectively, and for which the following holds instead of (22):

(41) Each ramification point of $\Gamma^{(n)}(S)$ is of order a) either $n+2$ or $2n+2$ if n is finite; b) $\aleph_0$ if $n = \aleph_0$.

In fact, if $r\in R(\Gamma^{(n)}(S)) = S\setminus\{a, b\}$, then $f^{-1}(r)$ is either a singleton or a two-point set, so r is the common end point of either one or two generators $G_{p,q}$ for a fixed $q\in\mathscr{C}^{(n)}$ (i.e. generators of D_q, see the figure). Since $\Gamma^{(n)}(S)$ is composed of n copies of D_q having the segment S is common, we have either n or $2n$ generators $G_{p,q}$ ending at r. Taking into account the two subsegments of S which have r as their common end point one can see that r is of order $n+2$ or $2n+2$ indeed. For $n = \aleph_0$ one can find a countable fan in $\Gamma_n(S)$ with r as its top. So (41) is shown.

The further part of the construction runs without essential changes. In particular, putting $X_1^{(n)} = S$ and defining $X_{i+1}^{(n)}$ as the dendroid obtained from $X_i^{(n)}$ by performing the operation $\Gamma^{(n)}$ on each free segment of $X_i^{(n)}$ we get an increasing sequence of dendroids. The resulting space $X^{(n)}$ is defined by

$$X^{(n)} = \mathrm{cl}(\bigcup\{X_i^{(n)}: i = 1, 2, \ldots\})$$

where cl denotes the closure. The inverse limit procedure shows as previously that $X^{(n)}$ is a smooth dendroid with required properties. Therefore we have the following

THEOREM 2. *For every number* $n = 1, 2, \ldots, \aleph_0$ *there exists a dendroid* $X^{(n)}$ *with properties* (1), (3)–(8) *of Theorem* 1 *and with property* (40).

Added in proof. Recently J. Nikiel answered Problems 1 and 2 in the negative.

References

[1] R. D. Anderson and G. Choquet, *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: an application of inverse limits*, Proc. Amer. Math. Soc. 10 (1959), pp. 347–353.
[2] L. M. Blumenthal and K. Menger, *Studies in Geometry*, W. H. Freeman and Co. 1970.
[3] J. J. Charatonik, *On ramification points in the classical sense*, Fund. Math. 51 (1962), pp. 229–252.
[4] — *A problem on ordinary points in a dendroid*, Topology and Appl. 3, Beograd 1977 (to appear).
[5] — *Open mappings of universal dendrites*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 28 (1980), pp. 489–494.
[6] — and W. J. Charatonik, *Inverse limits and smoothness of continua*, Acta Math. Hungar. 43 (1984) (to appear).
[7] — and C. Eberhart, *On smooth dendroids*, Fund. Math. 67 (1970), pp. 297–322.
[8] J. Grispolakis and E. D. Tymchatyn, *A universal smooth dendroid*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978), pp. 991–998.
[9] Z. Janiszewski, *Sur les continus irréductibles entre deux points*, These, Gauthier-Villars, Paris 1911.

[10] K. Kuratowski, *Topology*, vol. I, New York–London–Warszawa 1966.
[11] — *Topology*, vol. II, New York–London–Warszawa 1968.
[12] A. Lelek, *On plane dendroids and their end points in the classical sense*, Fund. Math. 49 (1961), pp. 301–319.
[13] K. Menger, *Grundzüge einer Theorie der Kurven*, Math. Ann. 95 (1926), pp. 277–306.
[14] — *Kurventheorie*, Chelsea Publ. Co. 1967.
[15] P. Urysohn, *Sur la ramifications des lignes cantoriennes*, C. R. Acad. Sci. Paris 175 (1922), pp. 481–484.
[16] — *Mémoire sur la multiplicités cantoriennes II*, Verhandelingen der Kon. Akademie van Wetenschappen, Amsterdam, 1 sectie, 13, No. 4 (1928), pp. 1–172.
[17] W. H. Young, *The Theory of Sets of Points*, University Press, Cambridge 1906.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW
pl. Grunwaldzki 2/4
50-384 Wrocław

Received 16 March 1981

A note on Robinson's non-negativity criterion

by

D. W. Dubois (Albuquerque, NM) and **T. Recio** (Santander)

Abstract. Let V be a real algebraic variety in $\boldsymbol{R}^n$ and $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ a system of polynomial inequalities over V, such that its solution set is Zariski dense on V. Then we prove that Robinson's criterion gives necessary and sufficient conditions for $f \geqslant 0$ to be a consequence of the given system if and only if $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ is *locally slack* on V in the sense of Stengle [S]. As a corollary we quickly obtain several recent results on positive function over the set V_c of central points of V, the image of V_c for finite birational morphisms and Efroymson's [E1] characterization of central points.

1. Introduction. Robinson's non-negativity criterion states that if a polynomial inequality $f \geqslant 0$ is a consequence of a finite system of polynomial inequalities $f_1 \geqslant 0, \ldots, f_r \geqslant 0$, then $f = \sum \varrho_\lambda^2 h_\lambda$, where the ϱ_λ are rational functions and the h_λ are products of some f_i (cf. [Ro]). Stengle [S] extended the result by showing that on a real variety V, if $f \geqslant 0$ is a consequence of $f_1 \geqslant 0, \ldots, f_r \geqslant 0$ then $a^2 f \equiv \sum \varrho_\lambda^2 h_\lambda$ $(\mathrm{mod}\, \mathscr{I}(V))$, where ϱ_λ and h_λ are as above and a is outside $\mathscr{I}(V)$. Motzkin [M] suggested that Robinson's criterion could be interpreted as the counter part for inequalities of the Hilbert Nullstellensatz; but as Stengle remarked, this comparison is inappropriate in that there exists in some cases a polynomial f which is expressible in the form $\sum \varrho_\lambda^2 h_\lambda$ but which is not positive definite over the closed semi-algebraic set $f_1 \geqslant 0, \ldots, f_r \geqslant 0$. The need for a study of this situation is noted in several recent papers, e.g. Stengle (loc. cit.), Gondard [G], Lorenz [L], where some counter-examples are also given to the sufficiency of the criterion. It is the purpose of this note to clarify the problem and then to apply our results to very recent theorems of Schwartz [Sch], Bochnak–Efroymson [B–E] and Adkins [A]. Our treatment is related to work of Dubois [D] and Brumfiel [B].

2. Locally slack systems on a semi-algebraic set. Locally slack systems of inequalities $\{f_1 \geqslant 0, \ldots, f_r \geqslant 0\}$ were first introduced by Stengle [loc. cit.] as a type of system of use in obtaining sufficiency for Robinson's criterion. However he was not able (see p. 96 [S]) to give a direct description of these systems in terms of the $f_1, \ldots, f_r$ (see 2.3 below). We are going to generalize his definition.