On embedding curves in two-dimensional polyhedra

by

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Abstract. A 3-book is a union of three discs with a common segment lying on their boundary. It is proved that every one-dimensional compact metric space not locally flat at points composing at most a zero-dimensional set can be embedded in a 3-book. In particular, for every curve there exists a curve of the same shape in a 3-book.

Introduction. It is well known that curves (one-dimensional continua) can be embedded in the 3-dimensional Euclidean space. Some curves cannot be embedded in 2-dimensional polyhedra. For instance the Menger universal curve has this property, since none of point of it has a neighbourhood which is flat (can be embedded in a plane). Some curves which are not flat can be embedded in 2-dimensional polyhedra (see [1], p. 45 and [2], p. 121). An n-book (compare [1], p. 43) is the union of n closed discs such that the intersection of those discs is a segment (called an edge) lying on their boundary and no two of them have any other common points. R. M. Bing [2] has noticed that a solenoid which is not flat can be embedded in a 3-book; a solenoid can be obtained as an intersection of a decreasing sequence of Möbius bands with added discs, everything lying in a 3-book.

In shape theory it is known ([6] or [3], p. 354) that for every curve X there exists a plane curve Y of the same shape as X if and only if X is movable. The results of this paper are connected with the question (due to J. Krasinkiewicz [4]) what shapes are embeddable in an n-book, $n \geqslant 3$.

We say that a family $\boldsymbol{U}$ of subsets of a space X is *isomorphic* to a family $\boldsymbol{V}$ of subsets of a space Y iff there is a bijection $f\colon \boldsymbol{U} \to \boldsymbol{V}$ such that for every subfamily $\boldsymbol{U}_0$ of $\boldsymbol{U}$ we have $\bigcap \boldsymbol{U}_0 = \emptyset$ if and only if $\bigcap f(\boldsymbol{U}_0) = \emptyset$.

§ 1. Preliminary construction. For every positive integer n we define the following sets:

$$I_n = \{(x, y, z) \in R^3 |\ x = 1/n,\ 0 \leqslant y \leqslant 1/n,\ z = 0\},$$

$$D_n = \{(x, y, z) \in R^3 |\ x = 1/n,\ 0 \leqslant y,\ z \leqslant 1/n\}.$$

Let $V = \{(x, y, z) \in R^3 |\ 0 \leqslant y \leqslant x \leqslant 1,\ z = 0\}$. Then the set $W = V \cup \bigcup_{n=2}^{\infty} D_n$ is homeomorphic to a subset of the 3-book,

$T = \{(x,y,z)|\ 0 \leqslant x \leqslant 1 \text{ and } ((y=0 \text{ and } 0 \leqslant z \leqslant 1) \text{ or } (z=0 \text{ and } -1 \leqslant y \leqslant 1))\}$.

For example W is homeomorphic to the subset $T_0 \subset T$ defined by

$$T_0 = \{(x,y,z)|\ x\cdot\sin(1/x) - x/2 \leqslant y \leqslant x\cdot\sin(1/x) - x/2 \text{ and } 0 < x \leqslant 1 \text{ and } (z=0 \text{ or } (y=0 \text{ and } 0 \leqslant z \leqslant x))\} \cup \{(0,0,0)\}$$

(see Fig. 1).

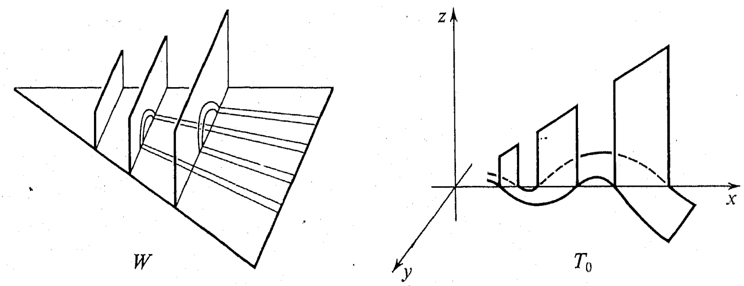

Fig. 1

Let us prove the following

(1.1) Lemma. *Let $\alpha = \{\alpha_i|\ i = 1, 2, \ldots, n\}$ be a family of mutually disjoint segments in I_1. Then there is a family $\{D_{i,j}|\ 1 \leqslant i < j \leqslant n\}$ of mutually disjoint discs in W such that $D_{i,j} \cap I_1$ is the sum of two segments, one of them contained in α_i and the other in α_j.*

Proof. Let f be an injection which maps the set $\{(i,j)|\ 1 \leqslant i < j \leqslant n\}$ in the set of positive integers. Then every disc $D_{i,j}$ can be defined as a union of three discs, two of which are contained in V between I_1 and $I_{f(i,j)}$ and the third in $D_{f(i,j)}$, (see Fig. 1).

Let Y be the union of two discs D and D_0 such that $D \cap D_0 = L$ is an arc lying in $\mathring{D} \cap \dot{D}_0$.

(1.2) Corollary. *Let $\{\alpha_{i,j}|\ i = 1, 2, \ldots, k,\ j = 0, 1, 2, \ldots, j(i)\}$ be a family of mutually disjoint segments contained in L. Let U be an open subset of Y which intersects L and is disjoint with $\bigcup \alpha_{i,j}$. Then there exists a family*

$$\{D_{i,j}|\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, j(i)\}$$

of mutually disjoint discs in Y such that

(1.3) $$D_{i,j} \subset D_0 \cup U,$$

(1.4) *one of the components of $D_{i,j} \cap L$ is the segment $\alpha_{i,j}$ contained in $\dot{D}_{i,j}$,*

(1.5) *$D_{i,j} \cap \alpha_{i,0}$ is a segment contained in $\dot{D}_{i,j} \cap \mathring{\alpha}_{i,0}$.*

(1.6) Corollary. *Let $\boldsymbol{A}$ be a finite family of sets with $\operatorname{ord}\boldsymbol{A} \leqslant 1$. Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be disjoint subfamilies of $\boldsymbol{A}$ such that $\operatorname{ord}\boldsymbol{B} \leqslant 0$. Then for any family $\{\alpha_A\}_{A \in \boldsymbol{C}}$ of mutually disjoint arcs which lie on the boundary $\dot{D}$ and for any positive ε there is a family $\boldsymbol{D} = \{D(A)\}_{A \in \boldsymbol{A}}$ of discs contained in Y such that*

(1.7) *$\boldsymbol{A}$ and $\boldsymbol{D}$ are isomorphic,*

(1.8) *if $A' \in \boldsymbol{A}$ intersects $A'' \in \boldsymbol{A}$ ($A' \neq A''$) then $D(A') \cap D(A'')$ is an arc which lies in $L \cap \dot{D}(A') \cap \dot{D}(A'')$,*

(1.9) *$D(A) \cap \dot{D} = \alpha_A$ for $A \in \boldsymbol{C}$,*

(1.10) *if $A \in \boldsymbol{B}$ then $\delta(D(A)) < \varepsilon$, $D(A) \subset \mathring{D}$ and L intersects the interior $\mathring{D}(A$ of $D(A)$.*

It is easy to prove the following

(1.11) Lemma. *Let L_0 be an arc in the interior $\mathring{D}_0$ of D_0 and let K be an arc on the boundary $\dot{D}$ of D. Then there is an embedding $h\colon D \to Y$ such that $h(L) \subset L_0$ and $h(x) = x$ for $x \in K$.*

Let D_0, D_1, D_2 be discs such that their union is the 3-book T, and let E be the edge of T ($D_0 \cap D_1 \cap D_2 = E$).

(1.12) Lemma. *Let X_1 and X_2 be 1-dimensional compacta such that $\dim(X_1 \cap X_2) = 0$. If $h_i\colon X_i \to D_1 \cup D_2$ ($i = 1, 2$) are embeddings such that*

$$h_1(X_1) \cap h_2(X_2) = \emptyset \quad \text{and} \quad h_i(X_1 \cap X_2) \subset \mathring{E} \text{ for } i = 1, 2,$$

then for any neighbourhood U of $X_1 \cap X_2$ in $X_1 \cup X_2$ there is an embedding $h\colon X_1 \cup X_2 \to T$ such that $h(x) = h_i(x)$ for $x \in X_i \setminus U$.

Proof. For every positive integer n, let $\boldsymbol{V}_n = \{V_{n,j}|\ j = 1, \ldots, j(n)\}$ be a closed-open covering of $X_1 \cap X_2$ and $\boldsymbol{V}_n' = \{V_{n,j}'|\ j = 1, \ldots, j(n)\}$ be a family of open subsets of $X_1 \cup X_2$ such that $j(1) = 1$, $V_{n,j} \subset V_{n,j}'$, $\boldsymbol{V}_{n+1}' \succ \boldsymbol{V}_n'$, $\operatorname{ord}\boldsymbol{V}_n' = 0$, $\boldsymbol{V}_{n+1} \succ \boldsymbol{V}_n$ and $\delta(V_{n,j}') < 1/n$ for $n \geqslant 2$. By induction we will define a sequences of embeddings $h_{i,n}\colon X_i \to T$ (for $i = 1, 2$) and a sequence $\boldsymbol{U}_n = \{U_{n,j}|\ j = 1, \ldots, j(n)\}$, of families of sets contained in T such that

(a) $\boldsymbol{U}_n$ and $\boldsymbol{V}_n$ are isomorphics and $\boldsymbol{U}_{n+1} \succ \boldsymbol{U}_n$,

(b) $h_i^{-1}(U_{1,1}) \subset U$ and $\delta(U_{n,j}) < 1/n$ for $n \geqslant 2$,

(c) $U_{n,j} = U_{n,j}^0 \cup U_{n,j}^1 \cup U_{n,j}^2$, where $U_{n,j}^k$ are the discs in D_k and $U_{n,j} \cap E = U_{n,j}^k \cap E$ is a segment, which we denote by $\alpha_{n,j}$ ($k = 0, 1, 2$),

(d) $h_{i,1} = h_i$ for $i = 1, 2$,

(e) $h_{1,n}(X_1) \cap h_{2,n}(X_2) = \emptyset$,

(f) $h_{i,n}(V_{n,j}) \subset \mathring{\alpha}_{n,j}$,

(g) for every n there exists $k \in \{0, 1, 2\}$ such that $h_{i,n}(X_i) \cap U_{n,j} \cap D_k \subset E$,

(h) $h_{i,n+1}(V'_{n,j}) \subset U_{n,j}$ and $h_{i,n+1}|(X \setminus \bigcup_{j=1}^{j(n)} V'_{n,j}) = h_{i,n}|(X \setminus \bigcup_{j=1}^{j(n)} V'_{n,j})$.

Suppose that we have defined the embeddings $h_{i,m}$ $(i = 1, 2)$ and families $\boldsymbol{U}_m$ for $m \leqslant n$. We infer that the interval $\alpha_{n,j}$ is not contained in $h_1(X_1) \cup h_2(X_2)$. Let us observe that there is a finite family $\{\beta^s_{i,j}\}$ of mutually disjoint segments contained in E such that $\bigcup_s \beta^s_{i,j} \supset h_{i,n}(V_{n+1,j})$ and $\alpha_{n,j} \supsetneq h_{1,n}(X_1) \cup h_{2,n}(X_2) \cup \bigcup_s \beta^s_{i,j}$ and $\beta^s_{i,j} \subset \mathring{\alpha}_{n,j'}$ if $V_{n+1,j} \subset V_{n,j'}$.

Let $\boldsymbol{U}_{n+1} = \{U_{n+1,j} |\ j = 1, 2, \ldots, j(n+1)\}$ be a family of mutually disjoint subsets of T which satisfies the conditions (b) and (c) and is such that

$$U_{n+1,j} \subset \mathring{U}_{n,j'} \quad \text{if} \quad V_{n+1,j} \subset V_{n,j'},$$

and

$$U_{n+1,j} \cap \big(h_{1,n}(X_1) \cup h_{2,n}(X_2) \cup \bigcup \beta^s_{i,j}\big) = \emptyset .$$

There is an open subset $\tilde{U}_{n,j} \subset U_{n,j}$ (for every j) which meets E and which is disjoint with $h_{1,n}(X_1) \cup h_{2,n}(X_2) \cup \bigcup \beta^s_{i,j} \cup \bigcup_j U_{n+1,j}$. Thus by Corollary (1.2) there is a finite family $\{D^s_{i,j}\}$ of mutually disjoint discs in T such that

(i) $D^s_{i,j} \subset U_{n,j'}$ if $\beta^s_{i,j} \subset U_{n,j'}$,

(ii) one of the components of $D^s_{i,j} \cap E$ is the segment $\beta^s_{i,j}$,

(iii) $(D^s_{i,j} \setminus \beta^s_{i,j}) \cap \big(h_{1,n}(X_1) \cup h_{2,n}(X_2)\big) = \emptyset$,

(iv) $D^s_{i,j} \cap U_{n+1,j'} = \emptyset$ if $j' \neq j$,

(v) $D^s_{i,j} \cap \alpha_{n+1,j}$ is a segment contained in $\mathring{\alpha}_{n+1,j}$, whose interior is contained in the interior of the disc $D^s_{i,j}$,

(vi) there is an $l \in \{0, 1, 2\}$, $l \neq k$, such that $D^s_{i,j} \cap U_{n+1,j}$ is contained in the disc $U_{n+1,j} \cap (D_k \cup D_l)$ (where k is from condition (g)) (see Fig. 2).

Let $\{B^s_{i,j}\}$ be a family of mutually disjoint discs in the disc $U_{n,j'} \setminus \mathring{D}_k$ such that $B^s_{i,j}$ is a neighbourhood of $\beta^s_{i,j}$ and $h^{-1}_{i,n}(B^s_{i,j}) \subset V'_{n,j'}$ if $\beta^s_{i,j} \subset \mathring{\alpha}_{n,j'}$ and the boundary $\dot{B}^s_{i,j}$ of $B^s_{i,j}$ is not contained in $h_{i,n}(X_i)$. Let $\gamma^s_{i,j}$ be an arc in $\dot{B}^s_{i,j}$ which contains the set $\dot{B}_{i,j} \cap h_{i,n}(X_i)$. By Lemma (1.11), there are embeddings $g^s_{i,j}\colon B^s_{i,j} \to B^s_{i,j} \cup D^s_{i,j}$ such that $g^s_{i,j}|\gamma^s_{i,j}$ is the identity and $g^s_{i,j}(\beta^s_{i,j}) \subset \alpha_{n+1,j}$.

We define the maps $h_{i,n+1}\colon X_i \to T$ as follows:

$$h_{i,n+1}(x) = \begin{cases} g^s_{i,j} \circ h_{i,n}(x) & \text{if } h_{i,n}(x) \in B^s_{i,j}, \\ h_{i,n}(x) & \text{if } h_{i,n}(x) \in T \setminus \bigcup_{j,s} B^s_{i,j}. \end{cases}$$

It is easy to see that conditions (a)-(h) are satisfied.

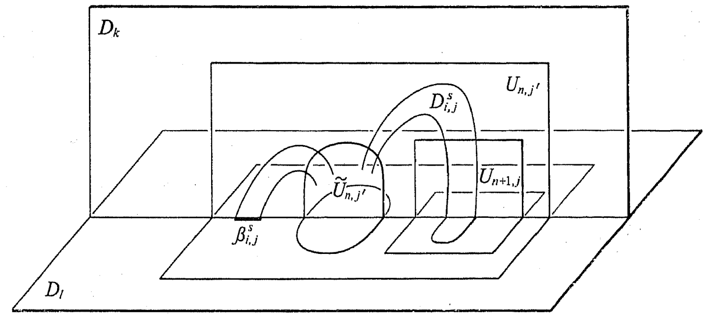

Fig. 2

The map $h\colon X_1 \cup X_2 \to T$ defined by

$$h(x) = \lim_{n\to\infty} h_{i,n}(x) \quad \text{if } x \in X_i$$

is a well-defined continuous map which satisfies the condition of our lemma.

§ 2. **The main result.** Let X be a 1-dimensional compactum. By X_F we denote the set of all points of X for which there are flat neighbourhoods; let $X_N = X \setminus X_F$. For a family $\boldsymbol{U}$ of subsets of X, by $\boldsymbol{U}_N$ we denote the family of all elements of $\boldsymbol{U}$ which intersect X_N; let $\boldsymbol{U}_F = \boldsymbol{U} \setminus \boldsymbol{U}_N$. Let us formulate the following

(2.1) Lemma. *If X is a 1-dimensional compactum with $\dim X_N \leqslant 0$ then there is a sequence $\{\boldsymbol{A}^k\}$ of finite families of closed subsets of X such that*

(a) *every element of $\boldsymbol{A}^k_F$ is flat,*

(b) $\operatorname{mesh} \boldsymbol{A}^k \leqslant 1/k$,

(c) $\operatorname{ord} \boldsymbol{A}^k_N \leqslant 0$, $\operatorname{ord} \boldsymbol{A}^1 \leqslant 1$ *and* $\operatorname{ord}(\boldsymbol{A}^{k+1} \cup \boldsymbol{A}^k_F) \leqslant 1$,

(d) *$\boldsymbol{A}^1$ is a covering of X,*

(e) *$\boldsymbol{A}^{k+1}$ is a covering of $\bigcup \boldsymbol{A}^k_N$ and $\boldsymbol{A}^{k+1}$ refines $\boldsymbol{A}^k_N$,*

(f) *the intersection $A' \cap A''$ of any two sets A', A'' of $\boldsymbol{A}^{k+1} \cup \boldsymbol{A}^k_F$ (or of $\boldsymbol{A}^1$) is a 0-dimensional or empty subset of $\operatorname{Fr} A' \cap \operatorname{Fr} A''$,*

(g) *any element of $\boldsymbol{A}^{k+1}$ intersects at most one element of $\boldsymbol{A}^k_F$,*

(h) *any element $A \in \boldsymbol{A}^{k+1}_N$ is disjoint with $\bigcup \boldsymbol{A}^k_F$.*

Proof. Since X_N is compact and $\dim X_N \leqslant 0$, there is a finite family $\boldsymbol{U}^0$ of open sets in X such that order $\boldsymbol{U}^0 \leqslant 0$ and mesh $\boldsymbol{U}^0 \leqslant 1$ and $\bigcup \boldsymbol{U}^0 \supset X_N$ and every member of $\boldsymbol{U}^0$ intersects X_N. Let $U_x \subset X_F$ be a flat neighbourhood of a point $x \in X_F$ (in X)

with the diameter $\delta(U_x)<1$. There is an open shrinking $\boldsymbol{U}^1$ of order $\leqslant 1$ of a finite subcover of the open cover $\boldsymbol{U}^0 \cup \{U_x |\ x \in X_F\}$ of X. Then there is a closed shrinking $\boldsymbol{A}^1$ of the cover $\boldsymbol{U}^1$ such that the intersection of any two elements A', A'' of $\boldsymbol{A}^1$ is a 0-dimensional or empty subset of $\operatorname{Fr} A' \cap \operatorname{Fr} A''$.

Suppose that we have defined the families $\boldsymbol{A}^1, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^k$ which satisfies conditions (a)-(h). Let $A \in \boldsymbol{A}_N^k$. For every $x \in \operatorname{Int} A \setminus X_N$ we find a flat neighbourhood $U_x^A \subset \operatorname{Int} A \setminus X_N$ with diameter $\delta(U_x^A)<1/(k+1)$. Observe that $X_N \cap \operatorname{Fr} A = \emptyset$ and $\dim(\operatorname{Fr} A) \leqslant 0$. Thus there is a family $\boldsymbol{U}_A^0$ of open subsets of A (in A) such that

(i) $\operatorname{ord} \boldsymbol{U}_A^0 \leqslant 0$ and $\operatorname{mesh} \boldsymbol{U}_A^0 < \dfrac{1}{k+1}$,

(ii) $\bigcup \boldsymbol{U}_A^0 \supset \operatorname{Fr} A \cup (A \cap X_N)$,

(iii) every member B of $\boldsymbol{U}_A^0$ intersects exactly one of the sets X_N or $\operatorname{Fr} A$, and B is flat if $B \cap \operatorname{Fr} A = \emptyset$,

(iv) any element $B \in \boldsymbol{U}_A^0$ intersects at most one element of $\boldsymbol{A}_F^k$.

There is an open shrinking $\boldsymbol{U}_A^1$ with order 1 of a finite subcover of the open cover $\boldsymbol{U}_A^0 \cup \{U_x^A |\ x \in \operatorname{Int} A \setminus X_N\}$. Then $\boldsymbol{A}_A^{k+1}$ is a closed shrinking of the covering $\boldsymbol{U}_A^1$ ($\boldsymbol{A}_A^{k+1}$ covers A) such that the intersection of any two elements A', A'' of $\boldsymbol{A}_A^{k+1}$ is a 0-dimensional or empty subset of $\operatorname{Fr} A' \cap \operatorname{Fr} A''$. The family $\boldsymbol{A}^{k+1}$, which is the sum of all the families $\boldsymbol{A}_A^{k+1}$ ($A \in \boldsymbol{A}_N^k$), satisfies conditions (a)-(h).

Denote by $\boldsymbol{A}_F$ the sum of all families $\boldsymbol{A}_F^k$. If the conditions (a)-(h) of Lemma (2.1) are satisfied then $\operatorname{ord} \boldsymbol{A}_F \leqslant 1$ and $\bigcup \boldsymbol{A}_F = X_F$.

(2.2) THEOREM. *If a one-dimensional compactum X is not locally flat at points composing the set of dimension $\leqslant 0$ then X can be embedded in the 3-book T.*

Proof. Let $\{\boldsymbol{A}^k\}$ be a sequence from Lemma (2.1). We can define a sequence $\{\boldsymbol{D}^k\}$ of families of discs in T, $\boldsymbol{D}^k = \{D(A)|\ A \in \boldsymbol{A}^k\}$, such that

(i) $\boldsymbol{A}^k$ and $\boldsymbol{D}^k$ are isomorphic,

(ii) $A \cap A' = \emptyset$ iff $D(A) \cap D(A') = 0$ for $A \in \boldsymbol{A}^k$, $A' \in \boldsymbol{A}^{k'}$,

(iii) if $A \in \boldsymbol{A}^k$ then $\delta(D(A))<1/k$,

(iv) if A and A' are elements of $\boldsymbol{A}^{k+1} \cup \boldsymbol{A}_F^k$ (or of $\boldsymbol{A}^1$) then the intersection $\dot{D}(A) \cap \dot{D}(A')$ is the empty set or a segment contained in $E \cap D(A) \cap D(A')$,

(v) if $A \in \boldsymbol{A}_N^k$ then the interior of $D(A)$ meets the edge E,

(vi) if $A \in \boldsymbol{A}_N^{k+1}$ then $D(A)$ is contained in $D(A')$ for some $A' \in \boldsymbol{A}_N^k$.

Using Lemma (1.1), we can define a family $\boldsymbol{D}^1$ satisfying the above conditions and such that $\delta(D(A))<\frac{1}{2}$ if $A \in \boldsymbol{A}_N^1$. If we have defined families $\boldsymbol{D}^1, \ldots, \boldsymbol{D}^k$ satisfying the above conditions and such that $\delta(D(A))<1/(k+1)$ if $A \in \boldsymbol{A}_N^k$, then by Corollary (1.6) there is a family $\boldsymbol{D}^{k+1}$ of discs in T satisfying the conditions (i)-(v) and such that $\delta(D(A))\leqslant 1/(k+2)$ for $A \in \boldsymbol{A}_N^{k+1}$.

Observe that if $A \in \boldsymbol{A}_F^k$ and $B \in \boldsymbol{A}^l$, where $l>k+1$, then $A \cap B = \emptyset$ and thus $D(A) \cap D(B) = \emptyset$. Thus, by (iv), if $A, B \in \boldsymbol{A}_F$, $A \neq B$, then $D(A) \cap D(B) = \dot{D}(A) \cap \dot{D}(B)$ is the empty set or a segment contained in E. Observe also that the order of family $\boldsymbol{D}_F = \{D(A)|\ A \in \boldsymbol{A}_F\}$ is $\leqslant 1$.

For every $A \in \boldsymbol{A}_F$ we find an embedding $h_A\colon A \to T$ such that

(i) $h_A(A) \subset \bigcup \{D(B)|\ B \in \boldsymbol{A}_F$ and $B \cap A \neq 0\} = D^*(A)$,

(ii) if $A, B \in \boldsymbol{A}_F$, $A \neq B$ then $h_A(A) \cap h_B(B) = \emptyset$,

(iii) if $A, B \in \boldsymbol{A}_F$ and $A \cap B \neq 0$, $A \neq B$, then $h_A(A \cap B) \subset D(A) \cap D(B)$.

Let $\{F_{\{A,B\}}|\ A, B \in \boldsymbol{A}_F,\ A \cap B \neq \emptyset,\ A \neq B\}$ be a family of mutually disjoint closed subsets of X such that

$$A \cap B \subset \operatorname{Int} F_{\{A,B\}} \subset A \cup B .$$

For every pair $A, B \in \boldsymbol{A}_F$, $A \neq B$, $A \cap B \neq \emptyset$, we change the embedding h_A on the set $\operatorname{Int} F_{\{A,B\}} \cap A$ (using Lemma (1.12)) to obtain an embedding $h_0\colon X_F \to T$ such that for every $A \in \boldsymbol{A}_F^k$ the image $h_0(A)$ is contained in the sum of $D^*(A)$ and discs with diameter $<1/k$ which intersect $D(A)$.

If $x \in X_N$, then $x = \bigcap_{k=1}^{\infty} A_x^k$, where $x \in A_x^k \in \boldsymbol{A}_N^k$. We define $h\colon X \to T$ by

$$h(x) = \begin{cases} \bigcap_{n=1}^{\infty} D(A_x^n) & \text{if } x \in X_N , \\ h_0(x) & \text{if } x \in X_F . \end{cases}$$

One can see that h is a continuous injection; thus h is an embedding.

It is known ([1], p. 44) that for $n>2$ there exists a universal curve $B(n)$ lying in the n-book. The Sierpiński universal plane curve [5] can be used to construct the curve $B(n)$. Suppose that the Sierpiński curve is obtained as a subset of a square $[0,1]^2$ by a standard method. The $B(n)$ is a union of n copies of the Sierpiński curve with a common segment $[0,1]\times\{0\}$. $B(n)$ is not locally flat at points of this segment.

(2.3) THEOREM. *There exists a universal space for the class of all one-dimensional compacta with a zero-dimensional set of points at which the space is not locally flat.*

Proof. It is easy to see that a compactum from this class can be embedded (as in the proof of Theorem (2.2)) in a 3-book in such a way that the edge of the book does not contain any segment of the compactum. It follows that a union of 3 copies of the intersection of the Sierpiński curve and the set $[0,1]\times[0,\frac{1}{2}]$ with a common set from $[0,1]\times\{\frac{1}{2}\}$ is the required universal curve. The intersection of the segment $[0,1]\times\{\frac{1}{2}\}$ and the Sierpiński curve (the set of points of the universal curve lying on the edge) is 0-dimensional.

Using the same method as in the proof of Theorem (2.2), one can prove the following

(2.4) THEOREM. *If X is a one-dimensional compactum and for every point $x \in X$ there exists a neighbourhood U which can be embedded in a 2-dimensional polyhedron, then X can be embedded in a 2-dimensional polyhedron.*

Now we will give an example of a continuum X which cannot be embedded in any 2-polyhedron and is such that the set of all points at which X is not locally flat forms an arc.

(2.5) EXAMPLE. Let $X = I \cup \bigcup_{n=1}^{\infty} S_n$ be a continuum which is a union of an arc I and a countable family of mutually disjoint solenoids S_n with $\delta(S_n) \to 0$ and such that $I \cap S_n$ is a point denoted by a_n, and the set $\{a_n|\ n = 1, 2, \ldots\}$ is dense in I. Then $X_N = I$ and X cannot be embedded in any 2-polyhedron, since otherwise I would be contained in an edge of a 2-polyhedron and then some solenoid S_n would be contained in a triangle.

§ 3. **Application.** Let us prove the following

(3.1) LEMMA. *Let k be an integer. If X is the inverse limit of an inverse sequence $\{X_i, p_i^j\}$ of connected finite 1-polyhedra with at most k points of order $\geqslant 3$, then X has at most k points, at which X is not locally flat.*

Proof. Denote by A_i the set of all points of X_i of order $\geqslant 3$. First assume that A_i is not empty for any i. Let $x = (x_i)$ be a point of the inverse limit $\varprojlim X_i$ such that

(3.2) there are an integer i and a closed neighbourhood U of x_i in X_i such that $p_i^j(A_j) \cap U = \emptyset$ for infinitely many $j \geqslant i$.

Let $j_1 < j_2 < j_3 < \ldots$ be a sequence of integers such that $p_i^{j_k}(A_{j_k}) \cap U = \emptyset$. The set $p_i^{-1}(U)$ is an inverse limit of the sets $(p_i^{j_k})^{-1}(U)$ which lies in a disjoint sum of segments $X_{j_k} \setminus A_{j_k}$; thus $p_i^{-1}(U)$ is flat.

If $y = (y_i) \in X = \varprojlim X_i$ does not satisfy condition (3.2), then

(3.3) for any integer i any neighbourhood V of y_i in X_i meets almost all sets $p_i^j(A_j)$, $j \geqslant i$.

Let $y^1 = (y_i^1)$, $y^2 = (y_i^2), \ldots, y^l = (y_i^l)$ be (different) points of X which do not satisfy condition (3.2). There is an index i such that $y_i^1, y_i^2, \ldots, y_i^l$ are different points in X_i. Let $V_1, V_2, \ldots, V_l$ be mutually disjoint neighbourhoods of points $y_i^1, y_i^2, \ldots, y_i^l$, respectively (in X_i). Thus by (3.3) $p_i^j(A_j)$ meets every set V_k, $k = 1, 2, \ldots, l$, for some integer j. Thus $l \leqslant k$.

The inverse limit of an inverse sequence and its subsequences are homeomorphic. Thus it remains to prove this lemma in the case where A_i is empty for every i. The proof in this case is similar to a part of the above proof.

By Theorem (2.2) we obtain the following

(3.4) COROLLARY. *The continuum X which satisfies the assumption of Lemma* (3.1) *is homeomorphic to a subset of the 3-book T.*

Every (pointed) 1-dimensional compactum X has the shape of the inverse limit of a (pointed) inverse sequence $\{X_i, p_i^j\}$ such that X_i is a disjoint finite union of bouquets of circles and p_i^j maps base points onto base points of bouquets for every $i < j$. The set of all points at which this limit is not locally flat has dimension $\leqslant 0$; thus, by Theorem (2.2), we have the following

(3.5) COROLLARY. *Every pointed 1-dimensional compactum X has the shape of a pointed subset of the 3-book T.*

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