in this way across the (n+1)st row until \( X_{n+1} \) has been constructed and an embedding of it into \( X_n \). Finally, construct \( X_{n+1,n+1} \) from \( X_{n+1,n} \), the way that \( X_{2,2} \) was constructed from \( X_{2,1} \).

Remark. H. Cook has shown that \( X_{n,n} \) is not hereditarily equivalent. It is an open question whether there exists a hereditarily equivalent (plane) continuum of positive span.


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Another universal metacompact developable
\( T_\gamma \)-space of weight \( m \)

by

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Abstract. Let \( m \) be an infinite cardinal number. We use the \( d \)-line constructed in [Ch2] in order to construct a simple universal metacompact developable \( T_\gamma \)-space of weight \( m \) analogous to a universal metric space of weight \( m \) constructed implicitly in the proof of the Nagata-Smirnov metrization theorem.

Let \( m \) be an infinite cardinal number. In [Ch2], we constructed a universal metacompact developable \( T_\gamma \)-space of weight \( m \). The construction was based on a method of constructing mappings into metacompact developable \( T_\gamma \)-spaces from [Ch1].

In section one of this paper we give another construction of a universal metacompact developable \( T_\gamma \)-space of weight \( m \). This construction is related to a method of constructing mappings into metacompact developable \( T_\gamma \)-spaces investigated in [Ch3]. It is simpler than the construction in [Ch2] and has its metric analogue.

In section two we generalize the construction from [Ch2] in order to obtain an orthocompact developable \( T_\gamma \)-space of weight \( 2^m \) containing all orthocompact developable \( T_\gamma \)-spaces of weight \( m \). The universal metacompact developable \( T_\gamma \)-space of weight \( m \) constructed [Ch2] is contained in this space in a natural way. We indicate some relations between the two constructions of universal spaces (Remark 2.7).

All our constructions are based on the \( d \)-line \( D \) (denoted by \( T(0) \) in [Ch2]).

In section three we construct a \( d \)-interval \( D^* \) and discuss the problem of extending mappings into \( D \) and \( D^* \).

We use the terminology and notation from [E]. All mappings are assumed to be continuous and all spaces are assumed to be \( T_\gamma \)-spaces. The last section requires the knowledge of [Ch2].

The \( d \)-line \( D \) [Ch2] (a similar, but more complicated space has been constructed earlier in [H]) is \( N^m \), where \( N \) is the set of natural numbers and \( N_\omega = N \setminus \{0\} \). The topology of \( D \) is generated by the subbase

\[ \mathcal{P} = \{ B(i); n, i \geq 1 \} \cup \{ B(i,j); n, i, j \geq 1 \} \cup \{ D \}, \]

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where

$$B_n(i) = \{ d \in D : d(n) \geq i \}$$

for \( n, i \geq 1 \),

$$B_{n,i}(j) = \{ d \in D : (d(n) \geq i) \land (d(n+i) \geq j) \}$$

$$= (D \setminus B_n(i)) \cup B_{n+i}(j)$$

for \( n, i, j \geq 1 \).

It is easy to see that \( D \) is a \( T_1 \)-space and each element of \( \mathcal{P} \) is an \( F_n \)-set in \( D \). Thus \( D \) is a developable \( T_1 \)-space of countable weight. The point \((0, 0, \ldots, \infty) \) is \( D \) and will be denoted by \( 0 \).

A collection \( \mathcal{P} \) of open subsets of a space \( X \) is said to be interior-preserving if the intersection of every \( \mathcal{P} \subset \mathcal{P} \) is open in \( X \).

A space \( X \) is said to be metacompact (metalindeff or orthocompact) if each open cover of \( X \) has a point-finite (point-countable or, respectively, interior-preserving) open refinement. It is well known that a developable space is metacompact (metalindeff or orthocompact) if it has a development consisting of point-finite (point-countable or, respectively, interior-preserving) covers. Observe that \( D \) (any space with a countable base consisting of \( F_n \)-sets) has a development consisting of two-element covers.

A space \( X \) is said to be semi-stratifiable [C] if each closed subset \( A \) of \( X \) can be represented as a \( G_{\delta} \)-set \( \bigcap_{k \geq 1} W_k(A) \) in such a way that \( A \subset A' \) implies \( W_k(A') \subset W_k(A) \) for \( k \geq 1 \). Developable spaces are semi-stratifiable.

In what follows \( m \) denotes a fixed infinite cardinal number, \( \mathcal{P}(m) \) is the powerset of \( m \) and \( \text{Fin}(m) \) is the set of finite subsets of \( m \).

The spaces constructed in this paper depend on \( m \). Since \( m \) is fixed, we shall often omit the symbol \( m \).

1. The space \( S(m) \). One of the most familiar universal metric spaces is the product of countably many hedgehogs \( J(m) \) [E, 4.4.9]. The Bing metrization theorem can be considered to be a corollary to the fact that, for any discrete collection \( \{ U_{a} : a \in m \} \) of functionally open subsets of a space \( X \), there exists an \( f : X \to J(m) \) such that \( U_{a} \) is the inverse image of the open sphere of \( J(m) \) corresponding to \( a \in S \).

If the collection \( \{ U_{a} : a \in m \} \) is locally finite, then \( J(m) \) should be replaced by \( K(m) \), which is a "hedgehog" whose spines are cubes of finite dimension.

More precisely, \( K(m) = \{ x \in J(m) : \| x \| \neq \| 0 \| \} \) is considered with the topology of uniform convergence.

It is easy to see that \( J(m) \) is naturally embedded in \( K(m) \) and that the Nagata-Smirnov metrization theorem can be proved by constructing an embedding into \( K(m) \).

Let \( S(m) = \{ x \in D^n : \| x \| \neq \| 0 \| \} \) and consider \( S \) with the topology generated by the products of open subsets of \( D \) with all but a finite number of factors equal to a neighbourhood of \( 0 \) in \( D \) (more precisely, the intersections of such products with \( S \)).

It is easy to check that \( S \) is a metacompact developable \( T_1 \)-space of weight \( m \).

For \( a \in m \), put \( H_a = \{ x \in S : x(a) \neq 0 \} \). Clearly, \( \{ H_a : a \in m \} \) is a point-finite collection of open subsets of \( S \). We have (see \[C2, Theorem 1])

1.1. Theorem. If \( X \) is a perfect space and \( \{ U_a : a \in m \} \) a collection of open subsets of \( X \) which is point-finite as an indexed collection, then there exists a mapping \( f : X \to S(m) \) such that \( f^{-1}(U_a) = U_a \) for \( a \in m \).

An immediate consequence of 1.1 is

1.2. Corollary. The space \( S(m)^m \) is universal for all metacompact developable \( T_1 \)-spaces of weight \( m \).

The proof of 1.1 will be based on the following lemma.

1.3. Lemma. If \( \{ V_i \}_{i \in I} \) is a decreasing sequence of open subsets of \( X \) and \( \bigcap_{i \in I} V_i = \emptyset \), then there exists a mapping \( f : X \to D \) such that \( f^{-1}(B_i) = V_i \) for \( n, i, j \geq 1 \).

Proof. We modify a reasoning from [Ch2]. We construct, by induction on \( n \geq 1 \), sequences \( \{ V_i(0) \}_{i \in I} \) of open subsets of \( X \) such that

(i) \( V_0(i) = V_i(0) \) for \( i \geq 1 \)

and, for \( n \geq 1 \),

(ii) \( \{ V_i(0) \}_{i \in I} \) is a decreasing sequence and \( \bigcap_{i \in I} V_i(0) = \emptyset \)

(iii) \( V_j(i,j) = V_{j+1}(i) \) for \( j \geq 1 \)

(iv) \( V_j(i,j) = \bigcap_{k \geq 1} V_{j+k}(i,j) \) is open in \( X \) for \( i, j \geq 1 \).

The sequence \( \{ V_i(0) \}_{i \in I} \) satisfying (ii) is defined by (i). Suppose that \( \{ V_i(0) \}_{i \in I} \) is given. For \( i \geq 1 \) let \( \{ V_j(i,j) \}_{j \in I} \) be a decreasing sequence of open subsets of \( X \) such that \( X \setminus V_j(i,j) = \bigcup_{k \geq 1} U_{j+k}(i,j) \).

Put

\( V_{j+1}(i) = \bigcap_{k \geq 1} V_{j+k}(i,j) \)

From the inductive assumptions it follows that (ii) and (iii) are satisfied. Moreover, from (ii) and (iii) it follows that, for \( i \geq 1 \), \( f^{-1}(B_j(i)) = V_j(i,j) \) and, consequently, \( V_j(i,j) = X \). If \( j \geq 1 \), then, by virtue of (iv), \( f^{-1}(B_0(i)) = V_0(i) \). This gives \( f : X \to D \) satisfying \( f^{-1}(B_0(i)) = V_0(i) \) and, consequently, \( f^{-1}(B_0(i,j)) = V_j(i,j) \). Thus \( f \) is continuous and (i) and (iii) imply that \( f^{-1}(B_1(i)) = V_1(i) \) and \( f^{-1}(D(0)) = \bigcap_{k \geq 1} V_{i+k}(i) \).

Proof of 1.1. Suppose that \( X \) is a perfect space and \( \{ U_a : a \in m \} \) is a collection of open subsets of \( X \) such that \( \{ a \in m : x \in U_a \} \) is finite for \( x \in X \).

Since \( X \) is a perfect space, one can construct, for \( k \geq 1 \), collections \( \{ E_k(k) : k \in m \} \) of closed subsets of \( X \) which are locally finite as indexed collections and satisfy \( \bigcup_{k \geq 1} E_k(k) = \emptyset \) for \( a \in m \) if \( X \) is developable and \( \{ \emptyset \} \) is a development of \( X \).

Then \( E_k(k) = X \setminus S(X, U_a, \emptyset) \) satisfy the above conditions.

For \( a \in m \) put \( V_a(i) = U_a \) and \( f_a(i) = U_a \setminus E(k) \) for \( i \geq 1 \). Let \( f : X \to D \)
be a mapping satisfying the requirements of Lemma 1.3 with respect to the sequence \( \{V_a\}_{a \in A} \).

Since \( f^{-1}(D \setminus \{0\}) = U_a \), it follows that \( f = \bigtriangleup f_a : X \to S \) and \( f^{-1}(H_a) = U_a \).

Thus it remains to prove that \( f \) is a continuous function.

From the definition of the topology of \( S \) it follows that it is sufficient to show that, for any (subbasic) neighbourhood \( B \) of \( 0 \) in \( D \), \( \{f^{-1}(B) : a \in m \} \) is interior-preserving in \( X \).

Take \( B = B_k(i, f) \). For \( a \in m \), \( X \setminus f^{-1}(B_k(i, i)) = V_a(i) \setminus V_a(f) = \bigcup E_k(k) \). Thus \( \{f^{-1}(B) : a \in m \} \) is locally finite in \( X \) and consequently \( \{f^{-1}(B) : a \in m \} \) is interior-preserving.

1.4. Remark. If \( X \) is additionally assumed to be collectionwise normal, then \( S(m) \) can be replaced by \( K(m) \) in 1.1.

1.5. Remark. Let \( Y(m) = \{s \in \mathbb{D}^m : \{a \in m : \gamma(a) \notin \delta B \} < 2^\omega \) for each neighbour-} 

hood \( B \) of \( 0 \) in \( D \) and consider \( Y \) with the topology generated in the same way as the topology of \( S \). It can be checked that \( Y \) is a quasi-developable \( \{B \} \) \( T_1 \)-space with a point-countable base of cardinality \( m \). If \( X \) is a semi-stratifiable space with a point-countable collection \( \{U_a : a \in m \} \) of open sets, then one can use \([Ch3, 44]\) in order to define sets \( E_k(k) \) which allow us to construct, as in the proof of 1.1, \( f : X \to Y \) such that \( U_a = f^{-1}(\{y \in Y : \gamma(y) \neq 0 \}) \). Thus \( T(m) \) contains topologically all metrizable developable \( T_1 \)-spaces of weight \( m \).

Unfortunately it is not a perfect space (even for \( m = \omega_0 \)).

2. The space \( Z(m) \). The existence of \( f : X \to S(m) \) satisfying the requirements of \( S(m) \) was based on the possibility of representing each \( U_a \) as the union of a countable collection \( \{E_k(k)\}_{k=1}^\omega \) of closed subsets of \( X \) such that \( \{E_k(k) : a \in m \} \) was locally finite in \( X \) for \( k \geq 1 \).

If \( \{U_a : a \in m \} \) is an interior-preserving collection of open subsets of a semi-stratifiable space \( X \) and \( P(a) = \bigcap \{U_a : a \in m \} \setminus \{U_a : a \notin \Delta \} \) for \( a \in m \), then each \( P(a) \) can be represented as the union of an countable collection \( \{E_k(a, k)\}_{k=1}^\omega \) of closed sets such that \( \{E_k(a, k) : a \in m \} \) is discrete in \( X \) for \( k \geq 0 \). We shall use this observation in order to generalize the construction of \( T(m) \) and \( T(m) \) from \([Ch2, Theorem 1, Remark 6]\).

Let \( Z(m) = \mathcal{P}(m) \times D \) and put

\[
G(a, i) = \{s \in Z : a \in m \text{ and } (a \neq i \Rightarrow d(1) \geq i)\} \quad \text{for } a \in m \text{ and } i \geq 0,
\]

\[
G(a, i) = \mathcal{P}(m) \times B_k(i) \quad \text{for } n, i \geq 1,
\]

\[
G(i, i) = \mathcal{P}(m) \times B_k(i) \quad \text{for } n, i \geq 1.
\]

Consider \( Z \) with the topology obtained by taking the sets defined above as a subbase of \( Z \).

One can modify the proof of the developability of \( T(m) \) from \([Ch2]\) in order to show that \( Z \) is an orthocompact developable \( T_1 \)-space. Clearly, the weight of \( Z \) is \( 2^m \).

For \( a \in m \), put \( G_a = G(a, 0) \). Clearly, \( \{G_a : a \in m \} \) is an interior-preserving collection of open subsets of \( Z \). We have \([Ch2, Theorem 1]\).

2.1. Theorem. If \( X \) is a semi-stratifiable space and \( \{U_a : a \in m \} \) an interior-preserving collection of open subsets of \( X \), then there exists \( \epsilon : X \to Z(m) \) such that \( g^{-1}(G_a) = U_a \) for \( a \in m \).

2.2. Corollary. The space \( Z(m) \) contains topologically all orthocompact developable \( T_1 \)-spaces of weight \( m \).

Proof of 2.1. Suppose that \( \{U_a : a \in m \} \) is an interior-preserving collection of open subsets of a semi-stratifiable space \( X \). Let \( P(a) \) and \( \{E_k(a, k) : a \in m \} \) for \( k \geq 0 \) be as in the introduction to this section.

Put \( V(i) = X \setminus \{E(a, k) : a \in m \text{ and } k < i\} \) and let \( f : X \to D \) be a mapping satisfying the requirements of 1.3 with respect to the sequence \( \{V(i)\}_{i=1}^\omega \). Define \( g(x) = (\{a \in m_k \mid f(x) \in E(a, k)\}) \in Z(m) \).

Clearly, \( g^{-1}(G_a) = U_a \) and the proof of the continuity of \( g \) reduces to a simple observation that

\[
g^{-1}(G_a(i, j)) = \bigcap \{U_a : a \in m \} \setminus \{E(a, k) : b \neq a \text{ and } k < i\}.
\]

2.3. Remark. If \( E(\emptyset, 0) = P(\emptyset) \), then \( g(x) = (\{a \in D : a \notin \emptyset \text{ or } d = 0\} \).

We shall close, we can always assume that \( E(\emptyset, 0) = p(\emptyset) \).

2.4. Remark. \( \{U_a : a \in m \} \) is point-finite as an indexed collection, then \( g(x) \in \mathcal{F}(m) \times D \).

2.5. Remark. If \( X \setminus U_a : a \in m \) is locally finite as an indexed collection then \( g(x) \in \{a \in D : a \notin \emptyset \text{ or } d = 0\} \).

2.6. Remark. A weak form of Lemma 1.3 (sufficient for proving 2.1) can be obtained from 2.1 by observing that \( \{U_a : a \in m \} \), where \( U_0 = X = U_0 = P(a) \) for \( 1 \leq a < \omega_0 \), is point-finite, \( p : T(m) \to D \) defined by \( p(a, d) = (\{a \in m \} \), is continuous and consequently \( f = p : X \to D \), where \( g : X \to T(m) \) is given by 2.1, satisfies \( f^{-1}(E_k(i)) = V(i) \) for \( i > 1 \).

2.7. Remark. The function \( f : T(m) \to S(m) \) defined by \( f(a, d) = (a, d) \), if \( a \neq d \) and \( f(a, d) = 0 \) if \( a = d \) is continuous. The restriction of \( f \) to \( \{a \in m : a \notin \emptyset \) or \( d = 0\} \) is a homeomorphic embedding of the subspace \( T(m) \) into \( S(m) \).

3. A developable \( T_1 \)-compactification of \( D \) and extensions of mappings. The results of \([Ch2]\) and the first section of this paper show that \( D \) can be considered to be a generalization of the real line (a \( d \)-line). It is easy to observe that \( \{B_k(i, i) : i \geq 1\} \) is a countable open cover of \( D \) with no finite subcover. We shall construct a developable \( T_1 \)-compactification \( D^o \) of \( D \) (a \( d \)-interval).

Let \( aN = N \cup \{a\} \) be the Alexandrov compactification of \( N \) and consider \( aN \) with a well-order generated by the natural homeomorphism of \( aN \) onto \( aN + 1 \).

Put \( D^o = \{d \in aN^o : d(1) = a \} \) implies \( d(n) = 0 \) for \( n \geq 1 \). Consider \( D_1 \)
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References