in this way across the $(n+1)$st row until $X_{n+1,n}$ has been constructed and an embedding of it into $X_{n,n}$. Finally, construct $X_{n+1,n+1}$ from $X_{n+1,n}$ the way that $X_{2,2}$ was constructed from $X_{2,1}$. ■

Remark. H. Cook has shown that $X_{\infty,\infty}$ is not hereditarily equivalent. It is an open question whether there exists a hereditarily equivalent (plane) continuum of positive span.

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Received 17 May 1982

Another universal metacompact developable T_1-space of weight $\mathfrak{m}$

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Abstract. Let $\mathfrak{m}$ be an infinite cardinal number. We use the d-line constructed in [Ch2] in order to construct a simple universal metacompact developable T_1-space of weight $\mathfrak{m}$ analogous to a universal metric space of weight $\mathfrak{m}$ constructed implicitly in the proof of the Nagata–Smirnov metrization theorem.

Let $\mathfrak{m}$ be an infinite cardinal number. In [Ch2], we constructed a universal metacompact developable T_1-space of weight $\mathfrak{m}$. The construction was based on a method of constructing mappings into metacompact developable T_1-spaces from [Ch1].

In section one of this paper we give another construction of a universal metacompact developable T_1-space of weight $\mathfrak{m}$. This construction is related to a method of constructing mappings into metacompact developable T_1-spaces investigated in [Ch3]. It is simpler than the construction in [Ch2] and has its metric analogue.

In section two we generalize the construction from [Ch2] in order to obtain an orthocompact developable T_1-space of weight $2^{\mathfrak{m}}$ containing all orthocompact developable T_1-spaces of weight $\mathfrak{m}$. The universal metacompact developable T_1-space of weight $\mathfrak{m}$ constructed [Ch2] is contained in this space in a natural way. We indicate some relations between the two constructions of universal spaces (Remark 2.7).

All our constructions are based on the d-line D (denoted by $T(0)$ in [Ch2]). In section three we construct a d-interval D^* and discuss the problem of extending mappings into D and D^*.

We use the terminology and notation from [E]. All mappings are assumed to be continuous and all spaces are assumed to be T_1-spaces. The last section requires the knowledge of [Ch2].

The d-line D [Ch2] (a similar, but more complicated space has been constructed earlier in [H]) is N^{N_+}, where N is the set of natural numbers and $N_+ = N\setminus\{0\}$. The topology of D is generated by the subbase

$$\mathscr{P} = \{B_n(i)\colon n, i\geqslant 1\} \cup \{B_n(i,j)\colon n, i, j\geqslant 1\} \cup \{D\},$$

where

$$B_n(i) = \{d \in D\colon d(n) \geqslant i\} \quad \text{for} \quad n, i \geqslant 1,$$
$$B_n(i,j) = \{d \in D\colon d(n) \geqslant i \Rightarrow d(n+1) \geqslant j\}$$
$$= (D \setminus B_n(i)) \cup B_{n+1}(j) \quad \text{for} \quad n, i, j \geqslant 1.$$

It is easy to see that D is a T_0-space and each element of $\mathscr{P}$ is an F_σ-set in D. Thus D is a developable T_1-space of countable weight. The point $(0, 0, \ldots) \in D$ will be denoted by 0.

A collection $\mathscr{U}$ of open subsets of a space X is said to be *interior-preserving* if the intersection of every $\mathscr{U}' \subset \mathscr{U}$ is open in X.

A space X is said to be *metacompact* (*metalindelöf* or *orthocompact*) if each open cover of X has a point-finite (point-countable or, respectively, interior-preserving) open refinement. It is well known that a developable space is *metacompact* (*metalindelöf* or *orthocompact*) iff it has a development consisting of point-finite (point-countable or, respectively, interior-preserving) covers. Observe that D (any space with a countable base consisting of F_σ-sets) has a development consisting of two-element covers.

A space X is said to be *semi-stratifiable* [C] if each closed subset A of X can be represented as a G_δ-set $\bigcap_{k \geqslant 1} W_k(A)$ in such a way that $A \subset A'$ implies $W_k(A) \subset W_k(A')$ for $k \geqslant 1$. Developable spaces are semi-stratifiable.

In what follows $\mathfrak{m}$ denotes a fixed infinite cardinal number, $\mathscr{P}(\mathfrak{m})$ is the power set of $\mathfrak{m}$ and $\mathrm{Fin}(\mathfrak{m})$ is the set of finite subsets of $\mathfrak{m}$.

The spaces constructed in this paper depend on $\mathfrak{m}$. Since $\mathfrak{m}$ is fixed, we shall often omit the symbol $\mathfrak{m}$.

1. The space $S(\mathfrak{m})$. One of the most familiar universal metric spaces is the product of countably many hedgehogs $J(\mathfrak{m})$ [E, 4.4.9]. The Bing metrization theorem can be considered to be a corollary to the fact that, for any discrete collection $\{U_\alpha\colon \alpha \in \mathfrak{m}\}$ of functionally open subsets of a space X, there exists an $f\colon X \to J(\mathfrak{m})$ such that U_α is the inverse image of the open spine of $J(\mathfrak{m})$ corresponding to α [S].

If the collection $\{U_\alpha\colon \alpha \in \mathfrak{m}\}$ is locally finite, then $J(\mathfrak{m})$ should be replaced by $K(\mathfrak{m})$, which is a "hedgehog" whose spines are cubes of finite dimensions.

More precisely, $K(\mathfrak{m}) = \{x \in I^{\mathfrak{m}}\colon |\{\alpha \in \mathfrak{m}\colon x(\alpha) \neq 0\}| < \aleph_0\}$ considered with the topology of uniform convergence.

It is easy to see that $J(\mathfrak{m})$ is naturally embedded in $K(\mathfrak{m})$ and that the Nagata–Smirnov metrization theorem can be proved by constructing an embedding into $K(\mathfrak{m})^{\aleph_0}$.

Let $S(\mathfrak{m}) = \{s \in D^{\mathfrak{m}}\colon |\{\alpha \in \mathfrak{m}\colon s(\alpha) \neq 0\}| < \aleph_0\}$ and consider S with the topology generated by the products of open subsets of D with all but a finite number of factors equal to a neighbourhood of 0 in D (more precisely, the intersections of such products with S).

It is easy to check that S is a metacompact developable T_1-space of weight $\mathfrak{m}$.

For $\alpha \in \mathfrak{m}$, put $H_\alpha = \{s \in S\colon s(\alpha) \neq 0\}$. Clearly, $\{H_\alpha\colon \alpha \in \mathfrak{m}\}$ is a point-finite collection of open subsets of S. We have (see [Ch2, Theorem 1]).

1.1. THEOREM. *If X is a perfect space and $\{U_\alpha\colon \alpha \in \mathfrak{m}\}$ a collection of open subsets of X which is point-finite as an indexed collection, then there exists a mapping $f\colon X \to S(\mathfrak{m})$ such that $f^{-1}(H_\alpha) = U_\alpha$ for $\alpha \in \mathfrak{m}$.*

An immediate consequence of 1.1 is

1.2. COROLLARY. *The space $S(\mathfrak{m})^{\aleph_0}$ is universal for all metacompact developable T_1-spaces of weight $\mathfrak{m}$.*

The proof of 1.1 will be based on the following lemma.

1.3. LEMMA. *If $\{V(i)\}_{i \geqslant 1}$ is a decreasing sequence of open subsets of a perfect space X and $\bigcap_{i \geqslant 1} V(i) = \emptyset$, then there exists a mapping $f\colon X \to D$ such that $f^{-1}(B_1(i)) = V(i)$, $f^{-1}(D \setminus \{0\}) = V(1)$ and $X \setminus f^{-1}(B_n(i,j)) \subset V(1) \setminus V(j)$ for $n, i, j \geqslant 1$.*

Proof. We modify a reasoning from [Ch2]. We construct, by induction on $n \geqslant 1$, sequences $\{V_n(i)\}_{i \geqslant 1}$ of open subsets of X such that

(i) $V_1(i) = V(i)$ for $i \geqslant 1$

and, for $n \geqslant 1$,

(ii) $\{V_n(i)\}_{i \geqslant 1}$ is a decreasing sequence and $\bigcap_{i \geqslant 1} V_n(i) = \emptyset$,

(iii) $V_n(j) \subset V_{n+1}(j) \subset V(1)$ for $j \geqslant 1$,

(iv) $V_n(i,j) = (X \setminus V_n(i)) \cup V_{n+1}(j)$ is open in X for $i, j \geqslant 1$.

The sequence $\{V_1(i)\}_{i \geqslant 1}$ satisfying (ii) is defined by (i). Suppose that $\{V_n(i)\}_{i \geqslant 1}$ is given. For $i \geqslant 1$ let $\{U_n(i,j)\}_{j \geqslant 1}$ be a decreasing sequence of open subsets of X such that $X \setminus V_n(i) = \bigcap_{j \geqslant 1} U_n(i,j)$.

Put

$$\text{(v)} \quad V_{n+1}(j) = V_n(j) \cup \bigcup_{i=1}^{j-1} (V_n(i) \cap U_n(i,j)).$$

From the inductive assumptions it follows that (ii) and (iii) are satisfied. Moreover, from (ii) and (iii) it follows that, for $i \geqslant j$, $V_n(i) \subset V_n(j) \subset V_{n+1}(j)$ and, consequently, $V_n(i,j) = X$. If $i < j$, then, by virtue of (v), $V_n(i,j) = U_n(i,j) \cup V_{n+1}(j)$. This shows that (iv) is satisfied and completes the inductive construction.

For $n \geqslant 1$ put $V_n(0) = X$ and define $f(x)(n) = \max\{i \geqslant 0\colon x \in V_n(i)\}$. This gives $f\colon X \to D$ satisfying $f^{-1}(B_n(i)) = V_n(i)$ and, consequently, $f^{-1}(B_n(i,j)) = V_n(i,j)$. Thus f is continuous and (i) and (iii) imply that $f^{-1}(B_1(i)) = V(i)$, $f^{-1}(D \setminus \{0\}) = V(1)$ and $X \setminus f^{-1}(B_n(i,j)) = X \setminus V_n(i,j) = V_n(i) \setminus V_{n+1}(j) \subset V(1) \setminus V(j)$.

Proof of 1.1. Suppose that X is a perfect space and $\{U_\alpha\colon \alpha \in \mathfrak{m}\}$ is a collection of open subsets of X such that $\{\alpha \in \mathfrak{m}\colon x \in U_\alpha\}$ is finite for $x \in X$.

Since X is a perfect space, one can construct, for $k \geqslant 1$, collections $\{E_\alpha(k)\colon \alpha \in \mathfrak{m}\}$ of closed subsets of X which are locally finite as indexed collections and satisfy $\bigcup_{k \geqslant 1} E_\alpha(k) = U_\alpha$ for $\alpha \in \mathfrak{m}$ (if X is developable and $\{\mathscr{U}_k\}_{k \geqslant 1}$ is a development of X, then $E_\alpha(k) = X \setminus \mathrm{St}(X \setminus U_\alpha, \mathscr{U}_k)$ satisfy the above conditions).

For $\alpha \in \mathfrak{m}$ put $V_\alpha(1) = U_\alpha$ and $V_\alpha(i) = U_\alpha \setminus \bigcup_{k<i} E_\alpha(k)$ for $i > 1$. Let $f_\alpha\colon X \to D$

be a mapping satisfying the requirements of Lemma 1.3 with respect to the sequence $\{V_\alpha(i)\}_{i\geq 1}$.

Since $f_\alpha^{-1}(D\setminus\{0\}) = U_\alpha$, it follows that $f = \underset{\alpha\in\mathfrak{m}}{\triangle} f_\alpha\colon X\to S$ and $f^{-1}(H_\alpha) = U_\alpha$. Thus it remains to prove that f is a continuous function.

From the definition of the topology of S it follows that it is sufficient to show that, for any (subbasic) neighbourhood B of 0 in D, $\{f_\alpha^{-1}(B)\colon \alpha\in\mathfrak{m}\}$ is interior-preserving in X.

Take $B = B_n(i,j)$. For $\alpha\in\mathfrak{m}$, $X\setminus f_\alpha^{-1}(B_n(i,j))\subset V_\alpha(1)\setminus V_\alpha(j)\subset \bigcup_{k<j} E_\alpha(k)$. Thus $\{X\setminus f_\alpha^{-1}(B)\colon \alpha\in\mathfrak{m}\}$ is locally finite in X and consequently $\{f_\alpha^{-1}(B)\colon \alpha\in\mathfrak{m}\}$ is interior-preserving.

1.4. Remark. If X is additionally assumed to be collectionwise normal, then $S(\mathfrak{m})$ can be replaced by $K(\mathfrak{m})$ in 1.1.

1.5. Remark. Let $Y(\mathfrak{m}) = \{y\in D^{\mathfrak{m}}\colon |\{\alpha\in\mathfrak{m}\colon y(\alpha)\notin B\}|<\aleph_0$ for each neighbourhood B of 0 in $D\}$ and consider Y with the topology generated in the same way as the topology of S. It can be checked that Y is a quasi-developable [Be] T_1-space with a point-countable base of cardinality $\mathfrak{m}$. If X is a semi-stratifiable space with a point-countable collection $\{U_\alpha\colon \alpha\in\mathfrak{m}\}$ of open sets, then one can use [Ch3, 4.4] in order to define sets $E_\alpha(k)$ which allow us to construct, as in the proof of 1.1, $f\colon X\to Y$ such that $U_\alpha = f^{-1}(\{y\in Y\colon y(\alpha)\neq 0\})$. Thus $Y(\mathfrak{m})^{\aleph_0}$ contains topologically all metalindelöf developable T_1-spaces of weight $\mathfrak{m}$. Unfortunately, it can be shown that $Y(\mathfrak{m})$ is not a perfect space (even for $\mathfrak{m} = \omega_0$).

2. The space $Z(\mathfrak{m})$. The existence of $f\colon X\to S(\mathfrak{m})$ satisfying the requirements of 1.1 was based on the possibility of representing each U_α as the union of a countable collection $\{E_\alpha(k)\}_{k\geq 1}$ of closed subsets of X such that $\{E_\alpha(k)\colon \alpha\in\mathfrak{m}\}$ was locally finite in X for $k\geq 1$.

If $\{U_\alpha\colon \alpha\in\mathfrak{m}\}$ is an interior-preserving collection of open subsets of a semi-stratifiable space X and $P(a) = \bigcap\{U_\alpha\colon \alpha\in a\}\setminus\bigcup\{U_\alpha\colon \alpha\notin a\}$ for $a\subset\mathfrak{m}$, then each $P(a)$ can be represented as the union of a countable collection $\{E(a,k)\}_{k\geq 0}$ of closed sets such that $\{E(a,k)\colon a\subset\mathfrak{m}\}$ is discrete in X for $k\geq 0$ [J, 4.8]. We shall use this observation in order to generalize the construction of $T(\mathfrak{m})$ and $T'(\mathfrak{m})$ from [Ch2, Theorem 1, Remark 6].

Let $Z(\mathfrak{m}) = \mathscr{P}(\mathfrak{m})\times D$ and put

$$G(a,i) = \{(b,d)\in Z\colon a\subset b \text{ and } (a\neq b \Rightarrow d(1)\geq i)\} \quad \text{for } a\subset\mathfrak{m} \text{ and } i\geq 0,$$
$$G_n(i) = \mathscr{P}(\mathfrak{m})\times B_n(i) \quad \text{for } n, i\geq 1,$$
$$G_n(i,j) = \mathscr{P}(\mathfrak{m})\times B_n(i,j) \quad \text{for } n, i, j\geq 1.$$

Consider Z with the topology obtained by taking the sets defined above as a subbase of Z.

One can modify the proof of the developability of $T(\mathfrak{m})$ from [Ch2] in order to show that Z is an orthocompact developable T_1-space. Clearly, the weight of Z is $2^{\mathfrak{m}}$.

For $\alpha\in\mathfrak{m}$, put $G_\alpha = G(\{\alpha\},0)$. Clearly, $\{G_\alpha\colon \alpha\in\mathfrak{m}\}$ is an interior-preserving collection of open subsets of Z. We have (see [Ch2, Theorem 1])

2.1. Theorem. *If X is a semi-stratifiable space and $\{U_\alpha\colon \alpha\in\mathfrak{m}\}$ an interior-preserving collection of open subsets of X, then there exists a $g\colon X\to Z(\mathfrak{m})$ such that $g^{-1}(G_\alpha) = U_\alpha$ for $\alpha\in\mathfrak{m}$.*

2.2. Corollary. *The space $Z(\mathfrak{m})^{\aleph_0}$ contains topologically all orthocompact developable T_1-spaces of weight $\mathfrak{m}$.*

Proof of 2.1. Suppose that $\{U_\alpha\colon \alpha\in\mathfrak{m}\}$ is an interior-preserving collection of open subsets of a semi-stratifiable space X. Let $P(a)$ and $\{E(a,k)\colon a\subset\mathfrak{m}\}$ for $k\geq 0$ be as in the introduction to this section.

Put $V(i) = X\setminus\bigcup\{E(a,k)\colon a\subset\mathfrak{m} \text{ and } k<i\}$ and let $f\colon X\to D$ be a mapping satisfying the requirements of 1.3 with respect to the sequence $\{V(i)\}_{i\geq 1}$. Define $g(x) = (\{\alpha\colon x\in U_\alpha\}, f(x))\in Z$.

Clearly, $g^{-1}(G_\alpha) = U_\alpha$ and the proof of the continuity of g reduces to a simple observation that

$$g^{-1}(G(a,i)) = \bigcap\{U_\alpha\colon \alpha\in a\}\setminus\bigcup\{E(b,k)\colon b\neq a \text{ and } k<i\}.$$

2.3. Remark. If $E(\emptyset,0) = P(\emptyset)$, then $g(X)\subset\{(a,d)\in Z\colon a\neq\emptyset \text{ or } d = 0\}$. Since $P(\emptyset)$ is closed, we can always assume that $E(\emptyset,0) = p(\emptyset)$.

2.4. Remark. If $\{U_\alpha\colon \alpha\in\mathfrak{m}\}$ is point-finite as an indexed collection, then $g(X)\subset \mathrm{Fin}(\mathfrak{m})\times D = T(\mathfrak{m})\subset Z(\mathfrak{m})$ (see [Ch2, Theorem 1]).

2.5. Remark. If $\{X\setminus U_\alpha\colon \alpha\in\mathfrak{m}\}$ is locally finite as an indexed collection then $g(X)\subset\{(a,d)\in Z\colon \mathfrak{m}\setminus a\in \mathrm{Fin}(\mathfrak{m})\}$ (see [Ch2, Remark 6]).

2.6. Remark. A weak form of Lemma 1.3 (sufficient for proving 2.1) can be obtained from 2.1 by observing that $\{U_\alpha\colon \alpha\in\omega_0\}$, where $U_0 = X$ and $U_\alpha = V(\alpha)$ for $1\leq\alpha<\omega_0$, is point-finite, $p\colon T(\mathfrak{m})\to D$ defined by $p(a,d) = (|a|, d)$ is continuous and consequently $f = p\circ g\colon X\to D$, where $g\colon X\to T(\omega_0)$ is given by 2.1, satisfies $f^{-1}(B_n(i)) = V(i)$ for $i\geq 1$.

2.7. Remark. The function $f\colon T(\mathfrak{m})\to S(\mathfrak{m})$ defined by $f(a,d)(\alpha) = p(a,d)$ if $\alpha\in a$ and $f(a,d)(\alpha) = 0$ if $\alpha\notin a$ is continuous. The restriction of f to $\{(a,d)\in T\colon a\neq\emptyset \text{ or } d = 0\}$ (see 2.3) is a homeomorphic embedding of this subspace of $T(\mathfrak{m})$ into $S(\mathfrak{m})$.

3. A developable T_1-compactification of D and extensions of mappings. The results of [Ch2] and the first section of this paper show that D can be considered to be a generalization of the real line (a d-line). It is easy to observe that $\{B_1(i,1)\colon i\geq 1\}$ is a countable open cover of D with no finite subcover. We shall construct a developable T_1-compactification D^* of D (a d-interval).

Let $\omega N = N\cup\{\omega\}$ be the Alexandroff compactification of N and consider ωN with a well-order generated by the natural homeomorphism of ωN onto ω_0+1.

Put $D^* = \{d\in \omega N^{N_+}\colon d(n+1) = \omega \text{ implies } d(n) = 0 \text{ for } n\geq 1\}$. Consider D

with the topology generated by the sets $B_n(i)$ and $B_n(i,j)$ defined in D^* for all $n, i, j \geqslant 1$ by the formulas defining the corresponding subbasic sets in D.

It is easy to see that D^* is a T_0-space and that each of the sets generating the topology of D is an F_σ-set. Thus D^* is a developable T_1-space of countable weight. Moreover, we have

3.1. PROPOSITION. *The space D^* is a T_1-compactification of D.*

Proof. Clearly, D is a dense subset of D^*. Suppose that D is not compact. Then there exists an open cover $\mathcal{U}$ of D^* such that no finite subcollection of $\mathcal{U}$ covers D^* and $\mathcal{U}$ consists of subbasic sets of D^* [E, 3.12.2].

For $n \geqslant 1$ let $d(n)$ be the lowest upper bound of $\{i \geqslant 1: B_n(i,j) \in \mathcal{U}$ for a $j \geqslant 1\}$ in ωN.

Observe that, since $B_n(i,j) \cup B_{n+1}(j,k) = D^*$ and $B_n(i) \cup B_n(i,j) = D^*$, it follows that

(*) if $B_n(i,j) \in \mathcal{U}$, then $d(n+1) < j$,

(**) if $B_n(i) \in \mathcal{U}$, then $d(n) < i$.

As a consequence of (*), we infer that $d(n) > 0$ implies $d(n+1) \neq \omega$, which means that $d = \{d(n)\}_{n \geqslant 1} \in D^*$.

Let B be an element of $\mathcal{U}$ containing d. Suppose that $B = B_n(i)$. Then, by virtue of (**), $d(n) < i$ and $d \notin B$. Thus B is of the form $B_n(i,j)$ and (*) shows that $d(n+1) < j$. From the definition of $d(n)$ we obtain $d(n) \geqslant i$ and, consequently, $d \notin B$. The contradiction shows that D^* is compact.

In [Ch2, Corollary 2] we proved that, if A is a closed subset of a perfect space X and g is a mapping of A into D, then there exists a mapping $f\colon X \to D$ which is an extension of g. The same reasoning can be used in order to prove the non-trivial part of the following characterization of perfect spaces (an analogous characterization of perfectly normal spaces is implicitly contained in the formulation of the Tietze extension theorem in [K, XII,5,Theorem 1]).

3.2. THEOREM. *A space X is perfect iff for each closed subset A of X and $g: A \to D^*$ there exists an extension $f\colon X \to D^*$ of g such that $f(X \setminus A) \subset D$.*

It is natural to expect that any mapping g of a closed subset A of a d-normal [Ch3] ($= D$-normal [Br2]) space X into D (D^*) can be extended to $f\colon X \to D$ (D^*). In [Ch2, Remarks 3.4] we have observed that such an extension exists if A is a G_δ-set (the same method can be applied to mappings into D^*). It turns out that the assumption that A is a G_δ-set is essential.

3.3. EXAMPLE. A normal space X containing a closed subset A such that there exists a mapping $g: A \to D$ which cannot be extended to any mapping $f\colon A \to D^*$.

Let $X = R_Q$ [E, 5.1.32]. Put $A = Q$ and let $\{q_m\}$: $m \geqslant 1$ be an enumeration of Q. The function $g: A \to D$ defined by $g(q_m) = (1, m, m, m, \ldots)$ is continuous because $g^{-1}(B)$ is either empty or co-finite for subbasic subsets of D.

Suppose that $f\colon X \to D^*$ is an extension of g. Consider $f^{-1}(B_1(1))$ and $f^{-1}(B_2(i))$ for $i \geqslant 1$. Each of these sets contains an open and dense subset of R and consequently the intersection of these sets contains a point $x \in R$. For $d = f(x)$ we have $d(1) \geqslant 1$ and $d(2) = \omega$, which contradicts the fact that $f(x) \in D^*$.

The possibility of extending mappings defined on a closed subset A of a d-normal space X can be characterized in terms of D-open sets [Br1] ($=$ inverse images of open subsets of D [Ch2, Remark 3]).

3.4. THEOREM. *For a closed subset A of a d-normal space X the following conditions are equivalent:*

(i) *any $g: A \to D$ has an extension $f\colon X \to D$,*

(ii) *any $g: A \to D$ has an extension $f\colon X \to D^*$,*

(iii) *if $\{V(i)\}_{i \geqslant 1}$ is a decreasing sequence of D-open subsets of A and $\bigcap_{i \geqslant 1} V(i) = \emptyset$, then there exists a decreasing sequence $\{W(i)\}_{i \geqslant 1}$ of D-open subsets of X such that $\bigcap_{i \geqslant 1} W(i) = \emptyset$ and $W(i) \cap A = V(i)$ for $i \geqslant 1$.*

Proof. (i) $\Rightarrow$ (ii) is obvious. Suppose that (ii) is satisfied and let $\{V(i)\}_{i \geqslant 1}$ be as in (iii). The proof of Lemma 1.3 shows that there exists a mapping $h\colon A \to D$ such that $h^{-1}(B_1(i)) = V(i)$. Consider $g: A \to D$ defined by $g(x) = (1, h(x))$ for $x \in A$. Let $f\colon X \to D^*$ be an extension of g and put $W(i) = f^{-1}(B_1(1) \cap B_2(i))$ for $i \geqslant 1$. It is easy to check (see the reasoning in 3.3) that $\{W(i)\}_{i \geqslant 1}$ has the required properties.

The proof of (iii) $\Rightarrow$ (i) is a modification of the proof of Theorem 2 in [Ch2].

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Received 30 June 1982