subset of $\Sigma\times\Sigma$. Apply the Kondo Uniformization theorem for coanalytic sets to obtain a coanalytic uniformization for D, i.e., get a coanalytic set $B\subseteq D$ such that B^x is a singleton whenever $D^x\neq\emptyset$. Let $A=(\Sigma\times\Sigma)-B$. Then,

(i) A is an analytic subset of $\Sigma\times\Sigma$.

(ii) For each $x\in\Sigma$, $A^x=\{y\in\Sigma\colon (x,y)\in A\}$ is either Σ or Σ minus a point.

Define a multifunction $F\colon\Sigma\to\Sigma$ by $F(x)=A^x$. Then,

(a) As each A^x is dense in Σ, F is $\boldsymbol{B}_\Sigma$-measurable, $\boldsymbol{B}_\Sigma$ being the Borel σ-field on Σ.

(b) For each $x\in X$, $F(x)$ is open in Σ.

(c) $\mathrm{Gr}(F)=A$ is analytic in $\Sigma\times\Sigma$.

However, F admits no $\boldsymbol{B}_\Sigma$-measurable selector. Indeed, A admits no coanalytic uniformization, *a fortiori*, no Borel uniformization. For if not, let $E\subseteq A$ be a coanalytic subset of $\Sigma\times\Sigma$ such that E^x is a singleton for each $x\in X$. As C is universal for the coanalytic subsets of $\Sigma\times\Sigma$ there is $x^*\in\Sigma$ such that $E=C^{x^*}$. Now, there is a unique $y^*\in\Sigma$ such that $(x^*,y^*)\in E$. It follows that $D^{x^*}=y^*$, and consequently, that $(x^*,y^*)\in B$. But $E\subseteq A$. So $(x^*,y^*)\in A=\Sigma\times\Sigma-B$, which leads to a contradiction.

Added in proof: Theorem 1.1 has been extended to an arbitrary measurable space (T,A) by the author in his doctoral dissertation: *Measurable sets in product spaces and their parametrizations*, Indian Statistical Institute, Calcutta 1981.

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Homology with models

by

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Abstract. A general type of topological homology theory is developed using the left derived functors of the left Kan extension or the equivalent André derived functors. The homology theory is based on a model category and a coefficient functor from the model category to the category of abelian groups. It is shown that if the model category contains a singleton and is closed under products with the unit interval, and the coefficient functor is homotopy invariant, then the homology theory satisfies the Eilenberg–Steenrod axioms. It is also shown that, in certain cases, these hypotheses can be weakened considerably. By appropriate choices of the model category both singular homology and an exact version of Vietoris–Čech homology are obtained as examples of the general theory.

1. Preliminaries and notation. Let $\mathfrak{M}$ be a small, full subcategory of a category $\mathfrak{X}$, $J\colon\mathfrak{M}\to\mathfrak{X}$ be the inclusion functor, and $\mathfrak{A}$ be an abelian category with colimits and enough projectives. Functor categories are denoted with brackets. Thus, $[\mathfrak{M},\mathfrak{A}]$ is the category whose objects are functors from $\mathfrak{M}$ to $\mathfrak{A}$ and whose morphisms are natural transformations between these functors. The left Kan extension along J,

$$\mathrm{Lan}_J\colon [\mathfrak{M},\mathfrak{A}]\to[\mathfrak{X},\mathfrak{A}]$$

is defined by

$$\mathrm{Lan}_J F(x)\to\varinjlim_{M\to X} F(M)$$

where F is an object of $[\mathfrak{M},\mathfrak{A}]$, X is an object of $\mathfrak{X}$, and the colimit is over the category of $\mathfrak{M}$-objects over X. (See [M2] or [H–S] for more details.) If $\alpha\colon M\to X$ is a morphism in $\mathfrak{X}$ (i.e., an $\mathfrak{M}$-object over X), then

$$i\langle\alpha\rangle\colon F(M)\to\mathrm{Lan}_J F(X)$$

denotes the injection into the colimit at α.

Since $\mathfrak{A}$ has enough projectives, so does $[\mathfrak{M},\mathfrak{A}]$. Thus, any object F of $[\mathfrak{M},\mathfrak{A}]$ has a projective resolution $P_*\to F\to 0$. The left derived functors of the left Kan extension are defined by

$$L_n\mathrm{Lan}_J F=H_n(\mathrm{Lan}_J P_*)\,.$$

Assume further that $\mathfrak{A}$ is AB-4 [G] and let (C_*F, ∂_*) be the André chain complex of F. Thus,

$$C_nF(X) = \coprod_{(\alpha_0, \ldots, \alpha_n)} F(M_0),$$

where $M_0 \xrightarrow{\alpha_0} \ldots \to M_n \xrightarrow{\alpha_n} X$ is a chain of morphisms with $\alpha_i, 0 \leqslant i \leqslant n-1$, in $\mathfrak{M}$ and α_n in $\mathfrak{T}$. The injection of $F(M_0)$ into $C_nF(X)$ at the component indexed by $(\alpha_0, \ldots, \alpha_n)$ is denoted by $\iota\langle\alpha_0, \ldots, \alpha_n\rangle$. Further details, including the definition of ∂_*, can be found in [A]. The André derived functors of F are defined by

$$A_nF = H_n(C_*F).$$

If f: $X \to Y$ is a morphism in $\mathfrak{T}$, then $C_nF(f)$: $C_nF(X) \to C_nF(Y)$ by

$$C_nF(X)\iota\langle\alpha_0, \ldots, \alpha_n\rangle = \iota\langle\alpha_0, \ldots, f\alpha_n\rangle,$$

and $A_nF(f)$ is obtained by taking homology.

Let G be a function from the objects of $\mathfrak{M}$ to the objects of $\mathfrak{A}$. (G need not be defined on morphisms, so it is not necessarily a functor.) The functor E_G: $\mathfrak{M} \to \mathfrak{A}$ defined by

$$E_G(M') = \coprod_{M \to M'} G(M) \text{ on objects } M' \text{ of } \mathfrak{M},$$

where the coproduct is over all $\mathfrak{M}$-objects over M', and by $E_G(f)\iota\langle\alpha\rangle = \iota\langle f\alpha\rangle$ on morphisms f: $M' \to M''$ of $\mathfrak{M}$, where $\iota\langle\alpha\rangle$ is the injection into the coproduct at the component indexed by α: $M \to M'$, is called an *elementary functor*. Thus, the objects of the André chain complex are elementary functors.

1.1. Lemma. *Let F be an object of $[\mathfrak{M}, \mathfrak{A}]$ and $E_{G_*} \to F \to 0$ be a resolution of F by elementary functors. Then, for any $n \geqslant 0$, $H_n(\mathrm{Lan}_J E_{G_*})$, A_nF and $L_n\mathrm{Lan}_J F$ are naturally equivalent. In particular, any chain morphism τ_*: $E_{G_*} \to C_*F$ compatible with augmentations induces a homology isomorphism.*

Proof. This follows from the fact that elementary functors are Lan_J-acyclic. See [A], [U1], and [U2]. ■

For a morphism f: $X \to Y$ in $\mathfrak{T}$, define $C_nF(X, f, X) = \mathrm{coker}\, C_nF(f)$. ∂_* passes to cokernels and $C_nF(X, f, Y)$ is a chain complex in $\mathfrak{A}$. The homology objects of this chain complex are denoted $A_nF(X, f, X)$.

Any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \text{in } \mathfrak{T}$$

induces a chain morphism

$$C_nF(\alpha, \beta)\colon C_nF(X, f, Y) \to C_nF(X', f', Y').$$

In homology this is $A_nF(\alpha, \beta)$.

If f is monic in $\mathfrak{T}$, then each $C_nF(f)$ is monic in $\mathfrak{A}$ and

$$0 \to C_nF(X) \xrightarrow{C_nF(f)} C_nF(X) \xrightarrow{\sigma_n(f)} C_nF(X, f, Y) \to 0$$

is a short exact sequence, where $\sigma_n(f)$ is the projection onto the cokernel. This short exact sequence yields a long exact sequence

$$\ldots \to A_nF(X, f, Y) \xrightarrow{\delta_n(f)} A_{n-1}F(X) \xrightarrow{A_{n-1}F(f)} A_{n-1}F(X) \to \ldots$$

and δ_* is natural.

If τ_* is a morphism of chain complexes, its mapping cone is denoted by $\mathrm{MC}(\tau_*)$.

1.2. Lemma. *If f: $X \to Y$ is monic in $\mathfrak{T}$ and $P_* \to F \to 0$ is a projective resolution of F in $[\mathfrak{M}, \mathfrak{A}]$, then*

$$H_*(\mathrm{MC}(\mathrm{Lan}_J P_*(f))), \quad H_*(\mathrm{MC}(C_*F(F))) \quad \textit{and} \quad A_nF(X, f, Y)$$

are naturally isomorphic.

Proof. This follows from the five lemma and the comparison theorem [M1]. ■

The following lemma will be needed in Section 5.

1.3. Lemma. *Let J'': $\mathfrak{M} \to \mathfrak{N}$, J': $\mathfrak{N} \to \mathfrak{T}$, and $J = J'J''$: $\mathfrak{M} \to \mathfrak{T}$ be inclusion functors. If F: $\mathfrak{M} \to \mathfrak{A}$ satisfies $L_n\mathrm{Lan}_{J''}F = 0$ for all $n > 0$, then*

$$L_n\mathrm{Lan}_J F = L_n\mathrm{Lan}_{J'}(\mathrm{Lan}_{J''}F).$$

Proof. If $P_* \to F \to 0$ is a projective resolution of F in $[\mathfrak{M}, \mathfrak{A}]$, then

$$\mathrm{Lan}_{J''}P_* \to \mathrm{Lan}_{J''}F \to 0$$

is a projective resolution of $\mathrm{Lan}_{J''}F$ in $[\mathfrak{N}, \mathfrak{A}]$. ■

2. The homology objects. In this section a sequence of homology objects, for both single spaces and pairs, and a boundary operator are defined. These homology objects are objects of the category of pro-abelian groups. The definition and properties of this category (which can be found in [Ma] and [Mo], and, in more generality, in [G]) will be reviewed here to establish the notation.

Henceforth, $\mathfrak{A}$ will be the category of abelian groups and Pro-$\mathfrak{A}$ will be the category whose objects are inverse systems $(A(d), \{p\langle d, d'\rangle\}, D)$, or just (A, p, D), where D is a directed set, $A(d)$ is an abelian group for all $d \in D$, and, for $d' \geqslant d$, $p\langle d', d\rangle$: $A(d') \to A(d)$ is a group homomorphism such that

$$p\langle d'', d\rangle = p\langle d', d\rangle p\langle d'', d'\rangle$$

when $d'' \geqslant d' \geqslant d$. A morphism f: $(A, p, D) \to (B, q, E)$ of inverse systems of abelian groups consists of a pair $(\bar{f}, \{f\langle e\rangle\})$ where $\bar{f}$: $E \to D$ is a function, and, for each $e \in E$, $f\langle e\rangle$: $A(\bar{f}(e)) \to B(e)$ is a group homomorphism such that, for e, $e' \in E$ with $e' \geqslant e$, there exist a $d \in D$ with $d \geqslant \bar{f}(e)$, $\bar{f}(e')$ and $g\langle e', e\rangle f\langle e'\rangle p\langle d, \bar{f}(e')\rangle = p\langle d, \bar{f}(e)\rangle f\langle e\rangle$. The morphisms of Pro-$\mathfrak{A}$ are obtained by identifying f and g: $(A, p, D) \to (B, g, E)$ if for every $e \in E$ there exist a $d \in D$ with $d \geqslant \bar{f}(e)$, $\bar{g}(e)$ and $f\langle e\rangle p\langle d, \bar{f}(e)\rangle = g\langle e\rangle p\langle d, \bar{g}(e)\rangle$.

A morphism $f\colon (A, p, D) \to (B, g, E)$ in Pro-$\mathfrak{A}$ is a special morphism if $D = E$ and $\bar{f} = 1$. Pro-$\mathfrak{A}$ has zero objects, kernals, cokernals, images, and exact sequences [Ma]. In particular, a sequence of special morphisms is exact if it is "pointwise" exact. [Ma, 4.4].

Let $\mathcal{T}op$ denote the category of topological spaces and continuous functions. For an object X of $\mathcal{T}op$, let $\mathrm{Cov}(X)$ denote the set of all collections of subsets of X whose interiors cover X. $\mathrm{Cov}(X)$ is a directed set under refinement and we write $\mathscr{U} \geqslant \mathscr{V}$ if $\mathscr{U}$ refines $\mathscr{V}$. $\mathscr{C}\mathcal{T}op$ is the category of covered topological spaces. Its objects are of the form, $(X, \mathscr{U})$ where X is a topological space and $\mathscr{U} \in \mathrm{Cov}(X)$. A morphism $f\colon (X, \mathscr{U}) \to (X, \mathscr{V})$ in $\mathscr{C}\mathcal{T}op$ is a continuous function $f\colon X \to Y$ such that $\mathscr{U} \geqslant f^{-1}(\mathscr{V})$. The forgetful functor $S\colon \mathscr{C}\mathcal{T}op \to \mathcal{T}op$, defined by $S(X, \mathscr{U}) = X$, $S(f) = f$, has a right adjoint $R\colon \mathcal{T}op \to \mathscr{C}\mathcal{T}op$ by $R(X) = (X, \{X\})$, $R(f) = f$. Let $\mathfrak{M}_0$ be a small, full subcategory of $\mathcal{T}op$ and $\mathfrak{M} = R(\mathfrak{M}_0)$. $\mathfrak{M}$ is called the model category. A *coefficient functor* is a functor $F\colon \mathfrak{M} \to \mathfrak{A}$. Note that $R|_{\mathfrak{M}_0}\colon \mathfrak{M}_0 \to \mathfrak{M}$ is an isomorphism of categories with inverse $S|_{\mathfrak{M}}$, and any $F\colon \mathfrak{M} \to \mathfrak{A}$ is equivalent, by composition with $S|_{\mathfrak{M}}$, to a functor $F_0\colon \mathfrak{M}_0 \to \mathfrak{A}$. Thus, it is only a slight abuse of terminology to call $\mathfrak{M}_0$ the model category and a functor $F_0\colon \mathfrak{M}_0 \to \mathfrak{A}$ the coefficient functor.

2.1. Definition. For each object $(X, \mathscr{U})$ in $\mathscr{C}\mathcal{T}op$, let

$$H_n(X, \mathscr{U}) = L_n \mathrm{Lan}_J F(X, \mathscr{U}).$$

Let $H_*\colon \mathcal{T}op \to \text{Pro-}\mathfrak{A}$ be the sequence of functors defined by

$$H_n(X) = \big(H_n(X, \mathscr{U}), p\langle \mathscr{U}, \mathscr{V}\rangle, \mathrm{Cov}(X)\big)$$

where, for $\mathscr{U} \geqslant \mathscr{V}$,

$$p\langle \mathscr{U}, \mathscr{V}\rangle\colon H_n(X, \mathscr{U}) \to H_n(X, \mathscr{V})$$

is induced by $l_X\colon (X, \mathscr{U}) \to (X, \mathscr{V})$ in $\mathscr{C}\mathcal{T}op$.

If $f\colon X \to Y$ in $\mathcal{T}op$, then $H_*(f)$ is defined by

$$H_n(f) = \big(f^{-1}, \{f\langle \mathscr{V}\rangle \mid \mathscr{V} \in \mathrm{Cov}(X)\}\big)$$

where $f^{-1}\colon \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$ and $f\langle \mathscr{V}\rangle\colon H_n(X, f^{-1}(\mathscr{V})) \to H_n(X, \mathscr{V})$ is the morphism in $\mathfrak{A}$ induced by $f\colon (X, f^{-1}(\mathscr{V})) \to (X, \mathscr{V})$ in $\mathscr{C}\mathcal{T}op$.

2.2. Note. By 1.1, an equivalent definition of $H_*(X)$ results from taking $H_n(X, \mathscr{U}) = A_n F(X, \mathscr{U})$ in 2.1. It will sometimes be convenient to use this alternate definition (as in the following definition)

2.3. Definition. Let (X, A) be a topological pair with inclusion map $j\colon A \to X$. In $\mathscr{C}\mathcal{T}op$, j induces morphisms $j\langle \mathscr{U}\rangle\colon (A, j^{-1}(\mathscr{U})) \to (X, \mathscr{U})$ for each $\mathscr{U} \in \mathrm{Cov}(X)$. If $H_n(j, \mathscr{U})$ denotes the group $A_n F\big((A, j^{-1}(\mathscr{U})), j\langle \mathscr{U}\rangle, (X, \mathscr{U})\big)$ (defined in Section 1), then $H_*(X, A)$ is defined by

$$H_n(X, A) = \big(H_n(j, \mathscr{U}), p\langle \mathscr{U}', \mathscr{U}\rangle, \mathrm{Cov}(X)\big)$$

where, for $\mathscr{U}' \geqslant \mathscr{U}$ in $\mathrm{Cov}(X)$,

$$p\langle \mathscr{U}', \mathscr{U}\rangle\colon H_n(j, \mathscr{U}') \to H_n(j, \mathscr{U})$$

is the morphism $A_n F(1_A, 1_X)$.

If $f\colon (X, A) \to (X, B)$ is a map of pairs, then $kf' = fj$ where $f' = f|_A$ and $k\colon B \to Y$ is the inclusion map. The pair (f', f) induces

$$f\langle \mathscr{V}\rangle\colon H_n(j, f^{-1}(\mathscr{V})) \to H_n(k, \mathscr{V})$$

for every $\mathscr{V} \in \mathrm{Cov}(X)$. $H_n(f)$ is defined to be

$$\big(f^{-1}, \{f\langle \mathscr{V}\rangle\colon \mathscr{V} \in \mathrm{Cov}(X)\}\big).$$

By definition, $H_n(A)$ is an object of Pro-$\mathfrak{A}$ indexed by $\mathrm{Cov}(A)$. The next lemma shows that, for a topological pair (X, A), $H_n(A)$ is naturally isomorphic to an object indexed by $\mathrm{Cov}(X)$.

2.4. Lemma. *Let (X, A) be a topological pair with inclusion $j\colon A \to X$. Let $H_n(A)_X$ denote the object*

$$\big(H_n(A, j^{-1}(\mathscr{U})), p\langle j^{-1}(\mathscr{U}), j^{-1}(\mathscr{U}')\rangle, \mathrm{Cov}(X)\big)$$

of Pro-$\mathfrak{A}$. *Then, there exists an isomorphism*

$$\varphi_n(X, A)\colon H_n(A) \to H_n(A)_X \quad \textit{in } \text{Pro-}\mathfrak{A}.$$

Proof. $\varphi_n(X, A) = \big(j^{-1}, \{1_{H_n(A, j^{-1}(\mathscr{U}))} \mid \mathscr{U} \in \mathrm{Cov}(X)\}\big)$. Its inverse in Pro-$\mathfrak{A}$ is $\psi_n(X, A)$ defined by

$$\psi_n(X, A) = \big(\bar{\psi}_n, \{\psi_n\langle \mathscr{V}\rangle \mid \mathscr{V} \in \mathrm{Cov}(A)\}\big)$$

where $\bar{\psi}_n\colon \mathrm{Cov}(A) \to \mathrm{Cov}(X)$ is defined by choosing, for each $\mathscr{V} \in \mathrm{Cov}(A)$, a $\bar{\psi}(\mathscr{V}) \in \mathrm{Cov}(X)$ for which $j^{-1}(\bar{\psi}(\mathscr{V})) = \mathscr{V}$. Then, each $\psi_n\langle \mathscr{V}\rangle$ is defined to be the identity on $H_n(A, \mathscr{V})$. ∎

Let $f\colon (X, A) \to (Y, B)$ be a map of topological pairs and $j\colon A \to X$ and $k\colon B \to Y$ be inclusions. Define $H_n(f)_{X,Y}\colon H_n(A)_X \to H_n(B)_Y$ by

$$H_n(f)_{X,Y} = \big(f^{-1}, \{f_{X,Y}\langle \mathscr{V}\rangle \mid \mathscr{V} \in \mathrm{Cov}(X)\}\big)$$

where $f_{X,Y}\langle \mathscr{V}\rangle\colon H_n(A, j^{-1}f^{-1}(\mathscr{V})) \to H_n(B, k^{-1}(\mathscr{V}))$ is induced by

$$f|_A\colon (A, j^{-1}f^{-1}(\mathscr{V})) \to (B, k^{-1}(\mathscr{V})).$$

With this definition $(X, A) \mapsto H_n(A)_X$ becomes a functor on the category of topological pairs. It is easily established that

$$\varphi_n(X, A)\colon H_n(A) \to H_n(A)_X$$

is a natural equivalence when $(X, A) \mapsto H_n(A)$ is also considered as a functor on the category of topological pairs.

2.5. Proposition. *Let (X, A) be a topological pair with inclusion $j\colon A \to X$. Then, there exist natural transformations $\partial_n(j)\colon H_n(X, A) \to H_{n-1}(A)$ and a long exact sequence*

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial_n(j)} H_{n-1}(A) \to \ldots$$

in Pro-$\mathfrak{A}$.

Proof. For $\mathscr{U} \in \mathrm{Cov}(X), j: (A, j^{-1}(\mathscr{U})) \to (X, \mathscr{U})$ is monic. Thus, the long exact sequence of André derived functors (from Section 1) is

$$\ldots \to H_n(A, j^{-1}(\mathscr{U})) \to H_n(X, \mathscr{U}) \to H_n(j, \mathscr{U}) \xrightarrow{\delta_n(j, \mathscr{U})} H_{n-1}(A, j^{-1}(\mathscr{U})) \to \ldots$$

So, in Pro-$\mathfrak{A}$, the long sequence of special morphisms

$$\ldots \to H_n(A)_X \to H_n(X) \to H_n(X, A) \xrightarrow{\delta_n(j)} H_{n-1}(A)_X \to \ldots$$

is exact, where $\delta_n(j)$ is the special morphism induced by the family $\{\delta_n(j, \mathscr{U}) |\ \mathscr{U} \in \mathrm{Cov}(X)\}$, because it is "pointwise" exact. Let $\varphi_n(X, A)$ be as in 2.4 and $\partial_n(j) = \varphi_n(X, A)^{-1} \partial_n'(j)$: $H_n(X, A) \to H_{n-1}(A)$. Then, ∂_n is natural and

$$\ldots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial_n(j)} H_{n-1}(A) \to \ldots$$

is exact in Pro-$\mathfrak{A}$. ∎

We can now verify that (H_*, ∂_*) satisfies the Eilenberg–Steenrod axioms, 1–4 and 7. [E–S]. (Axioms 5 and 6 are considered in the next two sections.) Axioms 1 and 2 require that each H_n be a functor and it is clear from the definitions that these are satisfied. Axiom 3, which requires that ∂_* be natural, is satisfied as a result of 2.5. Also, by 2.5, axiom 4, the exactness axiom, is satisfied. We consider axiom 7, the dimension axiom, next.

2.6. Lemma. *If a singleton space* $\{x\}$ *is an object of* $\mathfrak{M}_0$, *then* $H_n(\{x\}) = 0$ *for* $n>0$, *i.e.*, H_* *satisfies the dimension axiom.*

Proof. Since $(\{x\}, \{x\})$ is an object of $\mathfrak{M}$, $P_*(\{x\}, \{x\}) \to F(\{x\}, \{x\})$ is exact, and, thus, $H_n(\{x\})$ is the trivial group indexed by a singleton directed set, which is a zero object in Pro-$\mathfrak{A}$. ∎

3. The homotopy axiom. Let I denote the unit interval and suppose $\mathfrak{M}_0$ is closed under products with I, i.e., if M is an object of $\mathfrak{M}_0$, then $M\times I$ is an object of $\mathfrak{M}_0$. Define a functor T: $\mathfrak{M} \to \mathfrak{M}$ by $T(M, \{M\}) = (M\times I, \{M\times I\})$ and $T(f) = f\times 1_I$. Then, for $t \in I$, i_t: $1_{\mathfrak{M}} \to T$ is the natural transformation defined by $i_t(M)(m) = (m, t)$, for $m \in M$. As before, let $P_* \to F \to 0$ be a projective resolution of F in $[\mathfrak{M}, \mathfrak{A}]$.

3.1. Lemma. $P_*(i_0)$ *and* $P_*(i_1)$: $P_* \to P_*T$ *are homotopic.*

Proof. This follows from the comparison theorem [M1] since P_*T is exact on $\mathfrak{M}$. ∎

3.2. Lemma. *Let* $\mathscr{U} \in \mathrm{Cov}(X\times I)$. *Then, there exist a* $\mathscr{V} \in \mathrm{Cov}(X)$ *and a* $k \in N$ *such that for* $j = 0, \ldots, k-1$, *each*

$$V\times \left[\frac{j}{k}, \frac{j+1}{k}\right]$$

is contained in some $U \in \mathscr{U}$.

Proof. This follows from the compactness of I and the tube lemma. ∎

For $\mathscr{U} \in \mathrm{Cov}(X\times I)$, let $\mathscr{V}$ and k be as in 3.2 and let α: $(M, \{M\}) \to (X, \mathscr{V})$ be a morphism in $\mathscr{CT}op$. Define, for

$$j = 0, \ldots, k-1, g_j\colon\ M\times I \to M\times I \text{ by } g_j(m, t) = \left(m, \frac{j+t}{k}\right).$$

It follows from 3.2 that g_j is a morphism in $\mathscr{CT}op$ from $(M\times I, \{M\times I\})$ to $(M\times I, (\alpha\times 1_I)^{-1}(\mathscr{U}))$. Let $i_t'(M)$: $(M, \{M\}) \to (M\times I, (\alpha\times 1_I)^{-1}(\mathscr{U}))$ be the morphism induced in $\mathscr{CT}op$ by $i_t(M)$.

3.3. Lemma. $\mathrm{Lan}_J P_*(i_0'(M))$ *and* $\mathrm{Lan}_J P_*(i_1'(M))$ *are homotopic.*

Proof. $\mathrm{Lan}_J P_*(i_{j/k}'(M)) = \mathrm{Lan}_J P_*(g_j i_0(M)) = \mathrm{Lan}_J P_*(g_j) P_*(i_0(M))$ and

$$\mathrm{Lan}_J P_*(i_{(j+1)/k}'(M)) = \mathrm{Lan}_J P_*(g_j) P_*(i_1(M)).$$

By 3.1, $P_*(i_0(M))$ and $P_*(i_1(M))$ are homotopic. So, for $j = 0, \ldots, k-1$,

$$\mathrm{Lan}_J P_*(i_{j/k}'(M)) \quad\text{and}\quad \mathrm{Lan}_J P_*(i_{(j+1)/k}'(M))$$

are homotopic and the result follows. ∎

For $t \in I$, let $i_t(X)$: $X \to X\times I$ by $i_t(X)(x) = (x, t)$. Then, for $\mathscr{U} \in \mathrm{Cov}(X\times I)$ and $\mathscr{V} \in \mathrm{Cov}(X)$ as above, $i_t(X)$: $(X, \mathscr{V}) \to (X\times I, \mathscr{U})$ in $\mathscr{CT}op$.

3.4. Lemma. *There exists a homotopy* $D_*(X, \mathscr{U})$ *between*

$$\mathrm{Lan}_J P_*(i_0(X)) \quad \textit{and} \quad \mathrm{Lan}_J P_*(i_1(X))\colon \mathrm{Lan}_J P_*(X, \mathscr{V}) \to \mathrm{Lan}_J P_*(X\times I, \mathscr{U}),$$

and, if j: $A \to X$ *and* j_I: $A\times I \to X\times I$ *are inclusion maps, then*

$$\mathrm{Lan}_J P_*(j_I) D_*(A, j_I^{-1}(\mathscr{U})) = D_*(X, \mathscr{U}) \mathrm{Lan}_J P_*(j).$$

Proof. For each α: $(M, \{M\}) \to (X, \mathscr{V})$ in $\mathscr{CT}op$, let $D_*(\alpha)$ be the homotopy of 3.3 between $\mathrm{Lan}_J P_*(i_0'(M))$ and $\mathrm{Lan}_J P_*(i_1'(M))$ as morphism from

$$\mathrm{Lan}_J P_*(M, \{M\})$$

to

$$\mathrm{Lan}_J P_*(M\times I, (\alpha+1_I)^{-1}(\mathscr{U})).$$

Note that $D_*(\alpha)$ depends only on M and the cover $(\alpha\times 1_I)^{-1}(\mathscr{U})$ of $M\times I$. Define

$$D_n(X, \mathscr{U})\colon \mathrm{Lan}_J P_n(X, \mathscr{V}) \to \mathrm{Lan}_J P_{n+1}(X\times I, \mathscr{U})$$

by

$$D_n(X, \mathscr{U}) i\langle\alpha\rangle = \mathrm{Lan}_J P_{n+1}(\alpha\times 1_I) D_n(\alpha) i(1_M)$$

where, as before, $i\langle\alpha\rangle$ denotes the αth injection into the colimit.

A direct verification then shows that $D_*(X, \mathscr{U})$ is the required chain homotopy and that it is natural with respect to inclusions. ∎

For a topological pair (X, A), let $i_t(X, A)$ denote the map of pairs induced by $i_t(X)$.

3.5. LEMMA. *Let (X, A) be a topological pair and $j\colon A \to X$ be the inclusion map. For each $\mathscr{U} \in \mathrm{Cov}(X\times I)$ there exists a $\mathscr{V} \in \mathrm{Cov}(X)$ such that*

$$H_*(i_0(X, A)) = H_*(i_1(X, A))\colon H_*(j, \mathscr{V}) \to H_*(j_I, \mathscr{U}).$$

Proof. The mapping cone of $\mathrm{Lan}_J P_*(j)$ has homology isomorphic to $H_n(j, \mathscr{V})$ and the mapping cone of $\mathrm{Lan}_J P_*(j_I)$ has homology isomorphic to $H_n(j_I, \mathscr{U})$. The chain homotopy of 3.4, which is natural for inclusions, can be extended to a homotopy between these mapping cones and the existence of such a homotopy implies the result. ■

3.6. THEOREM. *Suppose $\mathfrak{M}_0$ is closed under products with unit interval and F is homotopy invariant. Let f and $g\colon (X, A) \to (X, B)$ be homotopic maps of topological pairs. Then,*

$$H_*(f) = H_*(g)\colon H_*(X, A) \to H_*(Y, B).$$

Thus, H_* satisfies the homotopy axiom.

Proof. Let $h\colon (X\times I, A\times I) \to (Y, B)$ be a homotopy between f and g. Then, $f = hi_0(X, A)$ and $g = hi_1(X, A)$. By the definitions of $H_*(X, A)$ and equality of morphisms in Pro-$\mathfrak{A}$, it is sufficient to show that for each $\mathscr{U} \in \mathrm{Cov}(X\times I)$ there exists a $\mathscr{V} \in \mathrm{Cov}(X)$ such that

$$H_*(i_0(X, A)) = H_*(i_1(X, A))\colon H_*(j, \mathscr{V}) \to H_*(j_I, \mathscr{U}).$$

This is established in 3.5. ■

4. The excision axiom. In this section the characterization of the homology objects in terms of André derived functors is used to verify that the excision axiom is satisfied.

Let (X, A) be a topological pair with inclusion $j\colon A \to X$ and let $\mathscr{U} \in \mathrm{Cov}(X)$. Then, $C_n((A, j^{-1}(\mathscr{U})), j, (X, \mathscr{U}))$ is defined in Section 1 to be

$$\mathrm{coker}\,(C_nF(j))\colon C_nF(A, j^{-1}(\mathscr{U})) \to C_nF(X, \mathscr{U}).$$

Henceforth, it will be denoted by $C_nF(j, \mathscr{U})$ for brevity.

4.1. LEMMA.

$$C_nF(j, \mathscr{U}) = \coprod_{\langle\alpha_0, \ldots, \alpha_n\rangle} F(M_0)$$

where the coproduct is over all chains of morphisms

$$(M_0, \{M_0\}) \xrightarrow{\alpha_0} \ldots \to (M_n, \{M_n\}) \xrightarrow{\alpha_n} (X, \mathscr{U}),$$

with $\alpha_0, \ldots, \alpha_{n-1}$ in $\mathfrak{M}$ and α_n in $\mathscr{C}\mathscr{T}op$, for which $\alpha_n(M_n) \not\subseteq A$.

Proof. Recall that $C_nF(j)\colon C_n(A, j^{-1}(\mathscr{U})) \to C_nF(X, \mathscr{U})$ is defined by

$$C_nF(j)\imath\langle\beta_0, \ldots, \beta_n\rangle = \imath\langle\beta_0, \ldots, j\beta_n\rangle.$$

Thus, the index set of $C_n(X, \mathscr{U})$ can be partitioned into the set of all $\langle\alpha_0, \ldots, \alpha_n\rangle$ for which $\alpha_n(M_n) \not\subseteq A$ and the set for which $\alpha_n(M_n) \subseteq A$. Then,

$$\mathrm{Im}\, C_nF(j) = \coprod_{\langle\alpha_0, \ldots, \alpha_n\rangle} F(M_0)$$

where the coproduct is over the first set in the partition. The result follows. ■

Let U be an open subset of X with $\overline{U} \subseteq \mathrm{Int}A$. Define $\bar{\tau}\colon \mathrm{Cov}(X-U) \to \mathrm{Cov}(X)$ by

$$\tau(\mathscr{V}) = \{V \cap (X-\overline{U})|\ V\in\mathscr{V}\} \cup \{\mathrm{Int}A\}.$$

By the previous lemma

$$C_nF(j, \bar{\tau}(\mathscr{V})) = \coprod_{\langle\alpha_0, \ldots, \alpha_n\rangle} F(M_0)$$

where $\alpha_n(M_n) \not\subseteq A$. But $\alpha_n(M_n)$ is contained in some element of $\bar{\tau}(\mathscr{V})$ because $\{M_n\}$ refines $\alpha_n^{-1}\bar{\tau}(\mathscr{V})$. Thus, $\alpha_n(M_n) \subseteq V\cap(X-U)$ for some $V\in\mathscr{V}$, and so α_n factors through the inclusion map $k\colon (X-U, A-U) \to (X, A)$. This justifies the following definition.

For $\mathscr{V} \in \mathrm{Cov}(X-U)$, define

$$\tau_n'\langle\mathscr{V}\rangle\colon C_n(j, \bar{\tau}(\mathscr{V})) \to C_n(i, \mathscr{V}),$$

where $i\colon A-U \to X-U$ is the inclusion map, by

$$\tau_n'\langle\mathscr{V}\rangle\imath\langle\alpha_0, \ldots, \alpha_n\rangle = \imath\langle\alpha_0, \ldots, \alpha_{n-1}, \beta_n\rangle$$

where $\alpha_n = k\beta_n$. Then, $\tau_*'\langle\mathscr{V}\rangle$ is a chain map and $\tau_n\langle\mathscr{V}\rangle$ is defined to be $H_n(\tau_*'\langle\mathscr{V}\rangle)$.

4.2. THEOREM. *$H_n(k)\colon H_n(X-U, A-U) \to H_n(X, A)$ is an isomorphism for all $n \geqslant 0$. Thus, H_* satisfies the excision axiom.*

Proof. The morphism $\tau_n\colon H_n(X, A) \to H_n(X-U, A-U)$ in Pro-$\mathfrak{A}$, defined by

$$\tau_n = (\bar{\tau}, \{\tau_n\langle\mathscr{V}\rangle|\ \mathscr{V}\in\mathrm{Cov}(X-U)\}),$$

is an inverse of $H_n(k)$ in Pro-$\mathfrak{A}$. ■

5. Summary of results and examples. The results of the previous sections can now be summarized.

5.1. THEOREM. *Let $\mathfrak{M}_0$ be a small, full subcategory of $\mathscr{T}op$ which is closed under products with the unit interval and has a singleton space as an object. Let $F\colon \mathfrak{M}_0 \to \mathfrak{A}$ be a homotopy invariant functor. Then, the construction of Section 2 yields a homology theory on $\mathscr{T}op$ with values in the category* Pro-$\mathfrak{A}$. ■

The hypotheses that $\mathfrak{M}_0$ be closed under products with the unit interval and contain a singleton space imply that $\mathfrak{M}_0$ contains all cubes, I^n. This is a severe restriction on the subcategories of $\mathscr{T}op$ that can serve as model categories. The following corollary will replace this restriction with a much weaker one.

In the statement of the corollary, the dependence of the homology theory on the model category and the coefficient functor is important. We will write $H_*(\ ; \mathfrak{M}_0, F)$ for the homology sequence with model category $\mathfrak{M}_0$ and coefficient functor F to emphasize this dependence.

Let $\mathfrak{M}_0$ be any small, full subcategory of $\mathcal{T}op$ which has a singleton as an object. Let $\overline{\mathfrak{M}}_0$ be the smallest full subcategory of $\mathcal{T}op$ which contains $\mathfrak{M}_0$ and is closed under products with the unit interval. Let $J''\colon R(\mathfrak{M}_0) \to R(\overline{\mathfrak{M}}_0)$ be the inclusion functor and $F\colon \mathfrak{M}_0 \to \mathfrak{A}$.

5.2. Corollary. *If for every object* $(\overline{M}, \{\overline{M}\})$ *of* $R(\overline{\mathfrak{M}}_0)$,

$$L_n \mathrm{Lan}_{J''} F(\overline{M}, \{\overline{M}\}) = 0\,, \quad \textit{for } n>0,$$

then

$$H_*(\ ; \mathfrak{M}_0, F) = H_*(\ ; \overline{\mathfrak{M}}_0, \mathrm{Lan}_{J''} F)\,.$$

In this case, if $\mathrm{Lan}_{J''}F$ *is homotopy invariant, then* $H_*(; \mathfrak{M}_0, F)$ *satisfies the axioms for a homology theory.*

Proof. This follows directly from 1.3. ■

5.3. Example (Singular Homology). Let $\mathfrak{M}_0$ be the full subcategory of $\mathcal{T}op$ whose objects are standard simplicies Δ^n, $n = 0, 1, \ldots,$ G be a fixed abelian group, and $F\colon \mathfrak{M} \to \mathfrak{A}$ be the constant functor with value G.

For a topological space X and an open cover $\mathcal{U}$ of X, the chain complex $\{C_*^{\mathrm{sing}}(X, \mathcal{U}), \partial_*\}$ of *$\mathcal{U}$ small singular simplicies* of X with coefficient group $G = F(\Delta^n, \{\Delta^n\})$ is defined by

$$C_n^{\mathrm{sing}}(X, \mathcal{U}) = \coprod \{F(\Delta^n, \{\Delta^n\}) |\ \alpha\colon (\Delta^n, \{\Delta^n\}) \to (X, \mathcal{U})\} \quad \text{in } \mathcal{C}\mathcal{T}op$$

and

$$\partial_n(X, \mathcal{U})\colon C_n^{\mathrm{sing}}(X, \mathcal{U}) \to C_{n-1}^{\mathrm{sing}}(X, \mathcal{U}) \quad \text{by} \quad \partial_n(X, \mathcal{U}) = \sum_{i=0}^{n} (-1)^i \sigma_i(X, \mathcal{U})$$

where

$$\sigma_i(X, \mathcal{U})\colon C_n^{\mathrm{sing}}(X, \mathcal{U}) \to C_{n-1}^{\mathrm{sing}}(X, \mathcal{U}) \quad \text{by} \quad \sigma_i \iota\langle\alpha\rangle = \iota\langle\alpha e_i\rangle$$

for $\alpha\colon (\Delta^n, \{\Delta^n\}) \to (X, \mathcal{U})$ in $\mathcal{C}\mathcal{T}op$ and $e_i\colon (\Delta^{n-1}, \{\Delta^{n-1}\}) \to (\Delta^n, \{\Delta^n\})$ the affine map of Δ^{n-1} onto the ith face of Δ^n. An augmentation $\varepsilon\colon C_0^{\mathrm{sing}} J \to F$ is defined by $\varepsilon(\Delta^n, \{\Delta^n\})\iota\langle\alpha\rangle = F(\alpha)$ for $\alpha\colon (\Delta^0, \{\Delta^0\}) \to (\Delta^n, \{\Delta^n\})$.

For $f\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ in $\mathcal{C}\mathcal{T}op$, $C_n^{\mathrm{sing}}(f)\colon C_n^{\mathrm{sing}}(X, \mathcal{U}) \to C_n^{\mathrm{sing}}(Y, \mathcal{V})$ is defined by $C_n^{\mathrm{sing}}(f)\iota\langle\alpha\rangle = \iota\langle f\alpha\rangle$ for $\alpha\colon (\Delta^n, \{\Delta^n\}) \to (X, \mathcal{U})$.

Each C_n^{sing} is an elementary functor on $\mathcal{C}\mathcal{T}op$, and the chain complex of groups $C_*^{\mathrm{sing}}(\Delta^m, \{\Delta^m\}) \to F(\Delta^m, \{\Delta^m\}) \to 0$ is well known to be exact for any standard simplex Δ^m. Thus, the chain complex of functors $C_*^{\mathrm{sing}} J \to F \to 0$ is a resolution of F by elementary functors. By 1.1, any natural transformation $C_*^{\mathrm{sing}} \to C_* F$ ($C_* F$ is the André complex) whose restriction is compatible with augmentations induces a homology isomorphism. One such natural transformation, τ_*, is given by $\tau_*(X, \mathcal{U})\iota\langle\alpha\rangle = \iota\langle 1, \ldots, 1, \alpha\rangle$ where $\alpha\colon (\Delta^n, \{\Delta^n\}) \to (X, \mathcal{U})$ and 1 denotes the identity on $(\Delta^n, \{\Delta^n\})$.

Thus, $H_n(\tau_*)$ defines a special isomorphism between the inverse system

$$\big(H_n^{\mathrm{sing}}(X, \mathcal{U}), \gamma\langle\mathcal{U}, \mathcal{V}\rangle, \mathrm{Cov}(X)\big) \quad \text{and} \quad H_n(X)\,,$$

where

$$H_n^{\mathrm{sing}}(X, \mathcal{U}) = H_n\big(C_*^{\mathrm{sing}}(X, \mathcal{U})\big)$$

and

$$\gamma\langle\mathcal{U}, \mathcal{V}\rangle = H_n^{\mathrm{sing}}(1_X)\colon H_n^{\mathrm{sing}}(X, \mathcal{U}) \to H_n^{\mathrm{sing}}(X, \mathcal{V})$$

when $\mathcal{U} \geqslant \mathcal{V}$.

For a topological pair (X, A) with inclusion $j\colon A \to X$, $C_*^{\mathrm{sing}}(X, A, \mathcal{U})$, $\mathcal{U} \in \mathrm{Cov}(X)$, is defined to be the cokernal of the monomorphism

$$C_*^{\mathrm{sing}}(j)\colon C_*^{\mathrm{sing}}(A, j^{-1}(\mathcal{U})) \to C_*^{\mathrm{sing}}(X, \mathcal{U})\,.$$

The natural transformation τ_* passes to cokernals, inducing a chain map

$$\bar{\tau}_*(X, A, \mathcal{U})\colon C_*^{\mathrm{sing}}(X, A, \mathcal{U}) \to C_* F\big((A, j^{-1}(\mathcal{U})), j, (X, \mathcal{U})\big)$$

which is natural for maps of pairs. By the five lemma, each $\bar{\tau}_*(X, A, \mathcal{U})$ induces a homology isomorphism. Thus, $H_n(\bar{\tau}_*)$ defines a special isomorphism between the inverse system

$$\big(H_n^{\mathrm{sing}}(X, A, \mathcal{U}), \gamma\langle\mathcal{U}, \mathcal{V}\rangle, \mathrm{Cov}(X)\big) \quad \text{and} \quad H_n(X, A)$$

where

$$H_n^{\mathrm{sing}}(X, A, \mathcal{U}) = H_n\big(C_*^{\mathrm{sing}}(X, A, \mathcal{U})\big)$$

and

$$\bar{\gamma}\langle\mathcal{U}, \mathcal{V}\rangle = H_n^{\mathrm{sing}}(1_A, 1_X)\colon H_*^{\mathrm{sing}}(X, A, \mathcal{U}) \to H_*^{\mathrm{sing}}(X, A, \mathcal{V})$$

when $\mathcal{U} \geqslant \mathcal{V}$.

Let sd be the subdivision chain map on the singular chain complex [S, p. 177 ff]. Then if $\mathcal{U} \geqslant \mathcal{V}$, there exist $n \in N$ such that

$$sd^n\colon C_*^{\mathrm{sing}}(X, A, \mathcal{U}) \to C_*^{\mathrm{sing}}(X, A, \mathcal{V})\,.$$

Furthermore, sd^n is a chain equivalence with homotopy inverse $C_*(1_X, 1_A)$. Thus, each $\bar{\gamma}\langle\mathcal{U}, \mathcal{V}\rangle$ is an isomorphism and,

$$\varprojlim_{\mathcal{U}} H_*^{\mathrm{sing}}(X, A, \mathcal{U}) \cong H_*^{\mathrm{sing}}(X, A)\,.$$

(Here, $H_*^{\mathrm{sing}}(X, A, \mathcal{U})$ denotes the standard singular homology sequence of the pair (X, A) which is identical to $H_*^{\mathrm{sing}}(X, A, \{X\})$ as defined above.) This implies that the limit over open covers preserves the exactness of the long singular homology sequence of a topological pair.

Note that the category $\mathfrak{M}_0$ of standard simplicies does not satisfy the hypotheses of Theorem 5.1 — $\Delta^n \times I$ is not a standard simplex. It does, however, satisfy the hypotheses of Corollary 5.2.

5.4. Example (Vietoris–Čech Homology). The model category, $\mathfrak{M}_0$, for Vietoris–Čech Homology consists of all finite sets $[n]$, $n \in N$, with the discrete topology and all (continuous) functions between them. $F\colon \mathfrak{M} \to \mathfrak{A}$ is, again, a constant functor. The Vietoris–Čech chain complex is defined by

$$C_n^V(X,\mathcal{U}) = \coprod \{F([n]) \mid \alpha\colon ([n],\{[n]\}) \to (X,\mathcal{U})\}$$

and $\partial_n\colon C_n^V(X,\mathcal{U}) \to C_{n-1}^V(X,\mathcal{U})$ by $\partial_n = \sum_{i=0}^{n}(-1)^i \partial_n^i$ where $\partial_n^i \iota\langle\alpha\rangle = \iota\langle\alpha e_n^i\rangle$ and $e_n^i\colon [n-1] \to [n]$ by $e_n^i(k)$ if $k<i$ and $e_n^i(k) = k+1$ if $k \geqslant i$.

Note that each C_n^V is an elementary functor. A chain contraction of $C_*^V([n],\{[n]\})$ is given by choosing a point $n_0 \in [n]$ and defining

$$\eta_k\colon C_k^V([n],\{[n]\}) \to C_{k+1}^V([n],\{[n]\}) \quad \text{by} \quad \eta_k\iota\langle\alpha\rangle = \iota\langle\bar{\alpha}\rangle$$

where

$$\alpha\colon ([k],\{[k]\}) \to ([n],\{[n]\}) \quad \text{and} \quad \bar{\alpha}\colon ([k+1],\{[k+1]\}) \to ([n],\{[n]\})$$

by $\bar{\alpha}(i) = \alpha(i)$ for $1 \leqslant i \leqslant k$ and $\bar{\alpha}(k+1) = n_0$. Thus, $C_*^V J \to F \to 0$ is a resolution of F by elementary functors. Hence, by 1.1, the natural transformation $\tau_*\colon C_*^V \to C_* F$ defined by $\tau_n\iota\langle\alpha\rangle = \iota\langle 1,\ldots,1,\alpha\rangle$, which is compatible with the augmentations, induces a homology isomorphism. Then, τ_* induces a special isomorphism from the inverse system $(H_n^V(X,\mathcal{U}), \gamma\langle\mathcal{U},\mathcal{V}\rangle, \mathrm{Cov}(X))$ to $H_n(X)$. Here,

$$H_n^V(X,\mathcal{U}) = H_n(C_*^V(X,\mathcal{U})) \quad \text{and} \quad \gamma\langle\mathcal{U},\mathcal{V}\rangle = H_n^V(1_X)\colon H_n^V(X,\mathcal{U}) \to H_n^V(X,\mathcal{V}),$$

when $\mathcal{U} \geqslant \mathcal{V}$.

As in 5.3, for a topological pair (X, A) with inclusion $j\colon A \to X$, $C_*^V(X,A,\mathcal{U})$, $\mathcal{U} \in \mathrm{Cov}(X)$, is defined to be the cokernal of

$$C_*^V(j)\colon C_*^V(A, j^{-1}(\mathcal{U})) \to C_*^V(X,\mathcal{U}) \quad \text{and} \quad H_n^V(X,A,\mathcal{U}) = H_n(C_*^V(X,A,\mathcal{U})).$$

Then, τ_* induces a special isomorphism from the inverse system

$$(H_n^V(X,A,\mathcal{U}), \bar{\gamma}\langle\mathcal{U},\mathcal{V}\rangle, \mathrm{Cov}(X))$$

to $H_n(X, A)$.

Unlike singular theory, Vietoris–Čech theory has no subdivision chain map. Thus, although the sequence

$$\ldots \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \ldots$$

is exact in Pro-$\mathfrak{A}$, applying the limit over covers of X does not necessarily produce an exact sequence in $\mathfrak{A}$ [E–S, X, 4].

As in 5.3, the hypotheses of 5.1 are not satisfied, but those of 5.2 are.

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