Forms and mappings. I: Generalities

by

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Abstract. This paper continues the problems of [4], [5] and [6]. The main question is the difference between homogeneous polynomial mappings and so called m-applications. This is investigated with the aid of the homomorphism h between the respective representing modules. Most of the results form a general tool for the subsequent parts of the paper. Section 5 yields some ideal theory and an explicit computation (for $m \leqslant 5$) of the modules $\tilde{\Gamma}^m(R^n)$ defined in [4].

In this paper all rings and algebras are assumed to be commutative with 1. The symbol of a ring will in general be omitted, for example at $\otimes$ and Hom. We will also assume that the degree m is positive.

1. Preliminaries. The following definitions are contained in [7], [3] and [4].

A polynomial law on the pair (M, N) of R-modules is a natural transformation $F = (F_A)$ of the functors $M\otimes$-, $N\otimes$-: R-Alg $\to$ Sets. It is called a *form of degree* m if $F_A(\underline{x}a) = F_A(\underline{x})a^m$ for any R-algebra A, $a \in A$ and $\underline{x} \in M\otimes A$. All such forms constitute an R-module denoted by $\mathcal{P}^m_R(M, N)$. This gives us, in a natural way, a functor $\mathcal{P}^m_R$: R-Mod$^0 \times R$-Mod $\to R$-Mod.

It is proved in [7] that $\mathcal{P}^m_R(R^n, R) \approx R[T_1, \ldots, T_n]_m$. In general, any form $F \in \mathcal{P}^m_R(M, N)$ has the shape

$$F_A(x_1\otimes a_1 + \ldots + x_n\otimes a_n) = \sum_{m_1+\ldots+m_n=m} F_{m_1,\ldots,m_n}(x_1, \ldots, x_n)\otimes a_1^{m_1} \ldots a_n^{m_n},$$

where $F_{m_1,\ldots,m_n}$: $M^n \to N$ are uniquely determined by F. In particular $F_m = F_R$ and $F_{1,\ldots,1} = PF$ is m-linear and symmetric.

For any mapping f: $M \to N$ define the nth defect $\Delta^n f$: $M^n \to N$ in the following way

$$(1.1) \qquad (\Delta^n f)(x_1, \ldots, x_n) = \sum_{H\subset[1,n]} (-1)^{n-|H|} f\big(\sum_{i\in H} x_i\big).$$

It can also be defined inductively as an $(n-1)$-fold iteration of Δ^2 (see [4], p. 221). Moreover, it follows from [4] that $PF = \Delta^m F_R$ for any form F of degree m. This gives us the natural transformation

$$T^m_R\colon \mathcal{P}^m_R(M, N) \to \mathrm{Appl}^m_R(M, N), \qquad T^m_R(F) = F_R,$$

where $\mathrm{Appl}_R^m(M, N)$ denotes the module of all m-applications $f\colon M \to N$, i.e., mappings satisfying the conditions:

(A1) $f(rx) = r^m f(x)$ for any $r \in R$ and $x \in M$,

(A2) The associated symmetric mapping $\Delta^m f$ is m-linear.

(As a consequence, $f(0) = 0$ and $\Delta^k f = 0$ for any $k > m$.) In the free case T_R^m gives us the following well-known mapping:

$$T^m\colon R[T_1, \ldots, T_n]_m \to \mathrm{Appl}_R^m(R^n, R), \qquad T^m(F)(x_1, \ldots, x_n) = F(x_1, \ldots, x_n).$$

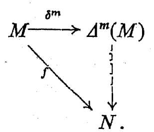
It is known from [3] that T_R^m is an isomorphism if $m \leqslant 2$ or $M = R$ or $m!$ is invertible in R. In general, it is neither injective nor surjective. Write $\mathrm{Ker}(T_R^m) = \tilde{\mathscr{P}}_R^m(M, N)$ and $\mathrm{Im}(T_R^m) = \mathrm{Hom}_R^m(M, N) \subset \mathrm{Appl}_R^m(M, N)$. The kernel is studied in [4] and [5], and the cokernel will be investigated in the present cycle of papers.

It follows from [7] that $\mathscr{P}^m(M, -)$ is represented by $\Gamma^m(M)$, the mth divided power of M. Moreover, it is evident that $\mathrm{Appl}^m(M, -)$ is represented by the module $\Delta^m(M)$ defined by the set of generators $\{\delta^m(x);\ x \in M\}$ and the relations

1° $\delta^m(rx) = r^m \delta^m(x)$ for any $r \in R$ and $x \in M$,

2° $\Delta^m \delta^m$ is m-linear,

and the correspondence is given by the following diagram:



(In [3], the module $\Delta^m(M)$ is denoted by $\Gamma_m(M)$ and has another presentation.)

T^m induces the natural homomorphism $h = h^m\colon \Delta^m(M) \to \Gamma^m(M)$ given by $h(\delta^m(x)) = x^{(m)}$ (see [3]). Write $\overline{\Gamma}^m(M) = \mathrm{Im}(h^m) = R\{x^{(m)};\ x \in M\}$. Since Hom is left exact it follows that $\tilde{\mathscr{P}}^m(M, -) = \mathrm{Ker}(T^m)$ is represented by $\tilde{\Gamma}^m(M) = \mathrm{Coker}(h^m) = \Gamma^m(M)/\overline{\Gamma}^m(M)$. On the other hand, $\mathrm{Hom}^m(M, -) = \mathrm{Im}(T^m)$ is representable if and only if the exact sequence

$$0 \to \overline{\Gamma}^m(M) \to \Gamma^m(M) \to \tilde{\Gamma}^m(M) \to 0$$

splits (see [4], Corollary 4.2). Moreover

Lemma 1.1. *$\mathrm{Hom}(\overline{\Gamma}^m(M), -)$ is the smallest representable functor containing* $\mathrm{Hom}^m(M, -)$.

Proof. A representable functor $F = \mathrm{Hom}(X, -)$ contains (isomorphically) $\mathrm{Hom}^m(M, -)$ if and only if there exists an exact sequence $0 \to \tilde{\mathscr{P}}^m(M, -) \hookrightarrow \mathscr{P}^m(M, -) \to F$, or, equivalently, an exact sequence $X \to \Gamma^m(M) \to \tilde{\Gamma}^m(M) \to 0$. In particular, $\mathrm{Hom}^m(M, -) \hookrightarrow \mathrm{Hom}(\overline{\Gamma}^m(M), -)$. In general, the image of X is $\overline{\Gamma}^m(M)$, and this gives us the unique monomorphism $\mathrm{Hom}(\overline{\Gamma}^m(M), -) \hookrightarrow F$ over $\mathrm{Hom}^m(M, -)$.

Corollary 1.2. *$\mathrm{Hom}(\overline{\Gamma}^m(M), N)$ is isomorphic to the following submodule of* $\mathrm{Appl}^m(M, N)$:

$$\overline{\mathrm{Hom}}^m(M, N) = \{g\colon M \xrightarrow{\delta^m} \Delta^m(M) \xrightarrow{h^m} \overline{\Gamma}^m(M) \xrightarrow{f} N;\ f \in \mathrm{Hom}(\overline{\Gamma}^m(M), N)\}.$$

Moreover, $g \in \mathrm{Hom}^m(M, N)$ if and only if f can be extended to $\Gamma^m(M)$. In particular, $\mathrm{Hom}^m(M, N) = \overline{\mathrm{Hom}}^m(M, N)$ for any injective N.

Let us consider the following extensions:

$$\mathrm{Hom}^m(M, N) \subset \overline{\mathrm{Hom}}^m(M, N) \subset \mathrm{Appl}^m(M, N).$$

The first of them can be embedded in the long exact sequence

$$0 \to \mathrm{Hom}^m(M, N) \to \overline{\mathrm{Hom}}^m(M, N) \to \mathrm{Ext}^1(\tilde{\Gamma}^m(M), N) \to \mathrm{Ext}^1(\Gamma^m(M), N) \to \ldots$$

(see [4], Corollary 4.3). Similarly, for the second extension:

$$0 \to \overline{\mathrm{Hom}}^m(M, N) \to \mathrm{Appl}^m(M, N) \to \mathrm{Hom}(\mathrm{Ker}(h^m(M)), N) \to \mathrm{Ext}^1(\overline{\Gamma}^m(M), N) \to$$
$$\to \mathrm{Ext}^1(\Delta^m(M), N) \to \ldots$$

Since Γ^m preserves projectives (see [7]), it follows that the first (the second) sequence reduces to the short sequence if M is projective (and $\mathrm{gl.dim}(R) \leqslant 1$). Then the question of the cokernel reduces to the computation of $\tilde{\Gamma}^m(M)$ or $\mathrm{Ker}(h^m(M))$. The value of $\tilde{\Gamma}^m(M)$ is found in [4] and [5], finding the value of $\mathrm{Ker}(h^m(M))$ is much more complicated.

The most interesting case is described in

Corollary 1.3. *The following conditions are equivalent:*

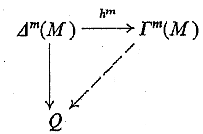
(1) $\mathrm{Ker}(h^m\colon \Delta^m(M) \to \Gamma^m(M)) = 0$,

(2) $\overline{\mathrm{Hom}}^m(M, -) = \mathrm{Appl}^m(M, -)$,

(2′) $\mathrm{Appl}^m(M, -)$ *is the smallest representable functor containing* $\mathrm{Hom}^m(M, -)$,

(3) $\mathrm{Hom}^m(M, Q) = \mathrm{Appl}^m(M, Q)$ *for any injective* Q,

(3′) *For any injective Q, one can complete any diagram of the form*



Proof. Evidently $(1) \Leftrightarrow (2) \Leftrightarrow (2') \Rightarrow (3) \Leftrightarrow (3')$. For the proof of $(3') \Rightarrow (1)$ use an injective module Q containing $\Delta^m(M)$.

Example 1.4. Let K be an algebraic extension of $\boldsymbol{Z}_p$. It follows from [6] (Theorems 2.7 and 4.1) that the above conditions are satisfied for any K-module M if and only if $K = \boldsymbol{Z}_p$ or $m \leqslant 2p$.

EXAMPLE 1.5. Let $m = 3$. It will be proved in Part II that the above conditions are satisfied for any flat R-module if R is a Dedekind domain, and for any R-module if $R = Z$ or no quotient field of R is isomorphic to Z_2.

In the next sections we state the results which will be used to determine $\mathrm{Ker}(h^m)$ in some cases, or, in particular, to prove theorems as in the above examples.

2. Fundamental properties. Let $X = \Gamma^m$, $\tilde{\Gamma}^m$, $\overline{\Gamma}^m$ or Δ^m, and let $F = \mathscr{P}^m$, $\tilde{\mathscr{P}}^m$, $\overline{\mathrm{Hom}}^m$ or Appl^m, respectively. Consequently, $F(M, N) \approx \mathrm{Hom}(X(M), N)$.

LEMMA 2.1. *X commutes with direct limits.*

Proof. It suffices to prove that the natural homomorphism $F(\varinjlim M_i, N) \xrightarrow{\varrho} \varprojlim F(M_i, N)$ is bijective. For the first three cases see [4], Corollary 4.5. Let $F = \mathrm{Appl}^m$. Observe that ϱ^{-1} exists for $F = \mathrm{Map}$, the functor of all mappings between modules. Moreover, the mapping reconstructed from m-applications is, obviously, an m-application; this completes the proof.

COROLLARY 2.2. *Suppose that h^m is mono, epi or iso on the subcategory of finitely generated (and free) R-modules. Then it is so on the category of all (flat) R-modules.*

The following diagram of modules and their homomorphisms

$$M \overset{i}{\underset{j}{\rightrightarrows}} N \xrightarrow{q} P$$

is called a *Grothendieck sequence* (see [7], p. 278, Definition 1) if q is surjective and

(2.1) $$\forall_{x,y \in N} \big(q(x) = q(y) \Leftrightarrow \exists_{z \in M} (x = i(z) \,\&\, y = j(z))\big).$$

An equivalent definition is given by the following conditions:

(1) $q = \mathrm{Coker}(i, j)$,

(2) $\forall_{x \in N} \exists_{t \in M}\ x = i(t) = j(t)$.

Evidently (1) and (2) follow from the previous definition. Conversely, if $q(x) = q(y)$ then $x - y = (i-j)(u)$ by (1) and hence $x - i(u) = y - j(u) = i(t) = j(t)$ by (2). This gives us (2.1) for $z = t+u$.

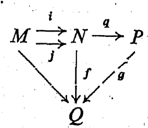
LEMMA 2.3. *If $X \neq \overline{\Gamma}^m$ then X preserves Grothendieck sequences.*

Proof. It can be proved directly that any X (without restrictions) preserves condition (2). Hence it suffices to prove that the sequence

$$0 \to F(P, Q) \xrightarrow{F(q,1)} F(N, Q) \overset{F(i,1)}{\underset{F(j,1)}{\rightrightarrows}} F(M, Q)$$

is exact for any i, j and q constituting a Grothendieck sequence, any Q and any $F \neq \overline{\mathrm{Hom}}^m$. Let $F = \mathrm{Appl}^m$. The only non-trivial part is the completion of the following commutative diagram:

(2.2)



where f is an m-application. It follows from (2.1) that g exists (as a mapping), and it is evidently an m-application. The case of $\mathscr{P}^m$ follows from [7] Théorème IV.4, and the case of $\tilde{\mathscr{P}}^m$ is a consequence of the two preceding cases.

Remark 2.4. Any exact sequence $K \xrightarrow{i} P \xrightarrow{q} M \to 0$ induces the following Grothendieck sequences:

$$P \oplus K \overset{(1,i)}{\underset{(1,0)}{\rightrightarrows}} P \xrightarrow{q} M,$$

and (by Lemma 2.3):

$$X(P \oplus K) \overset{X(1,i)}{\underset{X(1,0)}{\rightrightarrows}} X(P) \xrightarrow{X(q)} X(M)$$

for any $X \neq \overline{\Gamma}^m$. In particular,

(2.3) $$\Delta^m(M) \approx \Delta^m(P)/R\{\delta^m(x+y) - \delta^m(x);\ x \in P,\ y \in \mathrm{Ker}(q)\}$$

and similarly for other functors. This allows us to compute any value of X provided that the values of X on free modules are known. Lemma 2.3 is not true for $X = \overline{\Gamma}^m$ because in (2.2) g is not necessarily in $\overline{\mathrm{Hom}}^m$ even if f is (for example, it is possible that $\overline{\mathrm{Hom}}^m(N, Q) = \mathrm{Appl}^m(N, Q)$ and $\overline{\mathrm{Hom}}^m(P, Q) \neq \mathrm{Appl}^m(P, Q)$ as follows from Corollary 2.6 and Proposition 2.9 below).

COROLLARY 2.5. *Suppose that h_R^m is an isomorphism on the subcategory of finitely generated free R-modules. Then $h_R^m\colon \Delta_R^m \xrightarrow{\approx} \overline{\Gamma}_R^m = \Gamma_R^m$.*

Let us consider the simplest case of a cyclic module R/I. First of all, observe that $\Delta^m(R) = R\delta^m(1) \approx R$ and $\Gamma^m(R) = R1^{(m)} \approx R$ (see [3], Example 7.1) and hence

COROLLARY 2.6. $h_R^m(R)\colon \Delta_R^m(R) \xrightarrow{\approx} \Gamma_R^m(R)$.

It follows from (2.3) and [8], Proposition 8, that

$$h_R^m(R/I)\colon R/A_m(I) \xrightarrow{\text{nat}} R/D_m(I)$$

where $A_m(I)$ is generated by the values and $D_m(I)$ by the coefficients of the polynomials

(2.4) $$(X+y)^m - X^m = \sum_{i=1}^{m} \binom{m}{i} X^{m-i} y^i \quad (y \in I).$$

Evidently $h_R^m(R/I)$ is an isomorphism for $I = 0$ or R. Moreover, A_m and D_m commute with localizations (compare also Section 3), and hence we can assume that R is local.

PROPOSITION 2.7. *Let I be an ideal in a local ring (R, P), $\mathrm{char}(R) = \mathrm{char}(R/P) = p$ (possibly zero), and let q denote the p-primary part of m ($q = 1$ for $p = 0$). Then $A_m(I) = D_m(I) = I^{(q)} := (y^q;\ y \in I)$, and hence $h_R^m(R/I)$ is an isomorphism.*

Proof. First observe that $q = \min\left\{i > 0;\ \binom{m}{i} \neq 0 \text{ in } R\right\}$ (see for example [6], Lemma 2.2); clearly $\binom{m}{q}$ is invertible in R. Evidently $D_m(I) = I^{(q)}$ and $A_m(I)$ is generated by elements of the form $y^q + y^{q+1}a(y)$ for all $y \in I$. Since $y^m \in A_m(I)$ (put $X = 0$ in (2.4)), it follows by induction that $A_m(I)$ is also $I^{(q)}$.

Suppose now that (R, P) is a discrete valuation ring such that $\mathrm{char}(R) = 0$ and $\mathrm{char}(R/P) = p > 0$. Let V denote the valuation and let $e = V(p)$, $1 \leqslant e < \infty$, be the ramification index. Denote $A_m((r)) = (a_m(r))$ and $D_m((r)) = (d_m(r))$. The idea of the following Lemma 2.8 and Proposition 2.9 is based on [9], Proposition 3.

Lemma 2.8. *If R is as above, $i \geqslant 2$ and $V(r) \geqslant e$, then $V\left(\binom{m}{i} r^i\right) \geqslant V(mr)$ and the equality holds if and only if $p = 2$, $i = 2$, $V(r) = e$ and m is even.*

Proof. Let v_p denote the p-adic valuation on Q and let $k = v_p(i)$. Since $\binom{m}{i} = \frac{m}{i}\binom{m-1}{i-1}$, it follows that $v_p(m) - v_p\binom{m}{i} \leqslant v_p(m) - v_p\left(\frac{m}{i}\right) = k \leqslant p^k - 1 \leqslant i - 1$, and the equalities hold if and only if $p = 2$, $i = 2$ and m is even. Consequently

$$V(mr) - V\left(\binom{m}{i} r^i\right) = e\left(v_p(m) - v_p\binom{m}{i}\right) - V(r)(i-1) \leqslant (e - V(r))(i-1) \leqslant 0,$$

and the rest is immediate.

Proposition 2.9. *If R is as above and $V(r) \geqslant e$, then*

(1) $$V(d_m(r)) = V(m) + V(r);$$

(2) $$V(a_m(r)) = \begin{cases} V(m) + V(r) + 1 & \text{if } V(r) = e,\ R/P \approx Z_2 \text{ and } m > 2 \text{ is even}, \\ V(m) + V(r) & \text{otherwise}; \end{cases}$$

(3) $$\mathrm{Ker}(h_R^m(R/(r))) \approx \begin{cases} Z_2 & \text{if } V(r) = e,\ R/P \approx Z_2 \text{ and } m > 2 \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Observe that

$$V(d_m(r)) = \min\left\{V\left(\binom{m}{i} r^i\right);\ i = 1, \ldots, m\right\}$$

and

$$V(a_m(r)) = \min\left\{V\left(\sum_{i=1}^{m} \binom{m}{i} x^{m-i} y^i r^i\right);\ x, y \in R\right\}.$$

Hence Lemma 2.8 gives (1) and also (2) except the case where $p = 2$, $V(r) = e$ and m is even. The case of $m = 2$ is evident. For $m > 2$, the above sum has the form $mr(x^{m-1}y + ux^{m-2}y^2) + a$ where $u = \frac{1}{2}r(m-1)$ is invertible in R and $V(a) > V(m) + V(r)$. If $R/P \not\approx Z_2$ then there exist $x, y \in R$ such that $\bar{x}, \bar{y}, \overline{x + uy} \neq 0$ in R/P, and hence the element in the brackets is invertible. In this case, $V(a_m(r)) = V(m) + V(r)$ as required. Let $R/P \approx Z_2$. Since $m > 2$, it follows that $V(a_m(r)) \geqslant V(m) + V(r) + 1$ and the equality holds because of the case where $x = 1$ and y is the uniformizing parameter. Finally, (3) follows directly from (1) and (2).

The case of $V(r) < e$ is much more complicated. Hence the only complete globalization is the following:

Corollary 2.10. *Let R be a Dedekind domain. Suppose that any localization R_P is unramified provided that $\mathrm{char}(R_P) \neq \mathrm{char}(R/P)$ (for example, $R = Z$). Let I be an ideal in R and let t denote the number of prime ideals P in R satisfying $I \subset P$, $I \not\subset P^2$ and $R/P \approx Z_2$. Then*

$$\mathrm{Ker}(h_R^m(R/I)) \approx \begin{cases} Z_2 \times \ldots \times Z_2 \ (t \text{ times}) & \text{if } m > 2 \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $I \neq 0$. Write $I = P_1^{k_1} \ldots P_n^{k_n}$ for distinct maximal P_i, compute $D_m(I)$ and $A_m(I)$ locally (by Proposition 2.9), and divide $D_m(I)$ by $A_m(I)$ applying the Chinese Remainder Theorem.

Finally, let us study the finite generation of $X(M)$. If M is finitely generated over R then so is $\Gamma^m(M)$ (see [7]) and hence $\bar{\Gamma}^m(M)$. If, moreover, R is Noetherian, then so is $\bar{\Gamma}^m(M)$. For Δ^m we can prove only

Proposition 2.11. *Let M be an R-module generated over Z by $\{x_1, \ldots, x_k\}$, and let $m \geqslant 2$. Then $\Delta_R^m(M)$ is generated over Z by $\{\delta_R^m(n_1x_1 + \ldots + n_kx_k);\ 0 \leqslant n_i \leqslant m-1\}$. In particular, if M is a finitely generated R-module then so is $\Delta_R^m(M)$ provided that R is finitely generated as an abelian group.*

Proof. Since $\Delta_R^m(M)$ is a quotient of $\Delta_Z^m(M)$ it can be assumed that $R = Z$. Let $x, y \in M$ and $n \in Z$. The relation $(\Delta^m\delta^m)(nx, x, \ldots, x, y) = n(\Delta^m\delta^m)(x, \ldots, x, y)$ gives us the following equality:

$$\sum_{i=0}^{m-2} a_i \delta^m((n+i)x + y) + \sum_{j=0}^{m-1} b_j \delta^m(jx + y) + b\delta^m(x) = 0$$

where $a_i, b_j, b \in Z$, $a_{m-2} = \pm 1$ for $n > 0$ and $a_0 = \pm 1$ for $n < 0$. Using induction twice (for $n > 0$ and $n < 0$) we find that $\delta^m(nx + y)$ belongs to the submodule generated by $\delta^m(x)$ and $\delta^m(jx + y)$ for $j = 0, \ldots, m-1$. Applying this successively to $x = x_i$ and the respective y we complete the proof.

The second part of the above proposition will be improved (in Part II) in the simplest non-trivial case $m = 3$.

3. Change of the base ring. Let A be an R-algebra and let M be an R-module. Then [7], Théorème III.3, gives us the following natural graded A-algebra isomorphism

$$p\colon \Gamma_R(M) \otimes A \xrightarrow{\approx} \Gamma_A(M \otimes A), \quad p(x^{(m)} \otimes 1) = (x \otimes 1)^{(m)}.$$

This allows us to prove a generalization of [4], Theorem 6.1:

Proposition 3.1. *There exists a natural exact sequence*

$$\ldots \to \mathrm{Tor}_1^R(\Gamma_R^m(M), A) \to \mathrm{Tor}_1^R(\bar{\Gamma}_R^m(M), A) \to \bar{\Gamma}_R^m(M) \otimes A \xrightarrow{e} \bar{\Gamma}_A^m(M \otimes A) \to$$
$$\to \bar{\Gamma}_R^m(M) \otimes A \xrightarrow{q} \bar{\Gamma}_A^m(M \otimes A) \to 0$$

where e and q are induced by p above. Moreover, if $M \otimes A = \{x \otimes a;\ x \in M,\ a \in A\}$ (for example, if $A = R_S$ or R/I) then e is surjective and hence q is an isomorphism.

Proof. Evidently p induces e and hence we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\ldots\to \mathrm{Tor}_1^R(\Gamma_R^m(M), A) \to \mathrm{Tor}_1^R(\tilde\Gamma_R^m(M), A) \to & \bar\Gamma_R^m(M)\otimes A & \xrightarrow{i} & \Gamma_R^m(M)\otimes A & \to & \tilde\Gamma_R^m(M)\otimes A & \to 0 \\
& {\scriptstyle e}\downarrow & & {\scriptstyle p}\downarrow\approx & & {\scriptstyle q}\downarrow & \\
0\to & \bar\Gamma_A^m(M\otimes A) & \to & \Gamma_A^m(M\otimes A) & \to & \tilde\Gamma_A^m(M\otimes A) & \to 0
\end{array}$$

Observe that $\mathrm{Coker}(e)\approx\mathrm{Ker}(q)$ and $\mathrm{Coker}(q) = 0$ by the snake lemma. Moreover, $\mathrm{Ker}(i) = \mathrm{Ker}(pi) = \mathrm{Ker}(e)$. This completes the first part of the proof and the second is evident.

Observe that e is injective if A is a flat R-module. In particular

COROLLARY 3.2. *If $A = R_S$ where S is a multiplicative set in R then e is bijective. In other words,*

$$\bar\Gamma_R^m(M)_S \approx \bar\Gamma_{R_S}^m(M_S), \qquad \frac{x^{(m)}}{1} \leftrightarrow \left(\frac{x}{1}\right)^{(m)}.$$

If $A = R/I$ then e is surjective and

$$\mathrm{Ker}(e) = \mathrm{Ker}(i) = (\bar\Gamma_R^m(M) \cap I\Gamma_R^m(M))/I\bar\Gamma_R^m(M).$$

It is non-zero in general because of

EXAMPLE 3.3. Let (R, I) be a local Noetherian ring, $\dim(R)>0$, $A = R/I$ and $M = R^n$, $n>1$. It follows from [7] that $\Gamma_R^m(M)$ is free and $N = \tilde\Gamma_R^m(M)$ is finitely generated. Let $\ldots \to F_1 \to F_0$ be a minimal free resolution of N. Then $\mathrm{Ker}(e)\approx\mathrm{Tor}_1^R(N, R/I)\approx F_1/IF_1$, and hence

$$\begin{aligned}\mathrm{Ker}(e) = 0 &\Leftrightarrow F_1 = 0 \Leftrightarrow N \text{ is free} \Leftrightarrow N = 0 \text{ (by [5], Corollary 2.2)}\\ &\Leftrightarrow m\leqslant |R/I| \text{ (by [4], Theorem 6.4).}\end{aligned}$$

This is the trivial case where $\bar\Gamma^m = \Gamma^m$ over R and over R/I (see [4], Corollary 6.5).

We will prove similar properties of the functor Δ^m. First of all observe that $\mathrm{Appl}_A^m(M, N)\subset\mathrm{Appl}_R^m(M, N)$ for any R-algebra A and any A-modules M, N. (Moreover, the equality holds if $A = R/I$). Hence any module homomorphism $M \to N$ over $R \to A$ allows us to complete (in a unique way) the following diagram:

$$\begin{array}{ccc}
M & \xrightarrow{\delta_R^m} & \Delta_R^m(M) \\
\downarrow & & \downarrow \\
N & \xrightarrow{\delta_A^m} & \Delta_A^m(N).
\end{array}$$

COROLLARY 3.4. *Δ^m is an endo-functor of the category of pairs (R, M), where R is a (commutative) ring, and M is an R-module.*

Let $N = M\otimes A$. The above diagram gives us the following A-homomorphism:

$$d\colon \Delta_R^m(M)\otimes A \to \Delta_A^m(M\otimes A), \qquad d(\delta^m(x)\otimes 1) = \delta^m(x\otimes 1).$$

Moreover, it is evident that the following diagram is commutative

$$(3.1)\qquad \begin{array}{ccc}
\Delta_R^m(M)\otimes A & \xrightarrow{h_R^m\otimes 1} & \bar\Gamma_R^m(M)\otimes A \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle e} \\
\Delta_A^m(M\otimes A) & \xrightarrow{h_A^m} & \bar\Gamma_A^m(M\otimes A)
\end{array}$$

Observe that d is an epimorphism if $M\otimes A = \{m\otimes a;\ x\in M,\ a\in A\}$ (for example, if $A = R_S$ or R/I). In this case, d is invertible if and only if there exists an m-application $M\otimes A \to \Delta_R^m(M)\otimes A$ induced by δ_R^m. This is equivalent to the following condition:

(E) Any m-application $f\colon M \to N$ over R induces an m-application $g\colon M\otimes A\to \to N\otimes A$ over A such that the diagram

$$\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M\otimes A & \xrightarrow{g} & N\otimes A
\end{array}$$

is commutative, i.e., $g(x\otimes a) = f(x)\otimes a^m$ for $x\in M$ and $a\in A$.

PROPOSITION 3.5. *Condition* (E) *is fulfilled for any localization $R \to R_S$.*

Proof. Let $f\colon M \to N$ be an m-application over R. Define $g = f_S\colon M_S \to N_S$ as above, i.e., $g(x/s) = f(x)/s^m$. An easy verification shows that it is a correct definition and that g satisfies the required condition (A1). Moreover, it is evident that

$$(\Delta^m g)\left(\frac{x_1}{t}, \ldots, \frac{x_m}{t}\right) = \frac{(\Delta^m f)(x_1, \ldots, x_m)}{t^m}.$$

In particular, $\Delta^m g$ is m-additive. The following computation completes the proof of (A2):

$$\begin{aligned}(\Delta^m g)\left(\frac{r}{s}\cdot\frac{x_1}{t}, \frac{x_2}{t}, \ldots, \frac{x_m}{t}\right) &= (\Delta^m g)\left(\frac{rx_1}{st}, \frac{sx_2}{st}, \ldots, \frac{sx_m}{st}\right) = \frac{(\Delta^m f)(rx_1, sx_2, \ldots, sx_m)}{(st)^m}\\ &= \frac{r}{s}\cdot\frac{(\Delta^m f)(x_1, \ldots, x_m)}{t^m} = \frac{r}{s}(\Delta^m g)\left(\frac{x_1}{t}, \ldots, \frac{x_m}{t}\right).\end{aligned}$$

COROLLARY 3.6. *If $A = R_S$ where S is a multiplicative set in R then d is an isomorphism. In other words, there exists a natural isomorphism*

$$\Delta_R^m(M)_S\approx\Delta_{R_S}^m(M_S), \qquad \frac{\delta^m(x)}{1} \leftrightarrow \delta^m\left(\frac{x}{1}\right).$$

COROLLARY 3.7. $\mathrm{Ker}(h_R^m)_S\approx\mathrm{Ker}(h_{R_S}^m)$. *In particular,* $\mathrm{Ker}(h_R^m) = 0$ *if and only if* $\mathrm{Ker}(h_{R_P}^m) = 0$ *for any prime (maximal) ideal P in R.*

EXAMPLE 3.8. Let R be a domain of characteristic 0 or greater than m. Then $m!$ is invertible in $R_{(0)}$; consequently $(\mathrm{Ker}(h_R^m))_{(0)} = 0$, and hence $\mathrm{Ker}(h_R^m)$ is a torsion submodule of $\Delta_R^m(M)$. If R is a Dedekind domain and M is projective then so are $\Gamma^m(M)$ and $\bar{\Gamma}^m(M)$. In this case, $\Delta^m(M)\approx\bar{\Gamma}^m(M)\oplus\operatorname{Ker}(h^m)$.

Let $A = R/I$. We will find out when (E) is satisfied. Evidently we can assume that N is an R/I-module, i.e., that $IN = 0$. Then the diagram in (E) has the following shape:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow^{f} \\ M/IM & \overset{g}{\dashrightarrow} & N \end{array}$$

Corollary 3.9. *If M is an R/I-module then* (E) *is satisfied and consequently*

$$\Delta_{R/I}^m(M) = \Delta_{R/I}^m(M/IM)\approx\Delta_R^m(M)/I\Delta_R^m(M), \qquad \delta_{R/I}^m(x) \leftrightarrow \overline{\delta_R^m(x)}.$$

The mapping g completing the above diagram has the form $g(\bar{x}) = f(x)$, hence it is unique and it is evidently an m-application (over R or R/I). Unfortunately, the formula can be incorrect (it must be verified that $f(x) = f(y)$ for $x-y\in IM$; compare also Remark 2.4). In other words, we have only $\operatorname{Appl}_{R/I}^m(M/IM, N) = \operatorname{Appl}_R^m(M/IM, N)\hookrightarrow\operatorname{Appl}_R^m(M, N)$, and this can be imbedded into the following commutative diagram:

$$\begin{array}{ccccccc} 0\to\bar{\mathscr{P}}_{R/I}^m(M/IM, N) & \to & \mathscr{P}_{R/I}^m(M/IM, N) & \xrightarrow{T_{R/I}^m} & \operatorname{Appl}_{R/I}^m(M/IM, N) \\ \approx\downarrow q^* & & \approx\downarrow p^* & & \downarrow \\ 0\to\bar{\mathscr{P}}_R^m(M, N) & \longrightarrow & \mathscr{P}_R^m(M, N) & \xrightarrow{T_R^m} & \operatorname{Appl}_R^m(M, N). \end{array}$$

If $T_{R/I}^m$ is surjective then evidently $\operatorname{Appl}_{R/I}^m(M/IM, N) = \operatorname{Hom}_R^m(M, N)$. This is satisfied, for example, if $m = 3$, $R = \boldsymbol{Z}$, and R/I is a quotient field $\boldsymbol{Z}_p$ (Example 1.4). In this case $\operatorname{Hom}_R^m\subset\operatorname{Hom}_R^m = \operatorname{Appl}_R^m$ (Example 1.5), and the cokernel for $M = \boldsymbol{Z}^2$ is $\boldsymbol{Z}_2\otimes N$ (Example 4.4 in [4]), which is non-zero for $p = 2$ and $N \neq 0$.

Consider the epimorphism d: $\Delta_R^m(M)/I\Delta_R^m(M) \to \Delta_{R/I}^m(M/IM)$. It follows from the above that d is not always injective. Moreover, we can prove the following analogue of (2.3):

Corollary 3.10. $\operatorname{Ker}(d) = (K+I\Delta_R^m(M))/I\Delta_R^m(M)$ *where* $K = R\{\delta_R^m(x)-\delta_R^m(y);\ x-y\in IM\}$.

Proof. Evidently d induces the following epimorphism:

$$d'\colon\ \Delta = \Delta_R^m(M)/(K+I\Delta_R^m(M)) \to \Delta_{R/I}^m(M/IM), \qquad d'(\overline{\delta^m(x)}) = \delta^m(\bar{x}).$$

It suffices to prove that d' is bijective, and this means that we can complete the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\delta^m} & \Delta_R^m(M) \\ \downarrow & & \downarrow \text{nat} \\ M/IM & \dashrightarrow & \Delta \end{array}$$

This is possible since $\overline{\delta^m(x)} = \overline{\delta^m(y)}$ for $x-y\in IM$.

4. Defect decomposition. Let X: R-Mod $\to$ R-Mod be a functor satisfying $X(0) = 0$. The defect of X is the functor X^2: R-Mod$\times R$-Mod $\to$ R-Mod defined by Eilenberg and MacLane [2] in the following way:

$$X^2(M, N) = (\mathrm{id}-X(p)-X(q))(X(M\oplus N)), \qquad X^2(f, g) = X(f\oplus g),$$

where $p, q\in\operatorname{End}(M\oplus N)$ are the projections on M and N, respectively. It is easy to see that $X(M\oplus N) = X(M)\oplus X(N)\oplus X^2(M, N)$.

Put $X^1 = X$ and by induction

$$X^k(M_1, \ldots, M_{k-2}, -, -) = (X^{k-1}(M_1, \ldots, M_{k-2}, -))^2.$$

This defines in a natural way the functors X^k: R-Mod$\times\ldots\times R$-Mod $\to$ R-Mod satisfying $X^k(\ldots, 0, \ldots) = 0$ and the following generalized defect decomposition property:

$$X(M_1\oplus\ldots\oplus M_n) = \bigoplus_{k=1}^{n}\ \bigoplus_{1\leqslant j_1<\ldots<j_k\leqslant n} X^k(M_{j_1}, \ldots, M_{j_k}). \tag{4.1}$$

In particular,

$$X(R^n)\approx\bigoplus_{k=1}^{n}\binom{n}{k}X^k(R, \ldots, R). \tag{4.2}$$

Corollary 4.1. *If X preserves direct limits and $M = \bigoplus_{i\in I} M_i$ where I is ordered by $<$, then*

$$X(M) = \bigoplus_{k}\ \bigoplus_{i_1<\ldots<i_k} X^k(M_{i_1}, \ldots, M_{i_k}).$$

Easy induction shows that

$$X^k(M_1, \ldots, M_k) = \Big(\sum_{H\subset[1,k]}(-1)^{k-|H|}X(p_H)\Big)(X(M_1\oplus\ldots\oplus M_n)) \tag{4.3}$$

where $p_H\in\operatorname{End}(M_1\oplus\ldots\oplus M_k)$ denotes the projection on $\bigoplus_{i\in H} M_i$. X^k is a symmetric functor called the kth defect of X; X is a functor of degree m if $X^{m+1} = 0$.

Let H: $X\to Y$ be a natural transformation. Evidently H induces restrictions H^k: $X^k\to Y^k$, and hence preserves defect decomposition (4.1). In particular,

$$\operatorname{Ker}(H)^k = \operatorname{Ker}(H^k) = X^k\cap\operatorname{Ker}(H), \qquad \operatorname{Im}(H)^k = \operatorname{Im}(H^k) = Y^k\cap\operatorname{Im}(H),$$
$$\operatorname{Coker}(H)^k = \operatorname{Coker}(H^k)$$

(with the natural meaning of arguments).

Denote $(\Delta^m)^k = \Delta^{m,k}$, etc. It follows from (1.1) and (4.3) that

$$\Delta^{m,k}(M_1,\ldots, M_k) = R\{(\Delta^k\delta^m)(x_1, \ldots, x_k);\ x_i\in M_i\}\subset\Delta^m(M_1\oplus\ldots\oplus M_k).$$

The formula for $\Gamma^{m,k}$ follows from the graded algebra isomorphism $\Gamma(M)\otimes\Gamma(N)\approx\Gamma(M\oplus N)$ induced by multiplication (see [7], Théorème III.4). Identifying the above algebras, we obtain

$$\Gamma^{m,2}(M, N) = \bigoplus_{i=1}^{m-1}\Gamma^i(M)\otimes\Gamma^{m-i}(N)$$

and by induction

(4.4) $$\Gamma^{m,k}(M_1, \ldots, M_k) = \bigoplus_{\substack{m_1+\ldots+m_k=m \\ m_1,\ldots,m_k>0}} \Gamma^{m_1}(M_1)\otimes\ldots\otimes\Gamma^{m_k}(M_k).$$

This module can be characterized as a submodule of $\Gamma^m(M_1\oplus\ldots\oplus M_k)$ generated by elements $x_1^{(n_1)}\ldots x_s^{(n_s)}$ with $x_i\in M_1\cup\ldots\cup M_k$ and $n_1+\ldots+n_s = m$, depending properly (in non-zero divided powers) on any M_j. Moreover, h^m induces natural transformations $h^{m,k}$: $\Delta^{m,k}\to\Gamma^{m,k}$ such that $\overline{\Gamma}^{m,k} = \operatorname{Im}(h^{m,k}) = \Gamma^{m,k}\cap\overline{\Gamma}^m$ and $\tilde{\Gamma}^{m,k} = \Gamma^{m,k}/\overline{\Gamma}^{m,k}$. More explicitly,

$$\overline{\Gamma}^{m,k}(M_1, \ldots, M_k) = R\{(\Delta^k\gamma^m)(x_1, \ldots, x_k);\ x_i\in M_i\}$$

where $\gamma^m = h^m\delta^m$, i.e., $\gamma^m(x) = x^{(m)}$. [4; Lemma 3.1] shows that

(4.5) $$(\Delta^k\gamma^m)(x_1, \ldots, x_k) = \sum_{\substack{m_1+\ldots+m_k=m \\ m_1,\ldots,m_k>0}} x_1^{(m_1)}\ldots x_k^{(m_k)}.$$

As a corollary, Δ^m, Γ^m, $\overline{\Gamma}^m$ and $\tilde{\Gamma}^m$ are functors of degree m and the values of $\Gamma^{m,k}$, $\overline{\Gamma}^{m,k}$ and $\tilde{\Gamma}^{m,k}$ for $M_1 = \ldots = M_k = R$ coincide with the modules defined in [4], Section 8.

COROLLARY 4.2. $h^{m,m}$: $\Delta^{m,m}\xrightarrow{\approx}\overline{\Gamma}^{m,m} = \Gamma^{m,m} = (\ )\otimes\ldots\otimes(\ )$ *and* $\tilde{\Gamma}^{m,m} = 0$.

Proof. Observe that $h^{m,m}((\Delta^m\delta^m)(x_1, \ldots, x_m)) = x_1\ldots x_m = x_1\otimes\ldots\otimes x_m$ by (4.5) and (4.4). The inverse exists since $\Delta^m\delta^m$ is m-linear; the rest is evident.

An easy verification shows that many of the properties proved above for Δ^m, $\overline{\Gamma}^m$, Γ^m and $\tilde{\Gamma}^m$ are satisfied also by the defects. In particular, they commute with direct limits and localizations and preserve Grothendieck sequences (except $\overline{\Gamma}^{m,k}$). For example, if $M_i = P_i/K_i$ then

(4.6) $$\Delta^{m,k}(M_1, \ldots, M_k)\approx\Delta^{m,k}(P_1, \ldots, P_k)/K$$

where

$$\begin{aligned} K &= R\{(\Delta^k\delta^m)(x_1+y_1, \ldots, x_k+y_k)-(\Delta^k\delta^m)(x_1, \ldots, x_k);\ x_i\in P_i,\ y_i\in K_i\} \\ &= R\{(\Delta^k\delta^m)(x_1, \ldots, x_j+y_j, \ldots, x_k)-(\Delta^k\delta^m)(x_1, \ldots, x_k);\ x_i\in P_i,\ y_j\in K_j,\ j = 1, \ldots k\} \\ &= R\{(\Delta^k\delta^m)(x_1, \ldots, y_j, \ldots, x_k),\ (\Delta^{k+1}\delta^m)(x_1, \ldots, x_j, y_j, \ldots, x_k);\ x_i\in P_i,\ y_j\in K_j, \\ &\qquad j = 1, \ldots, k\}. \end{aligned}$$

COROLLARY 4.3. *If* $X = \Delta^m$, $\overline{\Gamma}^m$, Γ^m *or* $\tilde{\Gamma}^m$ *and* $\operatorname{Ann}(M_1)+\ldots+\operatorname{Ann}(M_k) = R$ *then* $X^k(M_1, \ldots, M_k) = 0$.

Proof. Since X^k commutes with localizations, it suffices to assume that R is a local ring. In this case $M_i = 0$ for some i.

Suppose that $A\subset B$ are classes of R-modules such that any $M\in B$ is a finite direct sum of modules from A. The defect decomposition shows that the investigation of h^m on B reduces to the study of $h^{m,k}$ on $A\times\ldots\times A$. This can be used (as well as Corollary 2.2) in the following two cases:

(1) $A = \{R\}$, B = the class of all finitely generated free R-modules (see (4.2));

(2) A (resp. B) = the class of all indecomposable (resp. finitely generated) R-modules (for a suitable ring R).

5. Ideals $I_m(R)$. The ideals defined below are closely related to the functors $\tilde{\Gamma}^m$ and will also be used in Part II.

For any $m\geqslant 2$ define $I_m(R) = (r-r^m;\ r\in R)$. Since $rs^m-r^ms = r(s^m-s)+ +s(r-r^m)$, it follows that $I_m(R) = (rs^m-r^ms;\ r, s\in R)$.

LEMMA 5.1. $I_m(R_S) = I_m(R)_S$, $I_m(R/J) = (I_m(R)+J)/J$.

Proof. The first formula results from the equalities

$$\frac{r}{s}-\left(\frac{r}{s}\right)^m = \frac{rs^m-r^ms}{s^{m+1}}, \quad \frac{r-r^m}{s} = \frac{1}{s}\left(\frac{r}{1}-\left(\frac{r}{1}\right)^m\right)$$

and the second is evident.

Observe that $I_m(R) = 0$ if and only if $r = r^m$ for any $r\in R$. Since $m\geqslant 2$, it follows that R is von Neumann regular.

LEMMA 5.2. *Let R be a domain. Then $I_m(R) = 0$ if and only if R is a finite field and $|R|-1|m-1$.*

Proof. $I_m(R) = 0$ means that $r^{m-1} = 1$ for any $0\neq r\in R$, and hence R is a finite field. R^* is a cyclic group and its order must divide $m-1$.

COROLLARY 5.3. *Let P be a prime ideal in R. Then $I_m(R)\subset P$ iff $I_m(R/P) = 0$ iff R/P is a finite field and $|R/P|-1|m-1$. In particular, $I_m(R) = R$ iff $I_m(R/M) = R/M$ for any maximal ideal M iff $|K|-1\nmid m-1$ for any finite quotient field K. (This is satisfied for example if $m!$ is invertible in R).*

COROLLARY 5.4. *Any Noetherian ring R has only a finite number of such maximal ideals M that $|R/M| = d$ (for any fixed d), and hence the set of its finite quotient fields is at most countable.*

Proof. It suffices to observe that any such M is a prime ideal minimal over $I_d(R)$ by Corollary 5.3.

We are ready to prove the following characterization:

PROPOSITION 5.5. $I_m(R) = \bigcap\{M\in\operatorname{Max}(R);\ |R/M|-1|m-1\}$ *(a radical ideal).*

Proof. Let (R, M) be a local ring. Any $x\in M$ gives an invertible element $1-x^{m-1}$, consequently $x = (x-x^m)/(1-x^{m-1})\in I_m(R)$. Hence, by Corollary 5.3,

$$I_m(R)\neq R \Leftrightarrow I_m(R) = M \Leftrightarrow |R/M|-1|m-1$$

as required. For arbitrary R, the inclusion $\subset$ follows from Corollary 5.3, and the inverse can be proved by localization since $(\bigcap M)_S\subset\bigcap M_S$.

COROLLARY 5.6. *If $m-1|n-1$ then $I_m(R)\supset I_n(R)$. In particular, $I_2(R)$ is the greatest ideal in the collection.*

COROLLARY 5.7. *If R is Noetherian then $R/I_m(R)$ is finite. (More precisely, it is a finite product of finite fields).*

Proof. Use Proposition 5.5, Corollary 5.4 and the Chinese Remainder Theorem. (Another proof follows from the von Neumann regularity of $R/I_m(R)$).

Suppose that R is a Noetherian ring and X is a finitely generated R-module. Then, by [5] (Theorem 2.3 and Theorem 1.4)

$$(5.1)\qquad \tilde{\Gamma}^m_R(X) \approx \bigoplus_{\substack{M \in \operatorname{Max}(R)\\ |R/M| < m}} \tilde{\Gamma}^m_{R/(M)}(X/(M)X)$$

where (M) denotes some power of M (depending on m and X). Observe that $R/(M)$ is local Artinian. Consequently, any finitely generated $R/(M)$-module has finite length (see for example [1]), and hence is finite as a set, because so is the only simple $R/(M)$-module R/M. This proves an analogue of Corollary 5.7 (see also [5], Corollary 2.2):

COROLLARY 5.8. *If R is Noetherian and X is a finitely generated R-module then $\tilde{\Gamma}^m_R(X)$ is finite. Consequently, $\tilde{\Gamma}^m_R(X)$ is a torsion module provided that R is infinite.*

If $m \leqslant 5$ and R is Noetherian then (5.1) holds for $(M) = M$ (see [5], Corollary 3.3). If, moreover, $X = R^n$ for $n > 1$ then all the direct summands are non-zero (see [4], Lemma 6.3) and hence

$$\operatorname{Ann}(\tilde{\Gamma}^m_R(R^n)) = \bigcap \{M \in \operatorname{Max}(R);\ |R/M| < m\} = \bigcap_{k=2}^{m-1} I_k(R) = \begin{cases} I_2(R), & m = 3, \\ I_3(R), & m = 4, \\ I_3(R) \cap I_4(R), & m = 5. \end{cases}$$

We are going to compute directly $\tilde{\Gamma}^m_R(R^n)$ for $m \leqslant 5$ (Corollary 5.10 below), which will give us the above formulas for arbitrary R. By (4.2) and Corollary 4.2 it suffices to determine $\tilde{\Gamma}^{m,k} = \tilde{\Gamma}^{m,k}_R(Re_1, \dots, Re_k)$ for $1 < k < m$, where $e_1, \dots, e_k$ form the standard basis of R^k. Let $(m_1, \dots, m_k)$ denote the base element $e_1^{(m_1)} \dots e_k^{(m_k)}$ of $\Gamma^{m,k}_R(Re_1, \dots, Re_k)$. Then (4.5) shows that $\overline{\Gamma}^{m,k} = \overline{\Gamma}^{m,k}_R(Re_1, \dots, Re_k)$ is generated by the following elements:

$$\sum_{\substack{m_1 + \dots + m_k = m\\ m_1, \dots, m_k > 0}} r_1^{m_1} \dots r_k^{m_k}(m_1, \dots, m_k), \qquad r_i \in R.$$

One of them, obtained for $r_1 = \dots = r_k = 1$, will be denoted by σ.

THEOREM 5.9. *In the above notation*

(1) $\overline{\Gamma}^{m,m-1} = R\sigma \oplus \bigoplus_{i=1}^{m-2} I_2(R)(1, \dots, 1, \overset{i}{2}, 1, \dots, 1)$, $\tilde{\Gamma}^{m,m-1} \approx (R/I_2(R))^{m-2}$ $(m \geqslant 3)$,

(2) $\overline{\Gamma}^{4,2} = R\sigma \oplus I_2(R)(2, 2) \oplus I_3(R)(3, 1)$, $\tilde{\Gamma}^{4,2} \approx R/I_2(R) \oplus R/I_3(R)$,

(3) $\overline{\Gamma}^{5,2} = R\sigma \oplus I_3(R)(2, 3) \oplus I_3(R)(3, 2) \oplus I_4(R)((4, 1) + (2, 3))$, $\tilde{\Gamma}^{5,2} \approx (R/I_3(R))^2 \oplus R/I_4(R)$,

(4) $\overline{\Gamma}^{5,3} = R\sigma \oplus I_3(R)(1, 3, 1) \oplus I_3(R)(3, 1, 1) \oplus I_2(R)(1, 2, 2) \oplus I_2(R)(2, 1, 2) \oplus I_2(R)(2, 2, 1)$, $\tilde{\Gamma}^{5,3} \approx (R/I_3(R))^2 \oplus (R/I_2(R))^3$.

Proof. It suffices to consider only $\overline{\Gamma} = \overline{\Gamma}^{m,k}$. Any generator of $\overline{\Gamma}$ can be expressed as an element of the right-hand side in the following way:

1° $\sum_{i=1}^{m-1} r_1 \dots r_i^2 \dots r_{m-1}(1, \dots, \overset{i}{2}, \dots, 1) = r_1 \dots r_{m-1}^2\sigma + \sum_{i=1}^{m-2} r_1 \dots \hat{r}_i \dots r_{m-2} \times$
$\times (r_i^2 r_{m-1} - r_i r_{m-1}^2)(1, \dots, \overset{i}{2}, \dots, 1)$,

2° $rs^3(1, 3) + r^2s^2(2, 2) + r^3s(3, 1) = rs^3\sigma + s(r^2s - rs^2)(2, 2) + (r^3s - rs^3)(3, 1)$,

3° $rs^4(1, 4) + r^2s^3(2, 3) + r^3s^2(3, 2) + r^4s(4, 1) = rs^4\sigma + r(rs^3 - r^3s)(2, 3) +$
$+ s(r^3s - rs^3)(3, 2) + (r^4s - rs^4)((4, 1) + (2, 3))$,

4° $rst^3(1, 1, 3) + rs^3t(1, 3, 1) + r^3st(3, 1, 1) + rs^2t^2(1, 2, 2) + r^2st^2(2, 1, 2) +$
$+ r^2s^2t(2, 2, 1) = rst^3\sigma + r(s^3t - st^3)(1, 3, 1) + s(r^3t - rt^3)(3, 1, 1) +$
$+ rt(s^2t - st^2)(1, 2, 2) + st(r^2t - rt^2)(2, 1, 2) + (t(r^2s^2 - rs) + rs(t - t^3))(2, 2, 1)$.

It suffices to prove that all the direct summands are contained in $\overline{\Gamma}$. Of course, this is evident for $R\sigma$.

(1) For any $i \leqslant m-2$ put $r_j = 1$ $(j \neq i)$ on the right-hand side of 1°.

(2) It follows from 2° that $s(r^2s - rs^2)(2, 2) + (r^3s - rs^3)(3, 1) \in \overline{\Gamma}$. Interchanging r and s and summing up, we obtain $(s-r)(r^2s - rs^2)(2, 2) = -rs(r-s)^2(2, 2) \in \overline{\Gamma}$. For $s = r-1$ this gives us $I_2(R)(2, 2) \subset \overline{\Gamma}$. Then also $I_3(R)(3, 1) \subset \overline{\Gamma}$.

(3)′ Suppose that 2 is invertible in R. Then $I_4(R) = I_2(R) = R$ by Corollary 5.3. Let us exchange r for $-r$ on the left-hand side of 3°. The summation shows that $r^2s^3(2, 3) + r^4s(4, 1) \in \overline{\Gamma}$. In particular $(2, 3) + (4, 1) \in \overline{\Gamma}$, as required, and hence $(r^2s^3 - r^4s)(2, 3) \in \overline{\Gamma}$. For $r = 1$ this gives us $I_3(R)(2, 3) \subset \overline{\Gamma}$, and the symmetric consideration shows that $I_3(R)(3, 2) \subset \overline{\Gamma}$.

(3)″ Suppose that 2 is non-invertible. Since both sides of (3) commute with localizations, it suffices to assume that R is a local ring. Hence any odd integer is invertible in R; in particular $I_2(R) = I_3(R) \supset I_4(R)$. Putting $s = 1$ and $r = \pm 2, \pm 3$ on the left-hand side of 3°, we obtain (since 3 is invertible) the following elements of $\overline{\Gamma}$:

$$\pm 2(1, 4) + 4(2, 3) \pm 8(3, 2) + 16(4, 1), \qquad \pm(1, 4) + 3(2, 3) \pm 9(3, 2) + 27(4, 1).$$

As a consequence, $\pm(1, 4) + (2, 3) \mp (3, 2) - 11(4, 1) \in \overline{\Gamma}$. These elements (and σ) give the following elements of $\overline{\Gamma}$:

$$2(3, 2) + 12(4, 1), \quad 2((2, 3) - 11(4, 1)), \quad 2((1, 4) - (3, 2)),$$

and, by symmetry, $2((2, 3) - (4, 1))$. Then $20(4, 1) \in \overline{\Gamma}$, hence $4(4, 1) \in \overline{\Gamma}$ (because 5 is invertible), and consequently (from above) $2(3, 2), 2(1, 4) \in \overline{\Gamma}$. By symmetry, $2(2, 3), 2(4, 1) \in \overline{\Gamma}$.

Put $s = 1$ and $r+1$ instead of r on the left-hand side of 3° and compute the defect. This gives us $2r(2, 3) + 3(r + r^2)(3, 2) + (4r + 6r^2 + 4r^3)(4, 1) \in \overline{\Gamma}$. Because of the above, $(r+r^2)(3,2)\in\overline{\Gamma}$. Hence $I_2(R)(3,2)\subset\overline{\Gamma}$, by symmetry $I_2(R)(2,3)\subset\overline{\Gamma}$, and the remaining inclusion follows from 3°.

(4)′ Suppose that 2 is invertible in R, and hence $I_2(R)=R$. Putting $-r$ instead of r in 4° and summing up, we obtain $r^2st^2(2,1,2)+r^2s^2t(2,2,1)\in\overline{\Gamma}$. Doing the same with s and the resulting element, we prove that $(2,2,1)$ — and by symmetry $(2,1,2)$ and $(1,2,2)$ — belong to $\overline{\Gamma}$. Hence the right-hand side of 4° for $s=t=1$ shows that $I_3(R)(3,1,1)$ — and obviously $I_3(R)(1,3,1)$ — are contained in $\overline{\Gamma}$.

(4)″ Suppose that 2 is non-invertible. As in (3)″ we can assume that 3 is invertible; hence $I_2(R)=I_3(R)$. Put $r+1$ instead of r and $s=t=1$ on the left-hand side of 4° and compute the defect. This gives us

$$3(r+r^2)(3,1,1)+2r\big((2,1,2)+(2,2,1)\big)\in\overline{\Gamma}. \tag{5.2}$$

Put $r=3$ and $s=t=1$ in 4°, cancel 3 and substract σ. This gives us

$$8(3,1,1)+2\big((2,1,2)+(2,2,1)\big)\in\overline{\Gamma}.$$

Comparing with (5.2) for $r=1$ we obtain $2(3,1,1)\in\overline{\Gamma}$, and hence $2\big((2,1,2)+$ $+(2,2,1)\big)\in\overline{\Gamma}$. In view of (5.2), $I_2(R)(3,1,1)$ — and also $I_2(R)(1,3,1)$, $I_2(R)(1,1,3)$ — are contained in $\overline{\Gamma}$. Then the right-hand side of 4° for $s=t=1$ shows that $I_2(R)\big((2,1,2)+(2,2,1)\big)\subset\overline{\Gamma}$. Since $I_2(R)\sigma\subset\overline{\Gamma}$, it follows that $I_2(R)(1,2,2)\subset\overline{\Gamma}$. The rest is given by symmetry.

COROLLARY 5.10. $\tilde{\Gamma}^3_R(R^n)\approx\binom{n}{2}(R/I_2(R))$,

$$\tilde{\Gamma}^4_R(R^n)\approx\left(\binom{n}{2}+2\binom{n}{3}\right)(R/I_2(R))\oplus\binom{n}{2}(R/I_3(R)),$$

$$\tilde{\Gamma}^5_R(R^n)\approx3\binom{n+1}{4}(R/I_2(R))\oplus2\binom{n+1}{3}(R/I_3(R))\oplus\binom{n}{2}(R/I_4(R)).$$

where sM denotes $M\oplus\ldots\oplus M$ (s times) and $\binom{n}{k}=0$ for $n<k$.

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