A hereditarily indecomposable, hereditarily non-chainable planar tree-like continuum

by

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Abstract. In [1] R. D. Anderson announced the existence of a hereditarily indecomposable tree-like continuum in the plane containing no chainable subcontinuum. Anderson's example was never published. In [9], Ingram, exploiting constructions from his earlier papers [6] and [8], produced a continuum satisfying all of the conditions of Anderson's example except planarity. In this paper we modify Ingram's construction to produce a planar example **.

The idea of the construction is like Ingram's in [9]: We begin with a hereditarily indecomposable continuum of positive span. Into every chainable subcontinuum, another hereditarily indecomposable continuum of positive span is inserted, yielding in the end a continuum of hereditarily positive span and hence hereditarily non-chainable. The difference between our construction and Ingram's is that he inserts the same continuum everywhere, whereas we choose our inserted continua from his planar family [8], in order to get the whole construction to fit in the plane. Moreover, Ingram inserts triods, with infinite rays converging to them, into his factor spaces, which makes these spaces non-planar. We will use finite approximations of such rays and indeed our factor spaces will be trees (and hence planar). In what follows we will assume that the reader is familiar with Ingram's papers [6], [7], [8] and [9].

§ 1. **Preliminaries.** A *continuum* is a compact connected metric space. A continuum X is said to be *tree-like* if it can be written as the inverse limit of a sequence of finite trees. Alternatively, X admits arbitrarily fine open covers whose nerves are finite trees (see [12]). Such covers are called *tree-chains*. A mapping f between trees is said to be *monotone* if for every point y in the range of f, $f^{-1}(y)$ is connected (and hence a continuum). This is equivalent to saying that for every continuum K in the range of f, $f^{-1}(K)$ is a continuum (see [10], p. 131). f is said to be *atomic* if for every continuum A in the domain of f such that $f(A)$ is non-degenerate, $A = f^{-1}(f(A))$. Atomic maps were introduced by Cook in [4]. They are known to be monotone (see [5]). From this fact it easily follows that atomic maps are *hereditarily monotone*, i.e., monotone when restricted to any subcontinuum of the domain. The continuum X is said to be *hereditarily indecomposable* if for every

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pair P, Q, of subcontinua of X with non-void intersection, either $P \subset Q$ or $Q \subset P$. A continuum X is said to have *positive span* if there exists another continuum C and maps $f, g\colon C \to X$ such that $f(C) = g(C)$, but for every $x \in C$, $f(x) \neq g(x)$. Continua without positive span are said to have *span* 0. A continuum X is said to be *arc-like* or *chainable* if it can be written as the inverse limit of a sequence of arcs. (Alternatively X admits arbitrarily fine open covers whose nerves are topological arcs. Such covers are called *chains* or *simple chains*). It is not difficult to show that chainable continua have span 0 (1). Thus any continuum having positive span is non-chainable. The notion of span was introduced by Lelek in [11]. We will not use it directly in this paper, but will rely heavily on Ingram's results concerning it in [9]. We now present several lemmas needed in the next two sections.

LEMMA 1. *Let I be the unit interval* $[0, 1]$ *with the usual metric and let* $D = \{x_1, x_2, \ldots\}$ *be a dense subset of distinct points in I. Let $\varepsilon > 0$. Then there is a monotone mapping m of I onto I such that the points of I whose inverse images under m are non-degenerate are precisely the points of D and such that m differs from the identity map on I by less than ε (in the uniform metric).*

Proof. We leave it to the reader to show the existence of maps lacking only closeness to the identity. (Hint: Do the construction one step at a time using inverse limits. An inverse limit of arcs with monotone bonding maps is an arc. See [3].)

Now let $\{x_1, x_2, \ldots, x_n\}$ be an initial subset of D which is $\varepsilon/2$-dense in I. Relabel the x_i's if necessary so that $x_1 < x_2 < \ldots < x_n$. Let $I_1, I_2, \ldots, I_n$ be a disjoint collection of closed subintervals of $[0, 1]$ (each of diameter less than ε) such that $x_i \in I_i$ for $i = 1, 2, \ldots, n$. m will map each interval I_i to x_i and each open interval $[x_i, x_{i+1}] - (I_i \cup I_{i+1})$ in a monotone fashion onto the interval (x_i, x_{i+1}) in such a way that precisely the points of D in (x_i, x_{i+1}) have non-degenerate pre-images. It is not difficult to verify that each point of I is moved less than ε by m. ■

The following results were proved by Cook in [4].

LEMMA 2. *Let Y be a hereditarily indecomposable continuum and let X be a continuum which admits an atomic mapping f onto Y such that for every $y \in Y$, $f^{-1}(y)$ is either a point or a hereditarily indecomposable continuum. Then X is hereditarily indecomposable.*

LEMMA 3. *If the continuum X is an inverse limit of continua with atomic bonding maps, then the projection map from X onto any of the factor spaces is atomic.*

The following result was proved by Ingram in [9].

LEMMA 4. *Let Y be a continuum with positive span and let X be a continuum which admits a monotone mapping onto Y. Then X has positive span.*

LEMMA 5 (hooking up triods). *Let $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$ be two (finite) increasing sequences of points in $I = [0, 1]$. Let $T_1, T_2, \ldots, T_n$ be disjoint simple triods in the interior of the cell $I \times I$. For each $i = 1, 2, \ldots, n$ let a_i and b_i be*

(1) One of the major unsolved questions in continua theory is whether continua of span 0 are chainable.

distinct endpoints of T_i. Then there are $2n$ disjoint arcs $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ in $I \times I$ such that for each $i = 1, 2, \ldots, n$ one endpoint of A_i is the point $(0, x_i)$ and the other is a_i, and one endpoint of B_i is the point $(1, y_i)$ and the other is b_i.

Proof. Let $S_1, S_2, \ldots, S_n$ be disjoint simple closed curves in the interior of $I \times I$ whose bounded complementary domains are disjoint and contain $T_1, T_2, \ldots, T_n$ respectively. Let h be a homeomorphism of $I \times I$ onto itself which fixes the boundary of the cell and sends each S_i onto a circle with center $(\frac{1}{2}, x_i)$. It is easy to draw the desired arcs in this transformed setting. Applying h^{-1} yields the desired arcs in the original setting. ■

§ 2. The inverse system. The desired example will be the inverse limit of a doubly indexed system of finite trees $X_{i,j}$ $(i, j = 1, 2, 3, \ldots)$ where the pairs (i, j) are ordered by the product partial order (i.e., $(i_1, j_1) < (i_2, j_2)$ if and only if $i_1 < i_2$ and $j_1 < j_2$). Bonding maps $g_{i,j}\colon X_{i,j+1} \to X_{i,j}$ and $f_{i,j}\colon X_{i+1,j} \to X_{i,j}$ will be defined for all $i, j = 1, 2, 3, \ldots$ The whole diagram will commute, so other bonding maps can be defined by composition. The spaces $X_{i,1}$ will be simple triods for all $i = 1, 2, \ldots$ and the maps $f_{i,1}$ will be "crooked Ingram maps" à la [8], modified slightly using Lemma 1, so that the diagram will commute. The $g_{i,j}$'s will be monotone maps used to insert triods. The maps $f_{i,j}$ will be modified crooked Ingram maps on the inserted triods. Now for the details of the construction.

For each $i = 1, 2, \ldots$ let f_i' be a "crooked Ingram map" of the triod $X_{i+1,1}$ onto $X_{i,1}$; i.e. f_i' is a composition of one of the maps f or g and a crookedness map as defined in [8]. Choose the f_i''s so that the inverse limit of the triods $X_{i,1}$ with bonding maps f_i' is a hereditarily indecomposable continuum $X_{\infty,1}$ of positive span and such that every proper subcontinuum of $X_{\infty,1}$ is a pseudo-arc.

Note that the mappings f_i' may be taken to be piecewise linear. We will choose them so that each triod $X_{i+1,1}$ can be written as a union of simple arcs meeting only at their endpoints and such that f_i' restricted to any one of these arcs is a homeomorphism. We will call these the *straight arcs* of $X_{i+1,1}$. The bonding maps $f_{i,1}\colon X_{i+1,1} \to X_{i,1}$ will be obtained from the maps f_i' by modifying them on the straight arcs of $X_{i+1,1}$ using Lemma 1. Note further that for any $x \in X_{i,1}$ and for any $j > i$, the pre-image of x in $X_{j,1}$ under the appropriate composition of maps f' is finite.

Now let $D_{1,1}$ be a countable dense subset of $X_{1,1}$ such that:

(i) No point of $D_{1,1}$ is an endpoint or the junction point of $X_{1,1}$, and

(ii) For every $x \in D_{1,1}$ and for every $i > 1$, each point of the pre-image of x in $X_{i,1}$ lies in the interior of a straight arc of $X_{i,1}$.

The elements of $D_{1,1}$ will be called the *primary insertion points* of $X_{1,1}$. (Other $X_{i,j}$'s will have primary and secondary insertion points.) We will assume that $D_{1,1}$ and the other sets $D_{i,j}$ of primary insertion points to be defined below are equipped with enumerations, so that we can speak of the "first", "second", etc. elements of $D_{i,j}$. Using Lemma 1, define the map $f_{1,1}\colon X_{2,1} \to X_{1,1}$ so that for any straight arc A of $X_{2,1}$, $f_{1,1}|A$ is monotone, and precisely the points of $D_{1,1}$ in the range of $f_{1,1}|A$ have non-degenerate pre-images in A. Moreover, define $f_{1,1}$ so that it differs from f'_1 by the (as yet unspecified) positive number ε_1. Thus for each $x \in D_{1,1}$, $f_{1,1}^{-1}(x)$ is a finite disjoint union of arcs in $X_{2,1}$, each arc lying in the interior of a straight arc of $X_{2,1}$. The midpoints of these arcs will be called the *secondary insertion points* of $X_{2,1}$. The *primary insertion points* of $X_{2,1}$ will be a countable dense subset $D_{2,1}$ of the interior of $f_{1,1}^{-1}(D_{1,1})$, disjoint from the set of secondary insertion points and such that:

(i) No point of $D_{2,1}$ is an endpoint or the junction point of $X_{2,1}$, and

(ii) For every $x \in D_{2,1}$ and for every $i > 2$, each point of the pre-image of x in $X_{i,1}$ (under the appropriate composition of maps f') lies in the interior of a straight arc of $X_{i,1}$.

The union of the sets of primary and secondary insertion points in $X_{2,1}$ will be denoted $E_{2,1}$ and will be called simply the set of *insertion points* of $X_{2,1}$. Note that $E_{2,1}$ is dense in $X_{2,1}$.

The general idea behind the insertion points is that they are the places where triods will eventually be inserted via the monotone maps $g_{i,j}$. We will always want $E_{i,j}$ to be dense in $X_{i,j}$. The primary insertion points are points where insertions are "started". They must be defined for the parts of $X_{i,j}$ which "can't be seen" in the previous factor spaces. These parts will be either arcs or triods which are collapsed to points by the bonding maps running out of $X_{i,j}$. The secondary insertion points are points where insertions are "continued", yielding hereditarily indecomposable continua of positive span as we pass to the inverse limit down a column.

We may now inductively define the maps $f_{i,1}$. Suppose that the sets $D_{n,1}$ and $E_{n,1}$ have been defined. Then using Lemma 1, define $f_{n,1}\colon X_{n+1,1} \to X_{n,1}$ by modifying f'_n on each straight arc A of $X_{n+1,1}$ in such a way that $f_{n,1}|A$ is monotone and precisely the points of $E_{n,1}$ in $X_{n,1}$ have non-degenerate pre-images in $X_{n+1,1}$. Moreover, define $f_{n,1}$ so that it differs from f'_n by less than the (as yet unspecified) positive number ε_n. Then for each insertion point x in $X_{n,1}$, $f_{n,1}^{-1}(x)$ will be a finite union of simple arcs, each lying in the interior of a straight arc in $X_{n+1,1}$. The midpoints of these arcs will be called the *secondary insertion points* of $X_{n+1,1}$. The set of *primary insertion points* will be a countable set $D_{n+1,1}$, disjoint from the set of secondary insertion points, dense in the interior of $f_{n,1}^{-1}(E_{n,1})$, and satisfying conditions analogous to (i) and (ii) above for $D_{2,1}$. The union of the sets of primary and secondary insertion points, denoted $E_{n+1,1}$, will be called simply the set of *insertion points* of $X_{n+1,1}$. $E_{n+1,1}$ will be dense in $X_{n+1,1}$.

It remains to specify the numbers ε_i. By Lemma 1 they may be chosen as small as we like. Using Brown's theorem [2] (2), choose the ε_i's so that the inverse limit of the sequence $(X_{n,1}; f_{n,1})$ is homeomorphic to the inverse limit of the sequence $(X_{n,1}; f'_n)$. Call this space $X_{\infty,1}$.

Now to define the second column. $X_{1,2}$ will look like $X_{1,1}$ except that a simple

(2) Because of the use of induction in the definition of the $f_{n,1}$'s, the hypotheses for Brown's theorem are not exactly satisfied. However, the proof of the theorem will go through in our setting.

triod will be inserted at the first point x of $D_{1,1}$ as follows: x divides $X_{1,1}$ uniquely into two finite trees whose union is $X_{1,1}$ and whose intersection is $\{x\}$. Form the disjoint union of the two trees. $X_{1,2}$ will be this union together with a simple triod, one of whose endpoints has been identified with the copy of x in one of the trees and another of whose endpoints has been identified with the copy of x in the other tree. The mapping $g_{1,1}\colon X_{2,1} \to X_{1,1}$ will be the natural monotone map collapsing the inserted triod to the point x. $X_{2,2}$ will be constructed from $X_{2,1}$ by inserting simple triods at each secondary insertion point in the pre-image of x under $f_{1,1}$. The mapping $g_{2,1}\colon X_{2,2} \to X_{2,1}$ will be the natural map collapsing the inserted triods to points. Continue down the second column in this way: $X_{n,2}$ will be constructed from $X_{n,1}$ by inserting triods at the secondary insertion points in the pre-image of x under the appriopriate composition of maps $f_{i,1}$. The mapping $g_{n,1}$: $X_{n,2} \to X_{n,1}$ will be the natural monotone map collapsing the inserted triods to points.

Now to define the mappings $f_{i,2}$ and the sets of primary and secondary insertion points in the $X_{i,2}$'s. First note that the secondary insertion points in any $X_{i,1}$, $i \geq 2$, at which triods were inserted, lie in the middle of little arcs in $X_{i,1}$ which collapse to points under $f_{i-1,1}$. The little arcs thus get split into two pieces in $X_{i,2}$. We will call these pieces the *arcs adjacent* to the corresponding inserted triods in $X_{i,2}$. Thus each $X_{i,2}$, $i \geq 2$, falls naturally into three parts: the inserted triods, the arcs adjacent to the inserted triods, and the rest of the space. Off the inserted triods and the arcs adjacent to them, each map $f_{n,2}\colon X_{n+1,2} \to X_{n,2}$ will look just like $f_{n,1}$, so that the following diagram commutes:

$$\begin{array}{ccc} X_{n,1} & \xleftarrow{\;g_{n,1}\;} & X_{n,2} \\ {\scriptstyle f_{n,1}}\big\uparrow & & \big\uparrow{\scriptstyle f_{n,2}} \\ X_{n+1,1} & \xleftarrow{\;g_{n+1,1}\;} & X_{n+1,2} \end{array}$$

Each arc adjacent to an inserted triod in $X_{n+1,2}$ will be mapped by $f_{n,2}$ *onto* the "corresponding" inserted triod in $X_{n,2}$, so that the above diagram continues to commute. Finally, each inserted triod in $X_{n+1,2}$ will map under $f_{n,2}$ onto the corresponding triod in $X_{n,2}$ so that the diagram continues to commute. We also need to define the maps on the three parts so that they match up at the overlaps, preserving continuity. The mappings on the inserted triods will be modified crooked Ingram maps defined as on the triods in the first column. Primary and secondary insertion points will also be defined for the inserted triods as they were for the triod in the first column. The mappings of the triods need not be exactly the same maps as in the first column. We will need some freedom in choosing the maps when we embed this system in the plane. Rather the maps need to be chosen in the same way as the maps in the first column. In particular they need to be chosen (using Brown's theorem) so that the inverse limit of any sequence of inserted triods running down the second column is a hereditarily indecomposable Ingram continuum [8] of positive span. In addition to the primary and secondary insertion points defined for the inserted triods, each $X_{n,2}$ will contain insertion points which map to the insertion points in $X_{n,1}$ under $g_{n,1}$. We will not give these insertion points any special name. We just think of them as the pre-images of as yet "unused" insertion points from $X_{n,1}$. Note that the bonding map $g_{n,1}$ is one-to-one on these points, so we may think of them as being "the same" as their images in $X_{n,1}$.

We have now inserted an uncountable family of hereditarily indecomposable continua of positive span into $X_{\infty,1}$. That is, $X_{\infty,2}$, the inverse limit of the second column, contains such a family. In the end we will want to choose the maps on the inserted triods so that the resulting family is planar. Below we will show that the induced map from $X_{\infty,2}$ to $X_{\infty,1}$ satisfies the hypotheses of Lemma 2, making $X_{\infty,2}$ hereditarily indecomposable. This will be true for all continua $X_{\infty,n}$, making our example $X_{\infty,\infty}$ hereditarily indecomposable.

We must now set up an induction for constructing each column of spaces $X_{i,n}$, $i = 1, 2, \ldots$ Each column will insert another family of Ingram continua into the inverse limit. The location of the insertion will be determined by an unused primary insertion point x in some previously defined factor space $X_{i(n),j(n)}$. A triod will be inserted at x (or more precisely, at its pre-image in $X_{i(n),n-1}$) and at all of the secondary insertion points associated with x running down the column, as was done in the construction of the second column. We need to arrange the induction so that all primary insertion points (and hence all secondary insertion points) in all factor spaces are eventually taken care of. Thus we need a numbering scheme, starting with 2 (the second column) which will tell us where to make the successive insertions. This will be a sequence of pairs of indices, $(i(2), j(2))$, $(i(3), j(3))$, $(i(4), j(4))$, ... satisfying the following conditions:

(1) $(i(2), j(2)) = (1, 1)$.

(2) For all pairs (i, j), $i, j = 1, 2, \ldots$, there are infinitely many natural numbers n such that $(i(n), j(n)) = (i, j)$.

(3) For all n, $j(n) < n$.

The first condition is just for consistency. It means that the construction already described for the second column was done in the right place, i.e. it involved an insertion at a primary insertion point in $X_{1,1}$. The second condition assures that there will be infinitely many columns reserved for the infinitely many primary insertion points in each factor space. The third condition is another consistency condition, guaranteeing that the space into which an insertion is to be made lies to the left of the nth column, i.e. has already been defined.

Now for the details of the induction. Suppose that all columns up to and including the nth column have been constructed. For all $i < i(n+1)$, let $X_{i,n+1} = X_{i,n}$ and let $g_{i,n}\colon X_{i,n+1} \to X_{i,n}$ be the identity map. For all $i < i(n+1)-1$, let $f_{i,n+1} = f_{i,n}$. The rest of the column, from $X_{i(n+1),n+1}$ down, will be constructed as the second column was from the first. $X_{i(n+1),n+1}$ will be constructed from $X_{i(n+1),n}$ by inserting a triod at the pre-image in $X_{i(n+1),n}$ of the first unused primary insertion point in $X_{i(n+1),j(n+1)}$. The spaces $X_{i,n+1}$, $i = i(n+1)+1, i(n+1)+2, \ldots$ will be constructed from the spaces $X_{i,n}$ by inserting triods at the pre-images in $X_{i,n}$ of the secondary insertion points in $X_{i,j(n+1)}$ associated with the given primary insertion point in $X_{i(n+1),j(n+1)}$. The mappings $g_{i,n}$ and $f_{i,n+1}$, $i = i(n+1), i(n+1)+1, i(n+1)+2, \ldots$ will be defined as they were for the second column, in particular so that the diagrams

$$\begin{array}{ccc} X_{i,n} & \xleftarrow{\;g_{i,n}\;} & X_{i,n+1} \\ {\scriptstyle f_{i,n}}\uparrow & & \uparrow{\scriptstyle f_{i,n+1}} \\ X_{i+1,n} & \xleftarrow[\;g_{i+1,n}\;]{} & X_{i+1,n+1} \end{array}$$

commute. Primary and secondary insertion points will also be defined in the inserted triods as they were in the second column. Finally, the map $f_{i(n+1)-1,n+1}$ will collapse the inserted triod in $X_{i(n+1),n+1}$ to a point and otherwise will look just like $f_{i(n+1)-1,n}$ so that the diagram

$$\begin{array}{ccc} X_{i(n+1)-1,n} & \xleftarrow{\;g_{i(n+1)-1,n}\;} & X_{i(n+1)-1,n+1} \\ {\scriptstyle f_{i(n+1)-1,n}}\uparrow & & \uparrow{\scriptstyle f_{i(n+1)-1,n+1}} \\ X_{i(n+1),n} & \xleftarrow[\;g_{i(n+1),n}\;]{} & X_{i(n+1),n+1} \end{array}$$

commutes (recall that $g_{i(n+1)-1,n}$ is the identity map).

§ 3. **The example.** In this section we will show that any space $X_{\infty,\infty}$ which is the inverse limit of an inverse system as described in § 2 is a hereditarily indecomposable, tree-like continuum of hereditarily positive span. In the next section we will show that by choosing the mappings of the inserted triods properly, we can make $X_{\infty,\infty}$ embeddable in the plane. In what follows $X_{\infty,n}$ will denote the inverse limit of the nth column of spaces $X_{i,n}$ with bonding maps $f_{i,n}$, and $p_{m,n}$ will denote the projection of $X_{\infty,n}$ onto the factor space $X_{m,n}$. $g_{\infty,n}$ will denote the map from $X_{\infty,n+1}$ to $X_{\infty,n}$ induced by the maps $g_{i,n}$ between the $(n+1)$st and nth columns.

PROPOSITION 1. *$X_{\infty,\infty}$ is hereditarily indecomposable.*

Proof. $X_{\infty,1}$ is hereditarily indecomposable by construction. We will show that each of the maps $g_{\infty,n}$ satisfies the hypotheses of Lemma 2, implying that all of the spaces $X_{\infty,n}$ are hereditarily indecomposable. Since $X_{\infty,\infty}$ is the inverse limit of these spaces, it follows easily that $X_{\infty,\infty}$ is hereditarily indecomposable.

Let C be a subcontinuum of some $X_{\infty,n+1}$ such that $g_{\infty,n}(C)$ is a non-degenerate subcontinuum of $X_{\infty,n}$. Let $z \in g_{\infty,n}^{-1}(g_{\infty,n}(C))$. We wish to show that $z \in C$. Suppose not. Then there is an m such that $p_{m,n+1}(z) \notin p_{m,n+1}(C)$. We may also assume that $g_{m,n}(p_{m,n+1}(C)) = p_{m,n}(g_{\infty,n}(C))$ is non-degenerate. Further, by hypothesis

$$g_{m,n}(p_{m,n+1}(z)) \in g_{m,n}(p_{m,n+1}(C)).$$

This means that $p_{m,n+1}(z)$ must lie in a triod T which meets $p_{m,n+1}(C)$ and is collapsed to a point by $g_{m,n}$. Since $p_{m,n+1}(z) \notin p_{m,n+1}(C)$, $p_{m,n+1}(C)$ does not contain all of T. But then $p_{m+1,n+1}(C)$ cannot meet any triod T' which maps into T under

$f_{m,n+1}$, because the non-degeneracy of $g_{m,n}(p_{m,n+1}(C))$ guarantees that $p_{m+1,n+1}(C)$ must contain a point outside any such T' and the intervals adjacent to it. If $p_{m+1,n+1}(C)$ contained a point in some T', it would have to contain an entire interval adjacent to T' by connectedness. But then $p_{m,n+1}(C) = f_{m,n+1}(p_{m+1,n+1}(C))$ would contain all of T. Since $p_{m+1,n+1}(z) \notin p_{m+1,n+1}(C)$, $g_{m+1,n}$ is one-to-one away from collapsing triods and $p_{m+1,n+1}(z)$ belongs to some triod T' or an arc adjacent to a T'; it follows that $g_{m+1,n}(p_{m+1,n+1}(z)) \notin g_{m+1,n}(p_{m+1,n+1}(C))$, i.e. $p_{m+1,n}(g_{\infty,n}(z)) \notin p_{m+1,n}(g_{\infty,n}(C))$, a contradiction. Therefore $g_{\infty,n}$ is atomic. It is straightforward to verify that if $x \in X_{\infty,n}$, then $g_{\infty,n}^{-1}(x)$ is either a point or a hereditarily indecomposable Ingram continuum. ■

PROPOSITION 2. *$X_{\infty,\infty}$ has hereditarily positive span.*

Proof. Let $\pi_{m,n}$ denote the projection of $X_{\infty,\infty}$ onto $X_{m,n}$. Let C be a non-degenerate subcontinuum of $X_{\infty,\infty}$ and choose a pair m, n such that $\pi_{m,n}(C)$ is non-degenerate. Since the insertion points are dense in $X_{m,n}$, $\pi_{m,n}(C)$ must contain an insertion point x_m in its interior. $f_{m,n}^{-1}(x_m)$ consists of a finite union of arcs, each containing a secondary insertion point as its midpoint. We will call any one of these arcs a *full pre-image* of x_m. x_m separates $X_{m,n}$ uniquely into two open sets which we may think of as "right" and "left". Consequently $f_{m,n}^{-1}(X_{m,n})-\{x_m\}$ can be written as a union of two disjoint open sets, namely the set L of points mapping to the left of x_m under $f_{m,n}$ and the set R of points mapping to the right. Since x_m lies in the interior of $\pi_{m,n}(C)$, $\pi_{m+1,n}(C)$ must have non-void intersection with both L and R. A straightforward connectedness argument will show that $\pi_{m+1,n}(C)$ must contain a full pre-image of x_m. Thus $\pi_{m+1,n}(C)$ contains a secondary insertion point x_{m+1} in its interior such that $f_{m,n}(x_{m+1}) = x_m$.

Proceeding inductively with the above argument, we may find a point $x = (\ldots, x_m, x_{m+1}, x_{m+2}, \ldots)$ in $\pi_{\infty,n}(C) \subset X_{\infty,n}$ such that the x_{m+i}'s are all secondary insertion points associated with the same primary insertion point (or more precisely, the x_{m+i}'s are all associated with the pre-image in some $X_{k,n}$, $k \leqslant m$, of a primary insertion point in some $X_{k,l}$, $l \leqslant n$). Thus there is an $n' \geqslant n$ such that $X_{\infty,n'}$ contains an Ingram continuum T of positive span which maps onto x under the natural induced map. Since this map is atomic (see proof of the previous proposition; compositions of atomic maps are atomic) and $\pi_{\infty,n}(C)$ is non-degenerate, $T \subset \pi_{\infty,n'}(C)$. By Lemma 3, $\pi_{\infty,n'}$ is atomic. Therefore $\pi_{\infty,n'}|C$ is monotone, so by Lemma 4, C has positive span. ■

§ 4. **Planarity.** We will show that $X_{\infty,\infty}$ can be made planar by thinking of it as the inverse limit of spaces $X_{n,n}$ with bonding maps $f_{n,n} \circ g_{n+1,n}$. We will show how to embed a cover with nerve $X_{n+1,n}$ inside a cover with nerve $X_{n,n}$ (corresponding to the map $f_{n,n}$) and a cover with nerve $X_{n+1,n+1}$ into a cover with nerve $X_{n+1,n}$ (corresponding to the map $g_{n+1,n}$) for each n. $X_{\infty,\infty}$ will then be the intersection of the unions of these covers.

We note that Ingram's family [8] can be realized in the plane using covers whose open sets have simple closed curves for boundaries and such that the boundaries of adjacent open sets meet in exactly two points. Our covers will satisfy this same condition. Moreover, since our bonding maps on the inserted triods differ from Ingram's by an arbitrarily small amount, we may use the same covers that he does in [8] for embedding the inserted triods.

We will describe the embedding of a cover corresponding to $X_{2,2}$ into a cover corresponding to $X_{1,1}$ and of a cover corresponding to $X_{3,3}$ into a cover corresponding to $X_{2,2}$ and then indicate an inductive procedure for producing all of the remaining finer covers.

$X_{2,1}$ will embed in $X_{1,1}$ exactly as Ingram's X_3 embeds in X_1 [3]. In fact the whole column $X_{n,1}$ will just be a collection of Ingram covers (for $n>2$ these covers will not figure directly into the description of $X_{\infty,\infty}$, but we will need to look at them to see how to embed other covers). Now the cover for $X_{2,2}$ will look just like the cover for $X_{2,1}$, except that some of the open sets will get replaced by little "triod chains" which fit inside those open sets. These correspond to the inserted triods in $X_{2,2}$ which map to points under $g_{2,1}$. To embed $X_{3,2}$ into $X_{2,2}$, first look to see how Ingram's cover $X_{3,1}$ embeds in $X_{2,1}$. $X_{3,2}$ will look exactly the same except at those open sets in $X_{2,1}$ which have become triod chains and those chains covering the associated arcs adjacent in $X_{2,2}$. So begin by putting these non-exceptional open sets into the open sets in $X_{2,2}$. This will leave a (finite) number of "loose ends" impinging on two of the end links of each inserted triod chain in $X_{2,2}$. Note that for each such triod chain in $X_{2,2}$ there will be an equal number of loose ends impinging on those two end links. In fact the loose ends come in pairs, and these pairs must be hooked up via triod chains and simple chains corresponding to the inserted triods and their associated adjacent intervals in $X_{3,2}$. First choose a triod chain for each pair of loose ends and embed these (at this point disjoint from the loose ends) in their corresponding triod chains in $X_{2,2}$. We may put as many as we need in any given triod chain in $X_{2,2}$ by going down far enough in Ingram's family of planar triod chain covers in [8]. Now all that remains is to hook the triod chains up to the loose ends using simple chains (corresponding to the arcs adjacent to the inserted triods in $X_{3,2}$). Note that the underlying spaces of the triod chains in $X_{2,2}$ are 2-cells. So Lemma 5 can be used to show how to do the hooking up. A little extra care may be needed to make sure that each of the simple chains passes through every link of the corresponding triod chain in $X_{2,2}$, thus guaranteeing the ontoness of the mapping of the adjacent intervals into the triods in $X_{2,2}$. $X_{3,3}$ will be constructed from $X_{3,2}$ the same way that $X_{2,2}$ was constructed from $X_{2,1}$.

Now proceed inductively to define the rest of the covers. To produce the cover $X_{n+1,n}$ embedded in $X_{n,n}$, begin with the cover $X_{n+1,1}$ (as we began with the cover $X_{3,1}$ above). Produce the cover $X_{n+1,2}$ from it and an embedding of it into the cover $X_{n,2}$ by inserting triods and simple chains as we did for $X_{3,2}$ above. Continue

[3] From now on we will use "$X_{i,j}$" to stand both for the factor space $X_{i,j}$ and for a cover whose nerve is $X_{i,j}$.

in this way across the $(n+1)$st row until $X_{n+1,n}$ has been constructed and an embedding of it into $X_{n,n}$. Finally, construct $X_{n+1,n+1}$ from $X_{n+1,n}$ the way that $X_{2,2}$ was constructed from $X_{2,1}$. ■

Remark. H. Cook has shown that $X_{\infty,\infty}$ is not hereditarily equivalent. It is an open question whether there exists a hereditarily equivalent (plane) continuum of positive span.

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Another universal metacompact developable T_1-space of weight $\mathfrak{m}$

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Abstract. Let $\mathfrak{m}$ be an infinite cardinal number. We use the d-line constructed in [Ch2] in order to construct a simple universal metacompact developable T_1-space of weight $\mathfrak{m}$ analogous to a universal metric space of weight $\mathfrak{m}$ constructed implicitly in the proof of the Nagata–Smirnov metrization theorem.

Let $\mathfrak{m}$ be an infinite cardinal number. In [Ch2], we constructed a universal metacompact developable T_1-space of weight $\mathfrak{m}$. The construction was based on a method of constructing mappings into metacompact developable T_1-spaces from [Ch1].

In section one of this paper we give another construction of a universal metacompact developable T_1-space of weight $\mathfrak{m}$. This construction is related to a method of constructing mappings into metacompact developable T_1-spaces investigated in [Ch3]. It is simpler than the construction in [Ch2] and has its metric analogue.

In section two we generalize the construction from [Ch2] in order to obtain an orthocompact developable T_1-space of weight $2^{\mathfrak{m}}$ containing all orthocompact developable T_1-spaces of weight $\mathfrak{m}$. The universal metacompact developable T_1-space of weight $\mathfrak{m}$ constructed [Ch2] is contained in this space in a natural way. We indicate some relations between the two constructions of universal spaces (Remark 2.7).

All our constructions are based on the d-line D (denoted by $T(0)$ in [Ch2]). In section three we construct a d-interval D^* and discuss the problem of extending mappings into D and D^*.

We use the terminology and notation from [E]. All mappings are assumed to be continuous and all spaces are assumed to be T_1-spaces. The last section requires the knowledge of [Ch2].

The d-line D [Ch2] (a similar, but more complicated space has been constructed earlier in [H]) is N^{N_+}, where N is the set of natural numbers and $N_+ = N\setminus\{0\}$. The topology of D is generated by the subbase

$$\mathscr{P} = \{B_n(i)\colon n, i\geqslant 1\} \cup \{B_n(i,j)\colon n, i, j\geqslant 1\} \cup \{D\},$$