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On d-paracompactness and related properties

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Abstract. We give a simple definition of d-paracompact spaces and use it in order to prove that the following classes of spaces are preserved by perfect mappings: the class of perfect preimages of metacompact (metalindelöf) Moore spaces and the class of spaces which, for an arbitrary open cover $\mathcal{U}$, admit a perfect $\mathcal{U}$-mapping onto a Moore space. These results were announced in [Ch6] and it was shown there that the class of perfect preimages of Moore spaces is not preserved by perfect mappings.

A method of constructing mappings onto developable spaces has been introduced in [P]. This method uses the concept of d-paracompact spaces. The definition of d-paracompactness in [P] is very technical. A simpler but still technical concept of a kernel-normal sequence of open covers has been introduced in [Br].

In the first section of this paper we give a simple definition of d-paracompactness which is analogous to certain characterizations of paracompactness and subparacompactness. We also define related concepts of d-regularity, d-normality and collectionwise d-normality. All these properties are weaker than developability ($= d$-metrizability). We prove that our definition of d-paracompactness is equivalent to the one given in [P] and show some simple facts about related concepts which are useful in further investigations of d-paracompactness.

In the second section we prove that, in the class of metacompact (metalindelöf) spaces, d-paracompactness shows some analogies with the weaker concept of subparacompactness. In particular, it is preserved (in both directions) by perfect mappings and, consequently, the class of perfect preimages of metacompact (metalindelöf) Moore spaces is preserved by perfect mappings (this solves Problem 3.1 of [Ch5]).

The third section is devoted to the investigation of the preservation of d-paracompactness by perfect mappings. We give a method of constructing perfect mappings from spaces which are not d-normal (d-normal but not d-paracompact) onto Moore spaces. According to the results of the second section, such Moore spaces cannot be metalindelöf. We prove that perfect images of d-paracompact p-spaces are d-paracompact p-spaces. This shows that the property of having, for an arbitrary open cover $\mathcal{U}$, a perfect $\mathcal{U}$-mapping onto a Moore space is an invariant of perfect mappings.

In the third section we are forced to use methods of the base of countable order theory. The fourth section contains some other applications of this theory in the investigation of covering properties.

The terminology and notation is from [E1]. In the last two sections we use the methods of the base of countable order theory of H. H. Wicke and J. M. Worrell, Jr. as presented in [ChČN]. All mappings are assumed to be continuous and onto. All spaces which are not explicitly said to be T_1-spaces are assumed to be Hausdorff, but p-spaces and Moore spaces are regular.

The symbol δ always denotes a countable collection of open covers of a given space X and, for a subset V of X,

$$\mathrm{int}_\delta V = \{x \in V\colon \mathrm{St}(x, \mathscr{D}) \subset V \text{ for a } \mathscr{D} \in \delta\}.$$

If $\mathscr{V}$ is a collection of subsets of X, then $\mathrm{int}_\delta \mathscr{V} = \{\mathrm{int}_\delta V\colon V \in \mathscr{V}\}$ [Br].

If $\mathscr{U}$ is an open cover of X and g maps X onto S, then g is said to be a *$\mathscr{U}$-mapping* if, for an open cover $\mathscr{H}$ of S, $\{g^{-1}(H)\colon H \in \mathscr{H}\}$ refines $\mathscr{U}$. If g is a closed mapping, then g is a $\mathscr{U}$-mapping iff the fibers of g refine $\mathscr{U}$.

A cover $\mathscr{U}$ of X is said to be d-normal if there exists a $\mathscr{U}$-mapping of X onto a developable T_1-space.

A space X is said to be *metalindelöf* ((σ-) *paralindelöf*) if each open cover of X has a point-countable ((σ-) locally countable) open refinement ([FR]).

1. Preliminaries. Pareek in [P] defined d-paracompactness and proved that a space X is *d-paracompact* iff for each open cover $\mathscr{U}$ of X there exists a $\mathscr{U}$-mapping of X onto a developable T_1-space. The definition in [P] is very technical. In order to investigate d-paracompactness, it is convinient to start with a simpler definition.

1.1. Definition. A space X is said to be *subparacompact* (*d-paracompact*) if for each open cover $\mathscr{V}$ of X there exist a collection δ and an (open) cover $\mathscr{W}$ refining $\mathrm{int}_\delta \mathscr{V}$.

We establish the equivalence of the two definitions of d-paracompactness by proving

1.2. Proposition. *A space X is d-paracompact iff for each open cover $\mathscr{U}$ of X there exists a $\mathscr{U}$-mapping of X onto a developable T_1-space.*

Proof. The "if" part is obvious. In order to prove the "only if" part assume that $\mathscr{U}$ is an open cover of a d-paracompact space X. By inductively applying 1.1 we can find a countable collection δ such that $\mathscr{U} \in \delta$ and, for each $\mathscr{V} \in \delta$, there exists a $\mathscr{W} \in \delta$ refining $\mathrm{int}_\delta \mathscr{V}$. Thus $\mathscr{U}$ is a member of the kernel-normal (in the terminology of [Br]) collection δ and, consequently, $\mathscr{U}$ is a d-normal cover [Br, Proposition 1].

Since both concepts of d-paracompactness are equivalent, we have

1.3. Theorem [P]. *A space X is a d-paracompact p-space iff for each open cover $\mathscr{U}$ of X there exists a perfect $\mathscr{U}$-mapping of X onto a Moore space.*

On the analogy of paracompactness and subparacompactness (see [Ch2]), we introduce the classes of d-regular, d-normal and collectionwise d-normal spaces.

1.4. Definition. A space X is said to be *d-normal* (*d-regular* [He]) if for each closed (one-point) set K and open set V containing K there exist an F_σ-set L and an open set W such that $K \subset W \subset L \subset V$.

1.5. Remark. By inductively applying the definition of d-normality we can obtain, for a closed set K and its neighbourhood V, an open F_σ-set W such that $K \subset W \subset V$. Thus d-normality is equivalent to D-normality introduced in [Br].

Observe that the set L can, equivalently, be replaced by $\mathrm{int}_\delta V$ for a certain δ (if $L = \bigcup_{n \geqslant 1} L_n$ then $\delta = \{\{V, X \setminus L_n\}\colon n \geqslant 1\}$). This suggests

1.6. Definition. A space X is *collectionwise d-normal* if for each discrete collection $\mathscr{K}$ of closed sets and its open expansion $\{V(K)\colon K \in \mathscr{K}\}$ there exist a δ and an open expansion $\{W(K)\colon K \in \mathscr{K}\}$ of $\mathscr{K}$ such that $K \subset W(K) \subset \mathrm{int}_\delta V(K)$ for $K \in \mathscr{K}$.

1.7. Remark. If $\mathrm{int}_\delta V$ is replaced by $\mathrm{Int}_\delta V = \bigcup \{W \subset V\colon W$ is open and $\mathrm{St}(W, \mathscr{D}) \subset V$ for a $\mathscr{D} \in \delta\}$, then 1.1 gives a characterization of paracompactness form [A], the concept of kernel-normality (see the proof of 1.2) gives a characterization of normal covers, 1.4 gives a characterization of normal spaces [E1, 1.5.14] and 1.6, a characterization of collectionwise normality (see [E1, p. 410]).

The following diagram describes the situation (see [Ch2]).

metrizable	⇒	paracompact	⇒	collectionwise normal	⇒	normal	⇒	regular
⇓		⇓		⇓		⇓		⇓
developable	⇒	d-paracompact	⇒	collectionwise d-normal	⇒	normal	⇒	d-regular
		⇓		⇓		⇓		
		subparacompact	⇒	collectionwise subnormal	⇒	subnormal		

A metacompact subparacompact space which is not d-normal can be obtained by removing the pair of non-isolated points from $A(\aleph_0) \times A(\aleph_1)$ [E1, 2.3.36].

This shows that the properties of the third row are essentially weaker than the properties of the second row.

We end this section by giving some results showing the analogies between the properties of the second and the first row.

1.8. Proposition. *A space X is d-paracompact iff X is a submetacompact* (*$=\theta$-refinable*) *collectionwise d-normal space* (see [E1, 5.3.3]).

From 1.8 and 1.3, it follows that the assumption that Y is perfect in Theorem 2.2 of [Ch5] can be replaced by the assumption that Y is a collectionwise d-normal space, which makes this result parallel to Theorem 1.2. B of [Ch5].

Another version of 1.8 is

1.9. Proposition. *If $\mathscr{V}$ is a point-finite open cover of a collectionwise d-normal space X, then there exist a collection δ consisting of point-finite open covers and a point-finite open cover $\mathscr{W}$ refining $\mathrm{int}_\delta \mathscr{V}$.*

Proof. Standard methods allow us to construct an open cover

$$\mathcal{W} = \{W(V):\ V \in \mathcal{V}\}$$

and a collection $\delta' = \{\mathcal{D}'_n:\ n \geqslant 1\}$ such that $W(V) \subset \operatorname{int}_{\delta'} V$ for $V \in \mathcal{V}$ [E1, 5.3.3]. Clearly, $\mathcal{W}$ is a point-finite cover. Thus, it is sufficient to replace each $\mathcal{D}'_n$ by a point-finite open cover $\mathcal{D}_n$ such that $\operatorname{St}(x, \mathcal{D}'_n) \subset V$ implies $\operatorname{St}(x, \mathcal{D}_n) \subset V$ for $V \in \mathcal{V}$.

Fix an $n \geqslant 1$ and put $E_n(V) = \{x:\ \operatorname{St}(x, \mathcal{D}'_n) \subset V\} = X \setminus \operatorname{St}(X \setminus V, \mathcal{D}'_n)$. Then $\{E_n(V):\ V \in \mathcal{V}\}$ is a locally finite collection of closed subsets of X.

Let $\mathcal{D}_n = \{D_n(x):\ x \in X\}$, where $D_n(x) = \bigcap \{V:\ x \in E_n(V)\} \setminus \bigcup \{E_n(V):\ x \notin E_n(V)\}$. It is easy to see that $\mathcal{D}_n$ is a point-finite open cover of X and that $x_0 \in E_n(V_0)$ implies that $\operatorname{St}(x_0, \mathcal{D}_n) \subset V_0$ (see [E1, 5.1.13]).

Using 1.9 and the proof of 1.2, we obtain (see [Br, Theorem 2])

1.10. Proposition. *If $\mathcal{U}$ is a point-finite open cover of a collectionwise d-normal space X, then there exists a $\mathcal{U}$-mapping of X onto a metacompact developable T_1-space.*

From 1.10, it follows that the assumption that Y is perfect in Theorem 2.2.B of [Ch4] can be replaced by the assumption that Y is a collectionwise d-normal space, which makes this result parallel to Theorem 1.2.B of [Ch4].

2. d-paracompact metalindelöf spaces. In the class of metalindelöf spaces, d-paracompactness behaves as subparacompactness.

2.1. Proposition. *For a space X the following conditions are equivalent:*

(i) *X is a d-paracompact metalindelöf space,*

(ii) *X is subparacompact and, for each open cover $\mathcal{V}$ of X, there exist an open cover $\mathcal{W}$ of X and, for $n \geqslant 1$, collections $\{E_n(W):\ W \in \mathcal{W}\}$ of closed subsets of X which are locally countable as indexed collections and satisfy*

$$(*)\quad \text{for each } W \in \mathcal{W},\ W \subset \bigcup_{n \geqslant 1} E_n(W) \subset V(W) \text{ for a } V(W) \in \mathcal{V},$$

(iii) *for each open cover $\mathcal{V}$ of X, there exist an open cover $\mathcal{W}$ and collections $\{E_n(W):\ W \in \mathcal{W}\}$ of closed subsets of X which are discrete as indexed collections and satisfy* $(*)$,

(iv) *same as* (iii) *with "discrete" replaced by "locally finite",*

(v) *each open cover $\mathcal{V}$ of X has an open σ-discretely decomposable refinement (this means that one can require that $W = \bigcup_{n \geqslant 1} E_n(W)$ in* (iii)).

Proof. (i) $\Rightarrow$ (ii). We can assume that $\mathcal{V}$ is a point-countable open cover of X, take a δ and an open shrinking $\mathcal{W}$ of $\operatorname{int}_\delta \mathcal{V}$ and define $E_n(W)$ as in the proof of 1.9.

(ii) $\Rightarrow$ (iii). This implication follows from [M, 3.5]. We give the proof for the sake of completeness.

Consider a collection $\{E(W):\ W \in \mathcal{W}\}$ of closed subsets of the subparacompact space X and assume that it is locally countable as an indexed collection. Take a cover $\mathcal{L} = \bigcup_{i \geqslant 1} \mathcal{L}_i$ such that each $\mathcal{L}_i$ is a discrete collection of closed subsets of X and $\mathcal{W}(L) = \{W \in \mathcal{W}:\ E(W) \cap L \neq \emptyset\}$ is countable for $L \in \mathcal{L}$. If $\{W_j(L): j \geqslant 1\}$ is an enumeration of $\mathcal{W}(L)$ for $L \in \mathcal{L}$, then the sets $E_{i,j}(W) = E(W) \cap \bigcup \{L \in \mathcal{L}_i:\ W = W_j(L)\}$ form, for $i, j \geqslant 1$, discrete collections indexed by $\mathcal{W}$ and satisfy $E(W) = \bigcup_{i,j \geqslant 1} E_{i,j}(W)$. This shows that (ii) implies (iii).

The equivalence of (iii) and (v) can be proved by induction as in 1.5.

The remaining implications can be proved either by observing that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) or, more directly, by using a method from [E1, 5.1.13] (as in the proof of 1.9) in order to show that (iv) $\Rightarrow$ (i).

The characterizations given in 2.1 allow to use standard methods in order to prove the following two theorems:

2.2. Theorem. *If $f:\ X \to Y$ is a closed mapping and X is a d-paracompact metalindelöf space, then Y is a d-paracompact metalindelöf space* (1).

Proof. From [Bu1], it follows that Y is subparacompact. Thus, in order to prove that Y satisfies (ii) of 2.1, it suffices to show that for any discrete collection $\mathcal{K}$ of closed subsets of Y and its open expansion $\mathcal{U} = \{U(K):\ K \in \mathcal{K}\}$ there exist an open expansion $\mathcal{H} = \{H(K):\ K \in \mathcal{K}\}$ of $\mathcal{K}$ and collections $\{F_n(K):\ K \in \mathcal{K}\}$ of closed subsets of Y which are locally countable as indexed collections and satisfy $K \subset H(K) \subset \bigcup_{n \geqslant 1} F_n(K) \subset U(K)$ for $K \in \mathcal{K}$.

We can assume that $\mathcal{U}$ is a point-countable expansion of $\mathcal{K}$ and that $U(K) \cap \bigcup \mathcal{K} \setminus \{K\} = \emptyset$ for $K \in \mathcal{K}$. Let $\mathcal{V} = \{f^{-1}(U):\ U \in \mathcal{U}\} \cup \{X \setminus \bigcup \mathcal{K}\}$ and let $\mathcal{W}$ and $\mathcal{E}_n$ satisfy (iii) with respect to $\mathcal{V}$ in X. Assume that $\mathcal{W}$ is a shrinking of $\mathcal{V}$. Then $H(K) = Y \setminus f(X \setminus W(K))$ and $F_n(K) = f(E_n(W(K)))$, where $W(K)$ is the element of $\mathcal{W}$ corresponding to $f^{-1}(U(K))$, satisfy our requirements.

2.3. Theorem. *If $f:\ X \to Y$ is a perfect mapping and Y is a d-paracompact metalindelöf space, then X is a d-paracompact metalindelöf space.*

Proof. Let $\mathcal{U}$ be an open cover of X. The space X is d-regular [He, 5.10] (see [E1, 3.7.23]). Thus, there exist an open cover $\mathcal{G}$ of X and, for $n \geqslant 1$, collections $\{L_n(G):\ G \in \mathcal{G}\}$ of closed subsets of Y such that, for each $G \in \mathcal{G}$, $G \subset \bigcup_{n \geqslant 1} L_n(G) \subset U(G)$ for a $U(G) \in \mathcal{U}$.

Since X is the perfect preimage of the space Y satisfying (iv) of 2.1, there exist an open cover $\mathcal{H}$ of X and collections $\{F_n(H):\ H \in \mathcal{H}\}$ of closed subsets of X which are locally finite as indexed collections and satisfy $H \subset \bigcup_{n \geqslant 1} F_n(H) \subset \bigcup \mathcal{G}(H)$ for a finite subcollection $\mathcal{G}(H)$ of $\mathcal{G}$.

For $H \in \mathcal{H}$ and $G \in \mathcal{G}(H)$ put $F_{n,m}(H, G) = F_n(H) \cap L_m(G)$. For every $n, m \geqslant 1$, the collection of such $F_{n,m}(H, G)$ is locally finite as an indexed collection. Moreover, $H \cap G \subset \bigcup_{n,m \geqslant 1} F_{n,m}(H, G) \subset U(G)$. This proves that X satisfies (iv) of 2.1.

(1) One can prove that in the class of orthocompact spaces d-paracompactness can be characterized by a condition obtained from (iii) of 2.1 by replacing discrete collections by closure-preserving collections and deduce that the metalindelöf property can be replaced by orthocompactness in 2.2.

From 2.2, 2.3 and 1.3, we obtain

2.4. COROLLARY. *The property of being a perfect preimage of a metacompact (metalindelöf) Moore space is an invariant of perfect mappings.*

Corollary 2.4 solves Problem 3.1 from [Ch5].

From 1.8, 2.3 and 1.3, we obtain (see [Ch5, Theorem 2.2]).

2.5. COROLLARY. *If a regular space Y is an open and compact image of a perfect preimage of a metric space, then Y is a perfect preimage of a metacompact Moore space iff Y is a collectionwise d-normal space.*

3. d-paracompact spaces. We start with two theorems showing that a perfect preimage of a d-paracompact space need not be d-paracompact.

3.1. THEOREM. *Let S be a separable Moore space containing a discrete closed subset of cardinality $\mathfrak{c}$. If every open subset of Z is the union of no more than $\mathfrak{c}$ closed subsets, then $S\times Z$ is d-normal iff Z is perfect.*

Proof. The "if" part is obvious (see [E2, 2.4.7]). Assume that $X = S\times Z$ is d-normal and let $Q = \{q_m\colon m\geqslant 1\}$ be a countable dense subset of S and R a discrete closed subset of cardinality $\mathfrak{c}$ disjoint from Q.

Let U be an open subset of Z, then $U = \bigcup\{K_r\colon r\in R\}$, where K_r is closed in Z for $r\in R$. We shall show that U is an F_σ-set.

Put $K = \bigcup\{\{r\}\times K_r\colon r\in R\}$ and $V = S\times U$. Clearly, K is closed in X and V is an open set containing K. Thus, there exist an open set W and closed sets L_n, $n\geqslant 1$, satisfying $K\subset W\subset \bigcup_{n\geqslant 1} L_n\subset V$.

Let $L_{n,m} = \{z\in Z\colon (q_m, z)\in L_n\}$. The sets $L_{n,m}$ are closed subsets of Z contained in U. If $z\in U$, then $z\in K_r$ for a certain $r\in R$ and, consequently, $(r, z)\in K\subset W$. Hence, there exist $n, m\geqslant 1$ such that $(q_m, z)\in L_n$ and it follows that $z\in L_{n,m}$.

3.2. THEOREM. *Let S be a separable Moore space containing a discrete closed subset of cardinality $\mathfrak{c}$. If Z has a collection γ of no more than $\mathfrak{c}$ open covers such that $z\in U$ and U open in Z implies $z\in \mathrm{St}(z, \mathscr{G})\subset U$ for a $\mathscr{G}\in\gamma$, then $S\times Z$ is d-paracompact iff Z is developable.*

Proof. The "if" part is obvious. Assume that $X = S\times Z$ is d-paracompact and choose Q and R in S as in the preceding proof.

By imitating the proof of the metrizability of collectionwise normal developable spaces [E1, 5.4.1], we can construct discrete collections $\mathscr{E}_r$ of closed subsets of Z and their open expansions $\{G_r(E)\colon E\in\mathscr{E}_r\}$, for $r\in R$, such that $z\in U$ and U open in Z implies that there exist $r\in R$ and $E\in\mathscr{E}_r$ satisfying $z\in E\subset G_r(E)\subset U$.

Put $\mathscr{K} = \{\{r\}\times E\colon r\in R \text{ and } E\in\mathscr{E}_r\}$ and $\mathscr{V} = \{((S\setminus R)\cup\{r\})\times G_r(E)\colon r\in R$ and $E\in\mathscr{E}_r\}$. Clearly, $\mathscr{K}$ is a discrete collection of closed subsets of Z and $\mathscr{V}$ is its open expansion. Thus, there exist an open expansion $\mathscr{W}$ of $\mathscr{K}$ and a $\delta = \{\mathscr{D}_n\colon n\geqslant 1\}$ such that $K\subset W(K)\subset \mathrm{int}_\delta V(K)$ for $K\in\mathscr{K}$.

Let $\mathscr{D}_{n,m}$ be the collection of the projections onto Z of the elements of $\{D\cap(\{q_m\}\times Z)\colon D\in\mathscr{D}_n\}$. We shall prove that $\{\mathscr{D}_{n,m}\colon n, m\geqslant 1\}$ is a development of Z. Suppose that $z\in U$ and U is open in Z. Then there exist $r\in R$ and $E\in\mathscr{E}_r$ such that $z\in E\subset G_r(E)\subset U$. For $K = \{r\}\times E\in\mathscr{K}$ we have $(r, z)\in K$ and, consequently, there exists an $m\geqslant 1$ such that $(q_m, z)\in W(K)$. Since $W(K)\subset \mathrm{int}_\delta V(K)$, there exists an $n\geqslant 1$ satisfying $\mathrm{St}((q_m, z), \mathscr{D}_n)\subset V(K)$ and from the definition of $V(K)$ we obtain $\mathrm{St}(z, \mathscr{D}_{n,m})\subset G_r(E)\subset U$.

3.3. EXAMPLE. Let S be a locally compact Moore space obtained by adding to a countable set Q of isolated points a set R of $\mathfrak{c}$ almost disjoint infinite subsets of Q [E1, 3.6.1]. If $Z = A(\aleph_1)$, then $X = S\times Z$ is not d-normal and the projection of X onto S is a perfect mapping. If Z is the two arrows space [E1, 3.10.C], then $X = S\times Z$ is a perfect (therefore d-normal) space which is not d-paracompact and is a perfect preimage of S. Moreover, X is a d-normal first countable space which is not collectionwise d-normal.

Example 3.3 shows that the conjecture in [P, p. 1041] that perfect preimages of Moore spaces are d-paracompact is false. In [Ch6] we use this example in the construction of examples showing that the property of being a perfect preimage of a Moore space is not an invariant of perfect mappings (see 2.4 and 3.5).

We do not know whether d-paracompactness is an invariant of perfect mappings. However, in the class of p-spaces we have

3.4. THEOREM. *If $f\colon X\to Y$ is a perfect mapping and X is a d-paracompact p-space, then Y is a d-paracompact p-space.*

From 3.4 and 1.3 we obtain

3.5. COROLLARY. *The property of having, for each open cover $\mathscr{U}$, a perfect $\mathscr{U}$-mapping onto a Moore space is an invariant of perfect mappings.*

Proof of 3.4. Let $f\colon X\to Y$ be a perfect mapping of a d-paracompact p-space X. We know [ChČN, 4.1] that Y is a subparacompact p-space. By virtue of 1.8 it suffices to show that Y is collectionwise d-normal. Our proof is similar to the proof of 4.1 in [ChČN].

Assume that $\mathscr{K}$ is a discrete collection of closed subsets of Y and $\mathscr{U} = \{U(K)\colon K\in\mathscr{K}\}$ an open expansion of $\mathscr{K}$. Then, in X, for $f^{-1}(\mathscr{K})$ and its open expansion $\mathscr{V}_1 = f^{-1}(\mathscr{U})$ there exist an open expansion $\mathscr{W}_1 = \{W_1(K)\colon K\in\mathscr{K}\}$ and δ_1 such that

$$f^{-1}(K)\subset W_1(K)\subset \mathrm{int}_{\delta_1} f^{-1}(U(K)).$$

Since f is a closed mapping, we can shrink $\mathscr{W}_1$ to an expansion $\mathscr{V}_2$ of $f^{-1}(\mathscr{K})$ consisting of inverse images of open subsets of Y. Again, there exist an open expansion $\mathscr{W}_2$ of $f^{-1}(\mathscr{K})$ and δ_2 such that

$$f^{-1}(K)\subset W_2(K)\subset \mathrm{int}_{\delta_2} V_2(K).$$

Take δ_3 to be a countable collection of covers of X witnessing the fact that X is a p-space.

Put $\delta = \delta_1\cup\delta_2\cup\delta_3$, represent δ as $\{\mathscr{D}_n\}_{n\geqslant 1}$ and let $\mathscr{B}_n$ be a base of X refining $\mathscr{D}_n$ for $n\geqslant 1$.

From our construction it follows that each sequence $\{B_n\}_{n\geq 1}$ such that $B_n \in \mathscr{B}_n$ and $\bar{B}_{n+1} \subset B_n$ for $n \geq 1$, satisfies (B denotes $\bigcap_{n\geq 1} B_n$)

(1) if $W_1(K) \cap B \neq \emptyset$, then $B_n \subset f^{-1}(U(K))$ for a certain $n \geq 1$,

(2) if $W_2(K) \cap B \neq \emptyset$, then $f^{-1}f(B_n) \subset W_1(K)$ for a certain $n \geq 1$,

(3) if $B \neq \emptyset$, then B is compact and each open set containing B contains a B_n for a certain $n \geq 1$.

Let $H(K)$ be an open subset of Y containing K such that $f^{-1}(H(K)) \subset W_2(K)$. We shall show that there exists a countable collection γ of open covers of Y such that $K \subset H(K) \subset \operatorname{int}_\gamma U(K)$ for $K \in \mathscr{K}$.

From [ChČN, 1.1, 2.8] it follows that it is sufficient to construct a sequence (a sieve) $\mathscr{G} = \{\langle \mathscr{G}_n, A_n, \pi_n \rangle\}_{n\geq 1}$ such that $\mathscr{G}_n = \{G(\alpha)\colon \alpha \in A_n\}$ is an open cover of Y, $\pi_n\colon A_{n+1} \to A_n$ for $n \geq 1$, $G(\alpha_n) = \bigcup \{G(\alpha)\colon \alpha \in \pi_n^{-1}(\alpha_n)\}$ for $\alpha_n \in A_n$ and each sequence (thread of $\mathscr{G}$) $\{G(\alpha_n)\}_{n\geq 1}$ such that $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for $n \geq 1$ satisfies

(4) if $H(K) \cap \bigcap_{n\geq 1} G(\alpha_n) \neq \emptyset$, then $G(\alpha_n) \subset U(K)$ for a certain $n \geq 1$.

We construct $\mathscr{G}$ by induction on $n \geq 1$ in such a way that $A_n \subset Y^n$ and π_n is the restriction of the projection of Y^{n+1} onto Y^n to A_{n+1}.

Starting with $A_0 = \emptyset$, $G(\emptyset) = Y$ and $\mathscr{B}_0(\emptyset) = X$ we can construct, by induction on $n \geq 1$, $A_n \subset Y^n$ and, for $\alpha \in A_n$, $G(\alpha)$ open in Y and a finite subcollection $\mathscr{B}_n(\alpha)$ of $\mathscr{B}_n$ in such a way that for $\alpha' = (y_1, \ldots, y_{n-1}) \in A_{n-1}$ and $y \in G(\alpha')$ the sequence $\alpha = (y_1, \ldots, y_{n-1}, y) \in A_n$ and the following conditions are satisfied:

(5) $G(\alpha) = Y \setminus f(X \cap \bigcup \mathscr{B}_n(\alpha))$,

(6) $y \in G(\alpha)$,

(7) the elements of $\mathscr{B}_n(\alpha)$ intersect $f^{-1}(y)$,

(8) the closures of the elements of $\mathscr{B}_n(\alpha)$ refine $\mathscr{B}_{n-1}(\alpha')$.

Clearly, our construction ensures that $\mathscr{G}$ is a sieve of Y. Let $(y_1, y_2, \ldots)$ be a sequence of elements of Y such that $\alpha_n = (y_1, \ldots, y_n) \in A_n$ for $n \geq 1$. It remains to check that $\{G(\alpha_n)\}_{n\geq 1}$ satisfies (4).

Suppose that this sequence does not satisfy (4). Then there exist a $K \in \mathscr{K}$ and $y \in H(K) \cap \bigcap_{n\geq 1} G(\alpha_n)$, such that $G(\alpha_n) \not\subset U(K)$ for $n \geq 1$. Thus (5), (8) and König's lemma [ChČN, 1.4] imply that there exist sequences $\{B_n\}_{n\geq 1}$ and $\{D_n\}_{n\geq 1}$ such that, for $n \geq 1$, B_n, $D_n \in \mathscr{B}_n(\alpha_n)$, $\bar{B}_{n+1} \subset B_n$, $\bar{D}_{n+1} \subset D_n$, $B_n \cap f^{-1}(y) \neq \emptyset$ and $D_n \not\subset f^{-1}(U(K))$.

From the compactness of $f^{-1}(y) \subset W_2(K)$ it follows that $B = \bigcap_{n\geq 1} B_n$ intersects $W_2(K)$. Thus (2) ensures the existence of an $n \geq 1$ such that $f^{-1}f(B_n) \subset W_1(K)$. Consider $Z = B \cup \bigcup_{m>n} (\bar{B}_m \cap f^{-1}(y_m))$. By virtue of (3), Z is compact. Therefore $f^{-1}f(Z)$ is a compact subset of $f^{-1}f(B_n) \subset W_1(K)$. But (7) implies that each D_n intersects $f^{-1}f(Z)$ and we obtain a contradiction with (1), because no D_n is contained in $f^{-1}(U(K))$.

4. Monotonic generalizations. The proof of 3.4 suggests the possibility of defining monotonic generalizations of d-paracompactness and subparacompactness.

We shall define such generalizations and use them in order to prove the following three theorems:

4.1. THEOREM. *Let X be a d-regular submetacompact space. If X is σ-paralindelöf, then X is d-paracompact. If X has a σ-locally countable base, then X is developable.*

4.2. THEOREM. *Let X be a semistratifiable space* (see 4.8). *If X is metalindelöf, then X is d-paracompact. If X has a point-countable base, then X is developable.*

4.3. THEOREM. *Let $\mathscr{U}$ be a well-ordered open cover of a submetacompact space X and let $P(U) = U \setminus \bigcup \{U' \in \mathscr{U}\colon U' < U\}$. Then every $P(U)$ is a G_δ-set iff there exists a δ such that $P(U) \subset \operatorname{int}_\delta U$ for $U \in \mathscr{U}$.*

Theorem 4.1 is a slight generalization of some results in [Bu2]. The second part of 4.2 follows from [H] and [C]. Both parts of 4.2 follow directly from

4.4. LEMMA. *If $\mathscr{V}$ is a point-countable open cover of a semistratifiable space X, then there exists a δ such that $V = \operatorname{int}_\delta V$ for $V \in \mathscr{V}$ (or, equivalently* (see the proof of 2.1), *$\mathscr{V}$ is σ-discretely decomposable).*

Theorem 4.3 is equivalent to [J, 1.16] (see [ChZ]). Observe that the second example constructed in 3.3 gives a perfectly subparacompact space X which is not d-paracompact.

In what follows, by a refinement of an open collection $\mathscr{V}$ we always mean a collection $\mathscr{W}$ refining $\mathscr{V}$ together with a correspondence $W \to V(W)$ such that $W \subset V(W)$ for $W \in \mathscr{W}$.

If $\mathscr{W}$ is a refinement of $\mathscr{V}$, then a sequence $\{\mathscr{B}_n\}_{n\geq 1}$ of bases of X is said to be a $(\mathscr{W}, \mathscr{V})$-sequence if each decreasing sequence $\{B_n\}_{n\geq 1}$ such that $B_n \in \mathscr{B}_n$ for $n \geq 1$ satisfies

$(\mathscr{W}, \mathscr{V})$ if $W \cap \bigcap_{n\geq 1} B_n \neq \emptyset$, then $B_n \subset V(W)$ for a certain $n \geq 1$.

4.5. DEFINITION. A space X is said to be *monotonically subparacompact* (*d-paracompact*) if for each open cover $\mathscr{V}$ of X there exist an (open) cover $\mathscr{W}$ refining $\mathscr{V}$ and a $(\mathscr{W}, \mathscr{V})$-sequence of bases of X.

Since $(\mathscr{W}, \mathscr{V})$ is a monotonic property in the sense of [ChČN], it follows that the existence of a $(\mathscr{W}, \mathscr{V})$-sequence of bases is equivalent to the existence of a $(\mathscr{W}, \mathscr{V})$-sieve and, in the class of submetacompact spaces, it is equivalent to the existence of a δ such that $W \subset \operatorname{int}_\delta V(W)$ (see [ChČN, 2.8] or [WW2]). In particular, we have

4.6. PROPOSITION. *A space X is subparacompact* (*d-paracompact or developable*) *iff X is submetacompact and monotonically subparacompact* (*d-paracompact or developable*).

From 4.6 it follows that 4.1 is a consequence of

4.7. THEOREM. *Let X be a d-regular space. If X is σ-paralindelöf, then X is* *monotonically d-paracompact. If X has a σ-locally countable base, then X is monotonically developable.*

Proof. Let $\mathscr{V}$ be an open cover of a d-regular σ-paralindelöf space X. Then there exists a refinement $\mathscr{W} = \bigcup_{i\geqslant 1} \mathscr{W}_i$ of $\mathscr{V}$ such that each $\mathscr{W}_i$ is a locally countable collection and each $W\in\mathscr{W}$ is contained in an F_σ-set contained in $V(W)$. Thus, for each $W\in\mathscr{W}$, there exists a collection $\delta(W)$ such that $W\subset \operatorname{int}_{\delta(W)} V(W)$.

If H is an open set intersecting countably many elements of $\mathscr{W}_i$, then we can use $\delta = \bigcup\{\delta(W)\colon W\in\mathscr{W}_i \text{ and } W\cap H\neq\emptyset\}$ in order to construct a $(\mathscr{W}_i, \mathscr{V})$-sieve on H. Since X can be covered by such sets H, it follows that, for each $i\geqslant 1$, X has a $(\mathscr{W}_i, \mathscr{V})$-sieve and consequently a $(\mathscr{W}_i, \mathscr{V})$-sequence of bases. The canonical function from ω onto $\omega\times\omega$ orders these bases into a $(\mathscr{W}, \mathscr{V})$-sequence.

Suppose now that $\mathscr{V} = \bigcup_{j\geqslant 1}\mathscr{V}_j$ is a base of X such that $\mathscr{V}_j$ is locally countable for $j\geqslant 1$. Since X is a d-regular space with a σ-locally countable base, each $V\in\mathscr{V}$ can be covered by a collection $\mathscr{W}(V) = \bigcup_{i\geqslant 1}\mathscr{W}_i(V)$ of open sets such that each $\mathscr{W}_i(V)$ is locally countable and each $W\in\mathscr{W}(V)$ is contained in an F_σ-set contained in V.

Let $\mathscr{W}_{i,j} = \bigcup\{\mathscr{W}_i(V)\colon V\in\mathscr{V}_j\}$ and $\mathscr{W} = \bigcup_{i,j\geqslant 1}\mathscr{W}_{i,j}$. Then each $\mathscr{W}_{i,j}$ is a locally countable (in a strong sense; each $W\in\mathscr{W}_{i,j}$ is indexed by a $V = V(W)$ such that $W\in\mathscr{W}_i(V)$ and may occur in $\mathscr{W}_{i,j}$ many times) refinement of $\mathscr{V}$. Thus, we can repeat the reasoning from the first part of the proof in order to construct a $(\mathscr{W}, \mathscr{V})$-sequence of bases. This sequence will be, by virtue of our construction, a monotonic development of X.

Let us now turn to the proof of 4.2. As we have observed, it suffices to prove 4.4. We shall use the following definition of semistratifiability:

4.8. Definition [C]. A space X is said to be *semistratifiable* if each point $x\in X$ has a sequence of neighbourhoods $\{H_n(x)\}_{n\geqslant 1}$ such that $x\in\bigcap_{n\geqslant 1} H_n(x_n)$ implies that $\{x_n\}_{n\geqslant 1}$ converges to x.

Since semistratifiable spaces are subparacompact [C], it follows that 4.4 is a consequence of

4.9. Proposition. *If $\mathscr{V}$ is a point-countable open cover of a semistratifiable space X, then X has a $(\mathscr{V}, \mathscr{V})$-sieve.*

Proof. We shall apply the method of proof of Theorem 2.8 in [Ch1] (we cannot apply this theorem because $\mathscr{V}$ need not be a $(\mathscr{V}, \mathscr{V})$-cover in the sense of 2.7 in [Ch1]).

Let $\{H_n(x)\}_{n\geqslant 1}$ be a sequence of neighbourhoods of x for $x\in X$ as in 4.8 and let $\{V_i(x)\colon i\geqslant 1\}$ be a fixed enumeration of $\{V\in\mathscr{V}\colon x\in V\}$ for $x\in X$. As in the proof of 3.4, we construct the sieve $\mathscr{G}$ so that $A_n\subset X^n$, π_n is the restriction of the projection and, for $\alpha' = (x_1, \ldots, x_{n-1})\in A_{n-1}$ and $x_n\in G(\alpha')$, $\alpha = (x_1, \ldots, x_{n-1}, x_n)\in A_n$ and $G(\alpha)$ satisfies

(1) $x_n\in G(\alpha)\subset H_n(x_n)$,

(2) $G(\alpha)\subset\bigcap\{V_i(x_k)\colon x\in V_i(x_k) \text{ and } i, k<n\}$.

In order to prove that $\mathscr{G}$ is a $(\mathscr{V}, \mathscr{V})$-sieve suppose that $(x_1, x_2, \ldots)$ is a sequence of elements of X such that $\alpha_n = (x_1, \ldots, x_n)\in A_n$ and $\bigcap_{n\geqslant 1} G(\alpha_n)$ intersects an element V of $\mathscr{V}$. We have to prove that $G(\alpha_n)\subset V$ for a certain $n\geqslant 1$.

Let $x\in V\cap\bigcap_{n\geqslant 1} G(\alpha_n)$. From (1) it follows that $\{x_n\}_{n\geqslant 1}$ converges to x. Take a $k\geqslant 1$ such that $x_k\in V$ and $i\geqslant 1$ such that $V = V_i(x_k)$. If $n\geqslant i+k$ is such that $x_n\in V$, then (2) gives $G(\alpha_n)\subset V$.

Before formulating the monotonic version of 4.3 we shall recall the definition of a monotonic G_δ-set ($=$ a set of interior condensation [WW1] or, shortly, a W_δ-set [ChČN]).

4.10. Definition. A subset P of X is a monotonic G_δ-set if there exists a sequence $\{\mathscr{B}_n\}_{n\geqslant 1}$ of bases of P in X such that the intersection of each decreasing sequence $\{B_n\}_{n\geqslant 1}$ such that $B_n\in\mathscr{B}_n$ is contained in P. Such a sequence will be called a *(W)-sequence of bases for P*. A space X is said to be *monotonically perfect* if the closed subsets of X are monotonic G_δ-sets.

4.11. Proposition. *Let $\mathscr{U}$ be a well-ordered open cover of X and let $P(U) = U\setminus\bigcup\{U'\in\mathscr{U}\colon U'<U\}$. Then every $P(U)$ is a monotonic G_δ-set iff there exists a $(\mathscr{P}, \mathscr{U})$-sequence of bases of X, where $\mathscr{P} = \{P(U)\colon U\in\mathscr{U}\}$.*

Proof. The "if" part is easy: if $\{\mathscr{B}_n\}_{n\geqslant 1}$ is a $(\mathscr{P}, \mathscr{U})$-sequence of bases of X, then $\{\mathscr{B}_n(U)\}_{n\geqslant 1}$, where $\mathscr{B}_n(U) = \{B\in\mathscr{B}_n\colon B\cap P(U)\neq\emptyset \text{ and } B\subset U\}$, is a (W)-sequence of bases for $P(U)$.

Suppose that each $P(U)$ has a (W)-sequence of bases $\{\mathscr{B}_n(U)\}_{n\geqslant 1}$. In order to construct a $(\mathscr{P}, \mathscr{U})$-sequence of bases which makes all the sets $P(U)$ monotonic G_δ-sets in a uniform way, we proceed as in the proof of Proposition 1 in [Ch3].

We can assume that, for each $U\in\mathscr{U}$, $\{\mathscr{B}_n(U)\}_{n\geqslant 1}$ is a decreasing sequence and $B\in\mathscr{B}_n(U)$ implies that $B\cap P(U)\neq\emptyset$ and $B\subset U$.

Let $\mathscr{B}_n = \bigcup\{\mathscr{B}_n(U)\colon U\in\mathscr{U}\}$. Clearly, each $\mathscr{B}_n$ is a base of X. We shall prove that $\{\mathscr{B}_n\}_{n\geqslant 1}$ is a $(\mathscr{P}, \mathscr{U})$-sequence of bases.

Take $P = P(U)\in\mathscr{P}$ and a decreasing sequence $\{B_n\}_{n\geqslant 1}$ such that $B_n\in\mathscr{B}_n$ and $P\cap\bigcap_{n\geqslant 1} B_n\neq\emptyset$. For each $n\geqslant 1$ there exists (exactly one) U_n such that $B_n\in\mathscr{B}_n(U_n)$. Since $B_{n+1}\subset B_n$, it follows that $U_{n+1}\leqslant U_n$ and consequently there exist a $U'\in\mathscr{U}$ and $m\geqslant 1$ such that $U_n = U'$ for $n\geqslant m$. Then $\bigcap_{n\geqslant m} B_n\subset P(U')$, which implies that $U' = U$ and, in particular, $B_m\subset U$.

Proposition 4.11 can be used in the proofs that the existence of primitive structures implies the existence of the corresponding monotonic structures in monotonically perfect spaces [WW1] (see [Ch3, Remark 1]).

Another interesting application of 4.11, besides 4.3 and its consequences, is

4.12. Corollary. *Monotonically perfect spaces are monotonically subparacompact.*

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