On expandability of models of arithmetic and set theory to models of weak second-order theories

by

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Abstract. In a previous paper we showed how to construct ω_1-like recursively saturated models of PA and ZF, in which all classes are definable. In this paper we strengthen this result slightly, and use it to answer two questions of Bell, Marek and Srebrny. That is to say, we show that there is no class of $L_{\infty\omega}$ sentences which defines the class of $\{\in\}$-reduct's of models of Kelley–Morse class theory (or even Σ_1^1-comprehension), and similarly for Σ_1^1 Peano Arithmetic. Both theorems remain true when one extends the language L to the language of models $\mathrm{HYP}_{\mathfrak{M}}$. This contrasts with the case of countable models, for which the Barwise Completeness Theorem applies.

In this article we answer negatively questions 2 and 3 of Marek and Srebrny [MS]. This is an application of a theorem from Kaufmann [Ka], for which Shelah [She] eliminated our hypothesis of $\diamondsuit_{\omega_1}$.

We will begin by summarizing some notation. Then we will recall the pertinent results of [MS] and generalize them in Theorem 1. Our main theorem is Theorem 2, which follows from [Ka] and [She]. We use this to prove Corollaries 2A and 2B, which answer negatively questions 2 and 3 of [MS].

Notation. PA is Peano Arithmetic. KM is Kelley–Morse set theory with classes; see for example Marek and Mostowski [MM] for a number of results on KM. Actually, in [MS] the authors consider KM to include the class choice schema. However, their Theorem 5.1 (see below), as well as the negative results that we present here, are true for either version of KM.

If $\mathfrak{M}$ is a model of KM, then $X \subseteq M$ is a *class* of $\mathfrak{M}$ if for all $m \in M$, $\{a \in X\colon \mathfrak{M} \models a \in m\}$ is definable in $\mathfrak{M}$ (with parameters); similarly for $\mathfrak{M} \models \mathrm{PA}$, where $<$ replaces $\in$ above. In particular, if $(\mathscr{F}, \mathfrak{M}) \models \mathrm{GB}$ (Gödel–Bernays set theory), then every $X \in \mathscr{F}$ is a class of $\mathfrak{M}$. For other notation see Barwise [B].

In [MS] one finds the following related theorems, all involving expandability of *countable* models to models of second-order theories.

THEOREM 3.3 [MS]. *There exists a single finitary sentence Φ such that for countable models $\mathfrak{M} \models P$ we have: $\mathfrak{M}$ is extendable to a model of* ZF *iff* $\mathrm{HYP}_{\mathfrak{M}} \models \Phi$.

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THEOREM 4.2. *Let $0<n\leqslant\omega$. There is a single finitary sentence Φ_n such that for countable models $\mathfrak{M}\models P$ we have: $\mathfrak{M}$ is expandable to a model of Δ^1_n-comprehension + (the class form of induction) iff $\mathrm{HYP}_{\mathfrak{M}}\models\Phi_n$.*

THEOREM 5.1. *There exists a single finitary sentence Φ such that for countable models $\mathfrak{M}\models\mathrm{ZFC}$, $\mathfrak{M}$ is KM-expandable if $\mathrm{HYP}_{\mathfrak{M}}\models\Phi$.*

In fact, these results all follow from the following general "soft" result. It has basically the same proof as the proofs given in [MS] of the results stated above. The reader may prefer to think of the case $S\subseteq T$, and of $\mathfrak{N}^{-I}$ as $\mathfrak{N}\restriction$ (language of S).

THEOREM 1. *Let S and T be recursively enumerable theories, and suppose I is an interpretation of S in T. Then there exists a finitary sentence Φ (in fact it is Π_1) such that for all countable $\mathfrak{M}\models S$:*

$$\mathfrak{M}=\mathfrak{N}^{-I}\quad \textit{for some } \mathfrak{N}\models T$$

iff $\mathrm{HYP}_{\mathfrak{M}}\models\Phi$.

Proof. Φ says that the following infinitary theory of $\mathrm{HYP}_{\mathfrak{M}}$ is consistent. One has a constant symbol $\overline{m}$ for each $m\in M$. The theory contains:

$$T;$$
$$\{I(\varphi)(\overline{m}_1,\ldots,\overline{m}_k):\ \mathfrak{M}\models\varphi[m_1,\ldots,m_k]\};$$
$$\{\forall x[(x=x)^I\leftrightarrow\bigvee_{m\in M}x=\overline{m}]\}.$$

By Barwise Completeness, such a Π_1 sentence Φ does indeed exist. ■

Notice that this answers a small part of question 1 from [MS]: that is, their Theorem 5.1 does hold for KM even without class choice.

Now question 3 (of [MS]) asks if the conclusions of Theorems 3.3, 4.2, and 5.1 (stated above) also hold for uncountable models. Question 2, which is attributed to J. Bell, asks if there is an infinitary sentence characterizing KM-expandability. We answer all of these questions negatively by proving Corollaries 2A and 2B below, which follow from the following theorem. We will prove this theorem after proving the corollaries. But first, we review some notation.

Notation. Σ^1_1-P is Peano arithmetic, with the class form of induction and the Σ^1_1-comprehension axiom. For any language L, L^* is the language of admissible sets $\mathscr{A}_{\mathfrak{M}}$, where $\mathfrak{M}$ is an L-structure. Finally, a model $\mathfrak{M}$ of ZF is ω_1-*like* if $\mathfrak{M}$ is uncountable but for all $a\in M$, $\{b\in M:\ \mathfrak{M}\models b\in a\}$ is countable.

THEOREM 2. *Let $\mathfrak{M}$ be any countable recursively saturated model of P or ZF. Then there exists a model $\mathfrak{N}$ with the following properties:*

(i) $\mathfrak{M}\prec_{\infty\omega}\mathfrak{N}$.

(ii) *$\mathfrak{N}$ is an ω_1-like end extension of $\mathfrak{M}$.*

(iii) *$\mathfrak{N}$ is recursively saturated.*

(iv) *Every class of $\mathfrak{N}$ is first-order definable with parameters in $\mathfrak{N}$.*

(v) $\mathrm{HYP}_{\mathfrak{M}}\prec_{\infty\omega}\mathrm{HYP}_{\mathfrak{N}}$.

COROLLARY 2A. *Let T be any consistent extension of Σ^1_1-P, or more generally, any consistent theory which interprets Σ^1_1-P. Then there is no class Φ of $L_{\infty\omega}$ sentences such that for all $\mathfrak{M}\models P$, $\mathfrak{M}$ is expandable to a model of T iff $\mathfrak{M}\models\Phi$. In fact, there is no class Φ of sentences of $L^*_{\omega\omega}$ such that for all $\mathfrak{M}\models P$, $\mathfrak{M}$ is expandable to a model of T iff $\mathrm{HYP}_{\mathfrak{M}}\models\Phi$.*

Proof. The former conclusion follows immediately from the latter, so we prove the latter. Let $\mathfrak{M}$ be any countable recursively saturated model of P, which is expandable to a model of T. Choose $\mathfrak{N}$ satisfying the conditions (i) through (v) of Theorem 2. Suppose Φ has the property that we are trying to refute. Then $\mathfrak{M}\models\Phi$, so since $\mathfrak{M}\prec_{\infty\omega}\mathfrak{N}$ by (v), $\mathfrak{N}\models\Phi$. A contradiction arises if we can show that $\mathfrak{N}$ is not expandable to a model of Σ^1_1-P.

To that end, suppose $\mathfrak{N}^*\models\Sigma^1_1$-$P$, where $\mathfrak{N}=\mathfrak{N}^*\restriction L$. Now it is well known that there exists a Σ^1_1 formula $\mathrm{SAT}(x_0,x_1,x_2)$ (with no parameters) having the following property:

For all formulas $\varphi(x_1,x_2)$ of L, if $\ulcorner\varphi\urcorner$ is the Gödel number of φ then

$$\Sigma^1_1\text{-}P\vdash\forall x_1\forall x_2[\varphi(x_1,x_2)\leftrightarrow\mathrm{SAT}(\ulcorner\varphi\urcorner,x_1,x_2)].$$

Using Σ^1_1-comprehension we may then form in $\mathfrak{N}^*$ the following class:

$$X=\{x:\ \neg\exists y\exists z[x=\langle y,z\rangle\wedge\mathrm{SAT}(y,x,z)]\}.$$

(Here, $\langle\cdot,\cdot\rangle$ is a canonical one-one pairing function in P.) Since X is a class of $\mathfrak{N}$, then by property (iv), we may choose a formula φ of L and a sequence a of parameters from $\mathfrak{N}$ such that

$$X=\{x\in N:\ \mathfrak{N}\models\varphi(x,a)\}.$$

We may assume that a is a single element, using the canonical pairing function. Now in $\mathfrak{N}$ let $b=\langle\ulcorner\varphi\urcorner,a\rangle$. Then

$b\in X$ iff

$\mathfrak{N}^*\models\neg\mathrm{SAT}(\ulcorner\varphi\urcorner,b,a)$ (by choice of b and X) iff

$\mathfrak{N}^*\models\neg\varphi(b,a)$ (by choice of SAT), iff

$b\notin X$ (by choice of φ),

and this is a contradiction. ■

The same proof shows:

COROLLARY 2B. *If we replace P by ZF and Σ^1_1-P by Gödel–Bernays set theory + Σ^1_1-comprehension, then Corollary 2A holds.* ■

We turn now to the task of proving Theorem 2. In [Ka] we showed that assuming $\diamondsuit_{\omega_1}$ this theorem holds, with (v) deleted and (i) replaced by $\mathfrak{M}\prec\mathfrak{N}$. Let us derive Theorem 2 from that result.

LEMMA 1. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be recursively saturated structures for a countable language, $\mathfrak{M} \subseteq \mathfrak{N}$. Then $\mathfrak{M} \prec_{\infty\omega} \mathfrak{N}$ iff* $\mathrm{HYP}_{\mathfrak{M}} \prec_{\infty\omega} \mathrm{HYP}_{\mathfrak{N}}$.

Proof. ($\Leftarrow$) is clear. For ($\Rightarrow$), first suppose $\mathfrak{M}$ and $\mathfrak{N}$ are countable. Then for all finite $\vec{m}$ from M, $(\mathfrak{M}, \vec{m}) \equiv_{\infty\omega} (\mathfrak{N}, \vec{m})$, so $(\mathfrak{M}, \vec{m}) \cong (\mathfrak{N}, \vec{m})$; therefore $(\mathrm{HYP}_{\mathfrak{M}}, \vec{m}) \cong (\mathrm{HYP}_{\mathfrak{N}}, \vec{m})$. Since every element of $\mathrm{HYP}_{\mathfrak{M}}$ is definable with parameters in $\mathfrak{M}$, this shows $\mathrm{HYP}_{\mathfrak{M}} \prec_{\infty\omega} \mathrm{HYP}_{\mathfrak{N}}$. The countability assumption is eliminated by Lévy absoluteness: see for example Barwise [B, II9.2]. ■

LEMMA 2. *Let* (Thm. 2)′ *denote Theorem* 2 *with* (v) *deleted and* (i) *replaced by*

(i′) $\mathfrak{M} \prec \mathfrak{N}$.

Then ZFC ⊢ (*Theorem* 2) ↔ (*Thm.* 2)′.

Proof. (→) is clear. For (←), first notice that (v) is superfluous, by Lemma 1. Since recursively saturated models are homogeneous (Schlipf [Sch, II1.8 (i)]), it suffices to show that $\mathfrak{M}$ and $\mathfrak{N}$ realize the same types. But this is well known; we thank Jim Schmerl for pointing this out, as it simplifies the proof of Theorem 2. To prove this, first choose $m \in M$ greater than all elements definable in $\mathfrak{M}$. Then given $\vec{n}$ from $\mathfrak{N}$, we find $\vec{a}$ in M such that $\vec{a}$ and $\vec{n}$ realize the same type. Just choose $\vec{a}$ in $\mathfrak{N}$ to realize the type $\{\varphi(\vec{x}) \leftrightarrow \varphi(\vec{n})\colon \varphi \in L\} \cup \{x_i < m\colon x_i \text{ occurs in } \vec{x}\}$. ■

Proof of Theorem 2. Actually, (Thm. 2)′ is just the theorem of [Ka]. Now $\diamondsuit_{\omega_1}$ is assumed for that result, but Shelah [She] has shown how to eliminate this added hypothesis. So by Lemma 2, we are done. ■

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Smooth dendroids without ordinary points

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Abstract. Smooth dendroids are constructed which are composed of end points and of ramification points only.

Let X be a metric continuum. If for every two points a and b of X there exists in X an arc (i.e. a continuous and one-to-one image of the closed unit interval $[0, 1]$ of reals) with end points a and b, then X is said to be *arcwise connected*. For an arcwise connected continuum X we accept the following three definitions. A point p of X is called an *end point* of X if p is an end point of every arc containing p and contained in X. A point p of X is called an *ordinary point* of X if there are in X exactly two arcs with p as the common end point and which are disjoint out of p. A point p of X is called a *ramification point* of X if there are in X three (or more) arcs with p as the common end point and which are disjoint out of p. In other words end points, ordinary points and ramification points of X are exactly points of order 1, 2 and $n \geqslant 3$ in the classical sense respectively (see [17], pp. 219–221; [9], Chapter IV, I, pp. 63–64; compare [12], pp. 301–302 and [3], pp. 229–230). Thus, given an arcwise connected continuum X, we can distinguish three disjoint sets of its points: the set $E(X)$ of end points of X, the set $O(X)$ of ordinary points of X, and the set $R(X)$ of ramification points of X, and we have

$$X = E(X) \cup O(X) \cup R(X).$$

It is easy to construct some particular examples of arcwise connected continua X with the property that some of these sets are empty.

A continuum is called *hereditarily unicoherent* if the intersection of any two its subcontinua is connected. A *dendroid* means an arcwise connected and hereditarily unicoherent metric continuum. A *dendrite* means a locally connected metric continuum that contains no simple closed curve. The concept of a dendroid is a generalization of one of a dendrite: every dendrite is a dendroid, and every locally connected dendroid is a dendrite (see [14], X, 2, Theorems 1 and 2, p. 306).

It is easy to observe, using the Menger n-spoke theorem ([14], VI, 1, pp. 213–214; [2], Theorem 13.20, p. 478; cf. [11], § 51, I, p. 277) that for locally connected con-