On span and weakly chainable continua

by

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Abstract. Two mappings $f, g\colon I \to Q$ of the closed unit interval into the Hilbert cube can be ε-uniformized provided there exist onto mappings $a, b\colon I \to I$ such that $f \circ a =_{\varepsilon} g \circ b$. Metric continua X of span zero are characterized by means of ε-uniformization of mappings of I into neighbourhoods of X in Q. This characterization is then used to obtain a variety of properties of continua of span zero.

1. Introduction. All spaces considered in this paper are metric unless otherwise stated. A continuum is a compact connected space. We write $f\colon X \twoheadrightarrow Y$ to indicate that f is a mapping of X onto Y. We let I denote the closed unit interval and Q the Hilbert cube. For $\varepsilon > 0$ and $A \subset X$ we let $S(A, \varepsilon)$ denote the open ε-ball around A in X. If $A \subset X$ we let $\mathrm{Cl}(A)$ denote the closure of A in X.

DEFINITION. We say two mappings $f, g\colon I \twoheadrightarrow I$ can be *uniformized* if there exist $a, b\colon I \twoheadrightarrow I$ such that $f \circ a = g \circ b$.

UNIFORMIZATION THEOREM (see Mioduszewski [20]). *If $f, g\colon I \twoheadrightarrow I$ are piecewise linear mappings then f and g can be uniformized.*

DEFINITION. If $f, g\colon I \to I$ are piecewise linear mappings and f is onto I then clearly there exist mappings $a, b\colon I \to I$ such that b is onto and $f \circ a = g \circ b$. We say g can be *uniformized with a piece of f*.

Note. If $f, g\colon I \twoheadrightarrow I$ are piecewise linear maps and $\varphi\colon I^2 \to R$ is defined by $\varphi(x, y) = f(x) - g(y)$ then f can be uniformized with g if and only if there exists a component K of $\varphi^{-1}(0)$ such that K meets all four sides of I^2. Note that K is a polyhedron. Also, g can be uniformized with a piece of f if and only if there exists a component K of $\varphi^{-1}(0)$ which meets both $I \times \{0\}$ and $I \times \{1\}$. Hence, f can be uniformized with g if and only if f can be uniformized with a piece of g and g can be uniformized with a piece of f.

DEFINITION. If $f, g\colon I \to Q$ are mappings and $\varepsilon > 0$ we say g can be *ε-uniformized with a piece of f* if there exist $a\colon I \to I$ and $b\colon I \twoheadrightarrow I$ such that $f \circ a =_{\varepsilon} g \circ b$,

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i.e., $f\circ a(t)\in S(g\circ b(t),\varepsilon)$ for each $t\in I$. We say f and g can be *ε-uniformized* if f can be ε-uniformized with a piece of g and g can be ε-uniformized with a piece of f.

Note. If $f,g\colon I\to Q$ are maps and $\varepsilon>0$ such that f can be ε-uniformized with a piece of g then

$$f(I)\subset S(g(I),\varepsilon).$$

DEFINITION. If X is a continuum let $\pi_i\colon X\times X\to X$ denote the ith coordinate projection for $i=1,2$. Let ΔX denote the diagonal in $X\times X$. Define [see 14] the *surjective span of X*, $\sigma^*(X)$, (resp. the *surjective semispan of X*, $\sigma_0^*(X)$) to be the least upper bound of all real numbers ε for which there exists a subcontinuum Z of $X\times X$ such that $\pi_1(Z)=X=\pi_2(Z)$ (resp. $\pi_1(Z)=X$) and $d(x,y)\geqslant\varepsilon$ for each $(x,y)\in Z$.

The *span of X* is $\sigma(X)=\sup\{\sigma^*(A)|\ A$ subcontinuum of $X\}$ and the *semispan of X* is $\sigma_0(X)=\sup\{\sigma_0^*(A)|\ A$ subcontinuum of $X\}$.

In this paper we characterize continua X with $\sigma_0^*(X)=0$ and $\sigma^*(X)=0$ in terms of uniformizations of mappings of I into neighbourhoods of X in Q. We characterize weak chainability of continua in terms of nice approximations of the continua by sequences of arcs. We prove that $\sigma_0^*(X)=0$ implies X is a weakly chainable atriodic tree-like continuum. We show that under some extra conditions the converse is also true.

2. A characterization of $\sigma^*(X)=0$.

THEOREM 1. *Let $X\subset Q$ be a continuum. Then $\sigma_0^*(X)=0$ if and only if for each pair of sequences $f_i)$, $g_i)$, $f_i, g_i\colon I\to Q$ of mappings such that $\lim_i g_i(I)=X$ and $\limsup_i f_i(I)\subset X$, and for each $\varepsilon>0$ there exists an integer n such that for each $i\geqslant n$ f_i ε-uniformizes with a piece of g_i.*

Proof. (⇒) Suppose $\sigma_0^*(X)=0$. Let $X_i=\{(x,y)\in I^2|\ d(f_i(x),g_i(y))<\varepsilon\}$. Then X_i is open in I^2 since f_i and g_i are continuous functions. If there is a component M_i of X_i such that M_i meets both $\{0\}\times I$ and $\{1\}\times I$ then there exists an arc $Y_i\subset M_i$ such that Y_i meets both $\{0\}\times I$ and $\{1\}\times I$ since M_i is an open and connected set in a Peano continuum. Let $h_i\colon I\twoheadrightarrow Y_i$ be a homeomorphism. Then $h_i(t)=(a_i(t),b_i(t))$ for each $t\in I$ where $a_i,b_i\colon I\to I$ are continuous functions and a_i is onto. For each $t\in I$

$$d(f_i\circ a_i(t),g_i\circ b_i(t))<\varepsilon$$

since $(a_i(t),b_i(t))\in Y_i\subset X_i$ and so f_i is ε-uniformizable with a piece of g_i.

Now suppose for each i no component of X_i meets both $\{0\}\times I$ and $\{1\}\times I$. Since I^2 is unicoherent and no component of X_i separates $I\times\{0\}$ from $I\times\{1\}$, X_i does not separate $I\times\{0\}$ from $I\times\{1\}$. Since $I^2\setminus X_i$ is compact some component K_i of $I^2\setminus X_i$ meets both $I\times\{0\}$ and $I\times\{1\}$. Let $Z_i=\{(f_i(x),g_i(y))|\ (x,y)\in K_i\}$. Then Z_i is a continuum in Q^2 such that $d(x,y)\geqslant\varepsilon$ for each $(x,y)\in Z_i$. Since Q is compact we may suppose the sequence $Z_i)$ converges to a continuum Z in the

hyperspace of continua in Q^2. Also, $Z\subset X\times X$. Let $\pi_2\colon Q^2\to Q$ be the projection onto the second coordinate. Then

$$\pi_2(Z)=\lim_i\pi_2(Z_i)=\lim_i g_i(I)=X.$$

For $(x,y)\in Z$ $d(x,y)\geqslant\varepsilon$. Thus, $\sigma_0^*(X)\geqslant\varepsilon$ which is a contradiction.

(⇐) Suppose $\sigma_0^*(X)>0$. By Lelek [cf. 13] there exists a continuum C and mappings $f,g\colon C\to X$ such that $g(C)=X$ and $d(f(x),g(x))\geqslant\varepsilon>0$ for each $x\in C$. By [22] Lemma 2.4 there exist maps $f_i, g_i\colon I\to Q$ such that $d(f_i(t),g_i(t))\geqslant\varepsilon-1/i$ for each $i=1,2,\ldots$ and each $t\in I$ and such that $\lim g_i(I)=g(C)=X$ and $\limsup_i f_i(I)=f(C)\subset X$.

If for each i f_i $1/i$-uniformizes with a piece of g_i then there exist maps $a_i,b_i\colon I\to I$ such that a_i is onto I and $f_i\circ a_i\,\overline{\overline{{}_{1/i}}}\,g_i\circ b_i$. Let $t_i\in I$ such that $a_i(t_i)=b_i(t_i)$. Then $f_i(a_i(t_i))\,\overline{\overline{{}_{1/i}}}\,g_i(b_i(t_i))$ which is a contradiction.

COROLLARY 2. *Let $X\subset Q$ be a continuum. Then $\sigma_0^*(X)=0$ if and only if for each pair of sequences $f_i)$, $g_i)$ of maps $f_i,g_i\colon I\to Q$ such that $\lim_i g_i(I)=\lim_i f_i(I)=X$ and each $\varepsilon>0$ there exists an integer n such that g_i ε-uniformizes with f_i for each $i\geqslant n$.*

Proof. (⇒) This is immediate from Theorem 1.

(⇐) Let $f_i)$ and $g_i)$ be two sequences of maps such that $f_i,g_i\colon I\to Q$, $\lim_i g_i(I)=X$ and $\limsup_i f_i(I)\subset X$. For each i let $f_i'\colon I\to Q$ be a map such that $f_i'(t)=f_i(4t-2)$ for $\frac{1}{2}\leqslant t\leqslant\frac{3}{4}$ and such that $\lim_i f_i'(I)=X$. By assumption for each $\varepsilon>0$ there exists an integer n such that $i\geqslant n$ implies g_i and f_i' can be ε-uniformized. Hence, f_i can be ε-uniformized with a piece of g_i.

COROLLARY 3. *Let X be a continuum in Q with $\sigma_0^*(X)=0$. There exists a sequence $f_i)$ of maps $f_i\colon I\to Q$ such that $\lim_i f_i(I)=X$ and for each $i,j=1,2,\ldots$ f_{i+j} $1/i$-uniformizes with f_i.*

Proof. Let $g_i)$ be a sequence of maps $g_i\colon I\to Q$ such that $\lim_i g_i(I)=X$. If no subsequence of $g_i)$ satisfies the theorem then there exists $\varepsilon>0$ such that for each sufficiently large i there exists j_i such that g_{i+j_i} does not ε-uniformize with g_i. Let $f_i=g_{i+j_i}$ for each sufficiently large i. Then Corollary 2 is violated.

THEOREM 4. *Let $X\subset Q$ be a continuum. Then $\sigma^*(X)=0$ if and only if for each pair of sequences $f_i)$, $g_i)$ of mappings $f_i,g_i\colon I\to Q$ such that $\lim_i f_i(I)=\lim_i g_i(I)=X$ and each $\varepsilon>0$ there exists n such that for each $i>n$ either f_i ε-uniformizes with a piece of g_i or g_i ε-uniformizes with a piece of f_i.*

Proof. The proof is similar to the proof of Theorem 1 and is omitted.

Davis [4] has proved that $(\sigma(X)=0)\Rightarrow(\sigma_0(X)=0)$. It would be very nice to know whether Davis' result extends to surjective span.

QUESTION 1. Does $(\sigma^*(X)=0)\Rightarrow(\sigma_0^*(X)=0)$?

QUESTION 2 [A. Lelek, problem 59, University of Houston Problem Book]. Does $(\sigma^*(X) = 0)$ (resp. $\sigma_0^*(X) = 0) \Rightarrow (\sigma(X) = 0)$ (resp. $\sigma_0(X) = 0)$?

3. A characterization of weak chainability. A finite sequence of sets $\mathscr{U} = \{U_1, \ldots, U_n\}$ is said to be a *weak chain* if $U_i \cap U_{i+1} \neq \emptyset$ for each $i = 1, \ldots, n-1$. A weak chain $\mathscr{V} = \{V_1, \ldots, V_m\}$ is a *refinement* of a weak chain $\mathscr{U} = \{U_1, \ldots, U_n\}$ and we write $\mathscr{V} \prec_w \mathscr{U}$ if there exists a function

$$f\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$$

such that for each $i \in \{1, \ldots, m\}$ $V_i \subset U_{f(i)}$ and for $i \in \{1, \ldots, m-1\}$ $|f(i) - f(i+1)| \leqslant 1$. We call f an *admissible map* of the weak chain $\mathscr{V}$ to the weak chain $\mathscr{U}$. A continuum X is said to be *weakly chainable* if there exists a sequence $\mathscr{U}_1, \mathscr{U}_2, \ldots$ of open covers of X which are weak chains such that the mesh of $\mathscr{U}_i$ is less than $1/i$ and $\mathscr{U}_{i+1} \prec_w \mathscr{U}_i$ for each $i = 1, 2, \ldots$

THEOREM (Fearnley [6] and Lelek [15]). *The continuum X is weakly chainable if and only if X is the continuous image of the pseudo-arc.*

DEFINITION. A continuum X is said to admit a *uniformizable approximation by arcs* if there exists a sequence $f_i)$ of mappings $f_i\colon I \to Q$ such that $\lim_i f_i(I) = X$ and for each $i, j = 1, 2, \ldots$ f_{i+j} $1/i$-uniformizes with a piece of f_i.

THEOREM 5. *A continuum X is weakly chainable if and only if X admits a uniformizable approximation by arcs.*

Proof. ($\Rightarrow$) Suppose X is a weakly chainable continuum. Let $T_1, T_2, \ldots$ be a sequence of finite open coverings of X (the members of T_i are open in Q) such that $\lim_i \operatorname{mesh} T_i = 0$. For each i let $P_i = \{U_1^i, \ldots, U_{n_i}^i\}$ be a weak chain in the cover T_i such that P_i covers X and for each i there exists an admissible map γ_i of the weak chain P_{i+1} to the weak chain P_i. Choose $x_j^i \in U_j^i$ for each $i = 1, 2, \ldots$ and $j = 1, \ldots, n_i$ such that $\{x_j^i|\ i = 1, 2, \ldots$ and $j = 1, \ldots, n_i\}$ is in general position in Q. Define $f_i\colon I \to Q$ such that $f_i\left(\dfrac{j}{n_i - 1}\right) = x_{j+1}^i$ for $j = 0, 1, \ldots, n_i - 1$ and f_i is linear on each interval of the form $\left[\dfrac{j}{n_i-1}, \dfrac{j+1}{n_i-1}\right]$. Then it is easy to see that $f_i)$ gives a uniformizable approximation of X by arcs. Take $a_i\colon I \to I$ to be the piecewise linear map which is defined by

$$a_i\left(\frac{j-1}{n_{i+1}-1}\right) = \frac{\gamma_i(j)-1}{n_i - 1} \quad \text{for} \quad j = 1, \ldots, n_{i+1}.$$

Then $f_{i+1} \circ a_i \underset{\varepsilon}{=} f_i$ for each $\varepsilon > 0$ and for each sufficiently large i.

($\Leftarrow$) Let $f_i)$ be a sequence of mappings $f_i\colon I \to Q$ such that $\lim_i f_i(I) = X$ and for each $i, j = 1, 2, \ldots$ f_{i+j} $1/i$-uniformizes with f_i. It is easy to see that for each i the Hausdorff distance from X to $f_i(I)$ is no more than $1/i$.

Let $T_i)$ be a sequence of finite open covers of X and $\delta_i)$ a sequence of positive numbers such that if for $U \in T_i$

$$U^- = \{x \in U|\ d(x, Q \setminus U) > \delta_i\} \quad \text{and} \quad U^+ = \left\{x \in U|\ d(x, Q \setminus U) > \frac{2\delta_i}{3}\right\}$$

then

(1) $\operatorname{mesh} T_{i+1} < \delta_i/3$,
(2) $\{U^-|\ U \in T_i\}$ is an open cover for X,
(3) T_{i+1} refines T_i.

For each i we have

$$\bigcup\{U \in T_i\} \supset \operatorname{Cl}(S(X, \delta_i)) \supset S(X, \delta_i) \supset \bigcup\{U \in T_{i+1}\}.$$

We may suppose without loss of generality that for each i

$$f_i(I) \subset \bigcup\{U^-|\ U \in T_i\}.$$

By taking a subsequence of $f_i)$, we may assume that for each $i = 1, 2, \ldots$ f_{i+1} uniformizes within $\delta_i/3$ with a piece of f_i.

Let $0 = t_0 < t_1 < \ldots < t_r = 1$ be such that $f_1([t_{i-1}, t_i]) \subset U_i^+$ for some $U_i \in T_1$ and for each $i = 1, \ldots, r$. Then $P_1 = \{U_1, \ldots, U_r\}$ is said to be a weak chain determined by f_1 in T_1. Then $X \subset \bigcup_{i=1}^{r} U_i$.

By hypothesis there exist piecewise linear mappings $a_1, b_1\colon I \to I$ such that a_1 is onto and $f_2 \circ a_1 \underset{\delta_1/3}{=} f_1 \circ b_1$. Let $0 = s_0 < s_1 < \ldots < s_{r_1} = 1$ be such that $b_1([s_{i-1}, s_i]) \subset [t_{j_i - 1}, t_{j_i}]$ for some $j_i \in \{1, \ldots, r\}$. Clearly, $|j_i - j_{i+1}| \leqslant 1$ for $i \in \{1, \ldots, r_1\}$. Let $\hat{P}_1 = \{V_1, \ldots, V_{r_1}\}$ where $V_i = U_{j_i}$ for $i \in \{1, \ldots, r_1\}$. Then $\hat{P}_1$ is a weak chain determined by $f_1 \circ b_1$ in T_1. Then $\alpha_1\colon \{1, \ldots, r_1\} \to \{1, \ldots, r\}$ defined by $\alpha_1(i) = j_i$ is an admissible map of the weak chain $\hat{P}_1$ to the weak chain P_1.

Let $0 = u_0 < u_1 < \ldots < u_{r_2} = 1$ be such that for each $i \in \{1, \ldots, r_2\}$

$$f_2 \circ a_1([u_{i-1}, u_i]) \subset O_i^+$$

for some $O_i \in T_2$ and $[u_{i-1}, u_i] \subset [s_{k_i - 1}, s_{k_i}]$ for some $k_i \in \{1, \ldots, r_1\}$. Then $P_2 = \{O_1, \ldots, O_{r_2}\}$ is a weak chain determined by $f_2 \circ a_1$ in T_2. If $i \in \{1, \ldots, r_2\}$ and $[u_{i-1}, u_i] \subset [s_{k_i-1}, s_{k_i}]$ for some $k_i \in \{1, \ldots, r_1\}$ then

$$S(f_2 \circ a_1([u_{i-1}, u_i]), \delta_1/3) \subset S(f_1 \circ b_1([s_{k_i-1}, s_{k_i}]), 2\delta_1/3) \subset S(V_{k_i}^+, 2\delta_1/3) \subset V_{k_i}.$$

Since $O_i \subset S(f_2 \circ a_1([u_{i-1}, u_i]), \delta_2/3)$ and $\delta_1/3 < \delta_2/3$, $O_i \subset V_{k_i}$.

Define $\beta_1\colon \{1, \ldots, r_2\} \to \{1, \ldots, r_1\}$ by $\beta_1(i) = k_i$. Then β_1 is an admissible map of the weak chain P_2 in T_2 to the weak chain $\hat{P}_1$ in T_1. The composition $\gamma_1 = \alpha_1 \circ \beta_1$ is an admissible map of the weak chain P_2 to the weak chain P_1.

By assumption there exist piecewise linear mappings $a, b\colon I \to I$ such that a is onto and $f_3 \circ a \underset{\delta_2/3}{=} f_2 \circ b$. Since $a_1\colon I \twoheadrightarrow I$ there exist by the Uniformization Theorem mappings $b_2, d\colon I \to I$ such that d is onto and $a_1 \circ b_2 = b \circ d$. Hence,

$$f_3 \circ a \circ d \underset{\delta_2/3}{=} f_2 \circ b \circ d = f_2 \circ a_1 \circ b_2.$$

Let $a_2 = a \circ d$. Let $\hat{P}_2$ be a weak chain determined by $f_2 \circ a_1 \circ b_2$ in T_2 and let P_3 be a weak chain determined by $f_3 \circ a_2$ in T_3 as above. The rest of the proof now proceeds easily by induction as above.

COROLLARY 6. *If X is a continuum with $\sigma_0^*(X) = 0$ then X is weakly chainable.*

Proof. This follows from Corollary 3 and Theorem 5.

COROLLARY 7. *If X is a homogeneous plane continuum then X is weakly chainable.*

Proof. By [22], either $\sigma_0(X) = 0$ or X is the union of two subcontinua A and B such that $\sigma_0(A) = \sigma_0(B) = 0$.

QUESTION 3. Does $(\sigma^*(X) = 0) \Rightarrow X$ is weakly chainable?

4. Continua with zero surjective span. If X is a continuum we let $C(X)$ denote the space of subcontinua of X with the Hausdorff metric. If $f\colon X \twoheadrightarrow Y$ is a mapping of continua we say f is *weakly confluent* if $K \in C(Y)$ implies there exists $L \in C(X)$ such that $f(L) = K$. We say $Y \in$ *class* W if $f\colon X \twoheadrightarrow Y$ is weakly confluent whenever f is any map of a continuum onto Y.

Davis [4] defined the notion of symmetric span zero which is weaker than span zero. He proved that continua with symmetric span zero are in class W. We prove a similar result for surjective span. It is known [4] that the dyadic solenoid has span positive but symmetric span zero. Since the dyadic solenoid is not the continuous image of the pseudo-arc by Fort [7] it follows that Corollary 6 fails for continua with symmetric span zero.

THEOREM 8. *If X is a continuum with $\sigma^*(X) = 0$ then $X \in$ class W.*

Proof. If $X \notin$ class W there exists (see Nadler [21]) a Whitney map $\mu\colon C(X) \to R$, (i.e. $(A \not\subseteq B$ in $C(X)) \Rightarrow (\mu(A) < \mu(B))$ and $\mu(\{x\}) = 0$ for all $x \in X$), $t_0 \in \mu(C(X))$ and Λ a continuum in $\mu^{-1}(t_0)$ such that $\bigcup \Lambda = X$ and $\Lambda \neq \mu^{-1}(t_0)$ (see [10]). Since $\mu^{-1}(t_0)$ is a continuum (see [21]) there exist $A, B \in \mu^{-1}(t_0) \setminus \Lambda$ with $A \neq B$. Since $A, B \in \mu^{-1}(t_0)$ $A \not\subset B$ and $B \not\subset A$ so there exist $x \in A \setminus B$ and $y \in B \setminus A$. By [21], 14.8.1 there exists a continuum Λ_1 in $\mu^{-1}(t_0)$ such that $A \in \Lambda_1$, $\Lambda \subset \Lambda_1$, $\Lambda_1 \setminus \Lambda$ is homeomorphic to $[0, 1)$ and $B \notin \Lambda_1$. (Take $C \in \Lambda$ such that $x \in C$. Let $\mathcal{L}$ be an arc in $\mu^{-1}(t_0)$ such that $x \in K$ for each $K \in \mathcal{L}$ and $A, C \in \mathcal{L}$. Let Λ_1 be irreducible in $\Lambda \cup \mathcal{L}$ with respect to containing A and Λ). Similarly, there exists a continuum Λ_2 in $\mu^{-1}(t_0)$ such that $B \in \Lambda_2$, $\Lambda \subset \Lambda_2$, $\Lambda_2 \setminus \Lambda$ is homeomorphic to $[0, 1)$ and $A \notin \Lambda_2$.

As in the proof of [10], Theorem 3.2 there exists a continuum Y and a mapping $f\colon Y \twoheadrightarrow X$ such that $\Lambda_1 \subset f(C(Y))$ but $B \notin f(C(Y))$. Let M_f be the mapping cylinder of f and let $\varphi\colon Y \times I \to M_f$ be the natural projection, i.e. $\varphi | Y \times (0, 1]$ is homeomorphism onto $\varphi(Y \times (0, 1])$ and $\varphi | Y \times \{0\} = f$ where Y is identified with $Y \times \{0\}$. For each $i = 1, 2, \ldots$ let A_i [see J. Grispolakis and E. D. Tymchatyn, *Continua which admit only certain classes of onto mappings*, Top. Proc. 3 (1978), 347–362, Theorem 3.5] be a ray compactified by $Y_i = \varphi(Y \times \{1/i\})$ such that if $h_i\colon [0, \infty) \twoheadrightarrow A_i$ is a homeomorphism then $\lim_n C(h_i([n, \infty))) = C(Y_i)$. Let $m_i < n_i$ be integers such that every subcontinuum of $h_i([m_i, n_i])$ is within a Hausdorff distance $1/i$ of some subcontinuum of Y_i and every subcontinuum of Y_i is within a Hausdorff distance $1/i$ of some subcontinuum of Y_i and every subcontinuum of Y_i is within a Hausdorff distance $1/i$ of $h_i([m_i, n_i])$. Then $h_i | [m_i, n_i]$ are maps such that $\lim_i h_i([m_i, n_i]) = X$ and $\lim_i C(h_i([m_i, n_i])) = f(C(Y))$. So

$$A \in \lim_i C(h_i([m_i, n_i])) \subset C(X) \setminus \{B\}.$$

Similarly, there exist mappings $g_i)$ $g_i\colon I \to Q$ such that $\lim_i g_i(I) = X$ and $B \in \lim_i C(g_i(I)) \subset C(X) \setminus \{A\}$.

Now if ε is sufficiently small then for sufficiently large i h_i does not ε-uniformize with a piece of g_i and vice versa. This is so because $h_i([m_i, n_i])$ (resp. $g_i(I)$) contains continua which are close in the sense of Hausdorff distance to A (resp. B) and $g_i(I)$ (resp. $h_i([m_i, n_i])$) does not. If h_i were to ε-uniformize with a piece of g_i then $g_i(I)$ would have to contain continua arbitrarily close to A.

Lelek [16] has proved that if $\sigma(X) = 0$ then X is atriodic and tree-like. Davis [4] has shown that $(\sigma(X) = 0) \Rightarrow (\sigma_0(X) = 0)$. We obtain Lelek's result under the weaker hypothesis $\sigma_0^*(X) = 0$.

THEOREM 9. *Let $X \subset Q$ be a continuum. If $\sigma_0^*(X) = 0$ then X is tree-like.*

Proof. Since $X \in$ class W by Theorem 8 and X is weakly chainable by Corollary 6 every subcontinuum of X is weakly chainable. Since the unit disk in the plane contains subcontinua which are not weakly chainable (see Fort [7]) X admits no weakly confluent map onto a plane disk. By Mazurkiewicz [18] $\dim X = 1$. To prove that X is tree-like it suffices by a theorem of Krasinkiewicz ([12], Theorem 3.1) to prove X admits no essential mapping onto the unit circle S^1.

Just suppose $g\colon X \to S^1$ is an essential mapping. Let $f\colon P \twoheadrightarrow X$ be a mapping of the pseudo-arc onto X. Let $h\colon [0, \infty) \to Q \setminus P$ be a homeomorphism such that $\mathrm{Cl}(h([0, \infty))) = h([0, \infty)) \cup P = P^*$. Then P^* is clearly a chainable continuum. Define an upper semi-continuous equivalence relation $\sim$ on P^* by $x \sim y$ in P^* if $x = y$ or $x, y \in P$ and $f(x) = f(y)$. Let $f^*\colon P^* \to X^* = P^*/\!\sim$ be the identification map. Let $g^*\colon X^* \to S^1$ be a continuous extension of g. Such an extension exists since $\dim X^* = 1$ and S^1 is an ANR (see Hu [11], p. 172).

Now, $g^* \circ f^*\colon P^* \to S^1$ is an inessential map so there exists a lifting $\varphi\colon P^* \to R$ of $g^* \circ f^*$ to the universal covering space R of S^1. Let $n_1 < n_2 < \ldots$ be integers such that if $B_i = f^* \circ h([n_i, n_{i+1}])$ then $\lim_i B_i = X$. For each i $g^* | B_i$ is inessential since B_i is an arc. For each i, let $\psi_i = \varphi \circ (f^* | h([n_i, n_{i+1}]))^{-1}\colon B_i \to R$. Then for each i ψ_i is a lifting of $g^* | B_i$ and $\psi_i(B_i) \subset \varphi(P^*)$, a compact set.

Let U be an open neighbourhood of X^* in Q such that g^* admits a continuous extension $g'\colon U \to S^1$. By Grispolakis [8] there exists a sequence $S_i)$ of simple closed curves in U such that $g' | S_i$ is essential for each i and $\limsup_i S_i \subset X$. For each i let $p_i \in S_i$ and let $f_i\colon (I, \dot{I}) \to (S_i, p_i)$ be a relative homeomorphism. If $\eta_i\colon I \to R$ is a lifting of $g' \circ f_i$ then diameter $\eta_i(I) \geqslant 1$ for each i. For each i let C_i be a simple closed curve in U such that $p_i \in C_i$ and C_i admits a $1/i$-local homeomorphism h_i onto S_i with the cardinality of $h_i^{-1}(p_i)$ equal to i. If $f_i'\colon (I, \dot{I}) \to (C_i, p_i)$ is a relative homeomorphism and $\eta_i'\colon I \to R$ is a lifting of $g' \circ f_i'$ then $\lim\limits_i$ diameter $\eta_i'(I) = \infty$.

For small $\varepsilon > 0$ and large i the map f_i' does not ε-uniformize with $f^* \circ h|[n_i, n_{i+1}]$ since the liftings ψ_i to R of $g' \circ f^* \circ h|[n_i, n_{i+1}]$ are bounded and the liftings η_i' of $g' \circ f_i'$ are unbounded. This contradicts Theorem 1 and completes the proof of the theorem.

THEOREM 10. *If X is a continuum with $\sigma_0^*(X) = 0$ then X is atriodic.*

Proof. Just suppose X contains a triod X_0. Then there exist continua A, B, C, M such that $X_0 = A \cup B \cup C$ and $M = A \cap B = A \cap C = B \cap C \neq X_0$. Let $d \in M$, $a \in A \setminus M$, $b \in B \setminus M$ and $c \in C \setminus M$. We may suppose $d(\{a, b, c\}, M) \geqslant 3$.

For each positive integer i define $f_i\colon I \to S(X_0, 1/i)$ such that

$$f_i\left(\frac{6j+k}{6i}\right) \in \begin{cases} S(a, 1/i) & \text{if } k = 1, \\ S(b, 1/i) & \text{if } k = 3, \\ S(c, 1/i) & \text{if } k = 5, \\ S(d, 1/i) & \text{if } k = 0, 2, 4, 6, \end{cases}$$

$$f_i\left(\left[\frac{6j}{6i}, \frac{6j+2}{6i}\right]\right) \subset S(A, 1/i),$$

$$f_i\left(\left[\frac{6j+2}{6i}, \frac{6j+4}{6i}\right]\right) \subset S(B, 1/i),$$

and

$$f_i\left(\left[\frac{6j+4}{6i}, \frac{6j+6}{6i}\right]\right) \subset S(C, 1/i)$$

for $j = 0, 1, \ldots, i-1$. Then it is easy to see that for each sufficiently large i f_i does not ε-uniformize with f_{i+j} for $j \geqslant 1$ where

$$\varepsilon = \tfrac{1}{3}\min\{d(a, b), d(a, c), d(b, c)\}.$$

In fact f_i can ε-uniformize with $f_{i+j}|L$ where L is a subarc of I of length at most $\dfrac{6i+4}{6(i+j)}$.

Let $g_i)$ be a sequence of mappings $g_i\colon I \to Q$ such that $\lim\limits_i g_i(I) = X$. By the proof of Corollary 3 there exists an integer n such that for each $i > n$ f_{i+j} $\frac{1}{2}\varepsilon$-uniformizes with a piece of g_i for each $j > i$. Hence, there exists a subinterval J of I and integers $k > j > 1$ such that f_{i+k} and f_{j+i} each $\frac{1}{2}\varepsilon$-uniformize with $f_i|J$. Hence (see [20]), f_{i+k} and f_{i+j} ε-uniformize. This is a contradiction. Hence, X is atriodic.

COROLLARY 11. *If X is a continuum with $\sigma_0^*(X) = 0$ then every subcontinuum of X is atriodic, tree-like and weakly chainable.*

We note some other conditions which imply that a space is atriodic and tree-like.

THEOREM 12. *If X is a weakly chainable continuum such that every subcontinuum of X is in class W then X is atriodic and tree-like.*

Proof. As in the proof of Theorem 9, X is one dimensional. It is well known (see [21], 14.73.18 and [10]) that a continuum in class W is not a triod. It follows by [24], 2.4 that no subcontinuum of X admits an essential mapping to any graph. By a theorem of Case and Chamberlin [2] X is tree-like.

COROLLARY 13. *Every hereditarily indecomposable and weakly chainable continuum is tree-like.*

Proof. It is a result of Cook [3] that every hereditarily indecomposable continuum is in class W.

THEOREM 14. *Let X be a continuum with $\sigma^*(X) = 0$. Then* $\dim X \leqslant 1$.

Proof. If $\dim X > 1$ then there exists an essential map σ of X onto the disk I^2. By [18] σ is weakly confluent. Let A' and B' be two disjoint copies of the spiral

$$S = \{\langle 1, \theta\rangle |\ 0 \leqslant \theta \leqslant 2\pi\} \cup \{\langle 1 + 1/\theta, \theta\rangle |\ \theta \geqslant 1\}$$

in the disk I^2 (where $\langle r, \theta\rangle$ denotes the polar coordinates of a point in the plane). Let A and B be continua in X such that $\sigma(A) = A'$ and $\sigma(B) = B'$. Define $p\colon S \to S^1$ by $p(\langle r, \theta\rangle) = \langle 1, \theta\rangle$. By abuse of notation let p also denote each of the natural projections of $A' \twoheadrightarrow S^1$ and $B' \twoheadrightarrow S^1$. By [7] $p \circ \sigma|A\colon A \to S^1$ and $p \circ \sigma|B\colon B \to S^1$ are essential maps.

Let $V_1, V_2, \ldots$ be a sequence of finite open covers of X such that mesh $V_j < 1/j$ for each j. Let $a_0 \in A$ and let $b_0 \in B$. We may suppose that for each j $a_0, b_0 \in N^1(V_j)$ and that $X \cup \bigcup\limits_{j=1}^{\infty} N^1(V_j)$ is a continuum in Q. (By $N^1(V_j)$ is meant the 1-dimensional skeleton of the nerve $N(V_j)$ of V_j).

Let H be the figure eight graph, i.e. H is the wedge of two disjoint circles. Define $h\colon A \cup B \twoheadrightarrow H$ such that $h|B \not\simeq *$, $h|A \not\simeq *$ and $h(A) \cap h(B) = \{h(a_0)\} = \{h(b_0)\}$. By Hu [11], p. 172 h can be extended to a map (which we again denote by h) of $U \cup N^1(V_1) \cup N^1(V_2) \cup \ldots$ to H where U is a small compact neighbourhood of $A \cup B$ in Q.

For each j let $U_j = \{V \in V_j |\ V \cap (A \cup B) \neq \emptyset\}$. By Mardešić and Segal [17] for each sufficiently large j $h|N(U_j) \not\simeq *$. Since H is one dimensional $h|N^1(U_j) \not\simeq *$ for each sufficiently large j. (Otherwise, h could be extended one simplex at a time to be inessential on $N(U_j)$, the nerve of U_j). For sufficiently large j there exist disjoint simple closed curves C_j, $D_j \subset N^1(U_j)$ such that $h|C_j \not\simeq *$, $h|D_j \not\simeq *$, C_j is contained in a small neighbourhood of A and D_j is contained in a small neighbourhood of B. (Since $h|A \not\simeq *$ there exists a minimal subcontinuum C_j of $N^1(U_j)$ in a small neighbourhood of A such that $h|C_j \not\simeq *$. Since $N^1(U_j)$ is a graph C_j is a simple closed curve). We may suppose h was chosen such that $h(C_j) \cap h(D_j) = h(a_0)$.

Let (P, π) be the universal covering space of H. Then P is an infinite tree. Let $z_0 \in \pi^{-1}(h(a_0))$. We may suppose P and H are metrized so that $x, y \in P$ with $\pi(x) = \pi(y)$ implies $d(x, y)$ is an integer and π is a local isometry of P onto H.

For each j let $f_j\colon I \twoheadrightarrow N^1(V_j)$ be a map such that $f_j(0) = a_0$ and $f_j(1) = b_0$. Let $\varphi_j\colon I \to P$ be the lifting of $h \circ f_j$ such that $\varphi_j(0) = z_0$. Let diameter $\varphi_j(I) \leqslant N_j$

where N_j is an integer. Let $h'_j\colon S^1 \to N^1(U_j)$ and $k'_j\colon S^1 \to N^1(U_i)$ be maps such that $h'_j(1) = a_0$, $k'_j(1) = b_0$ and h'_j and k'_j are homotopic to (N_j+1)-fold covering maps of S^1 onto C_j and D_j respectively. Define $h_j, k_j\colon I \to N^1(V_j)$ by

$$h_j(t) = \begin{cases} h'_j(e^{4\pi it}) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ f_j(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

and

$$k_j(t) = \begin{cases} f_j(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ k'_j(e^{4\pi it}) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easy to see that if $\psi_j\colon I \to P$ is a lifting of $h \circ h_j$ and $\xi_j\colon I \to P$ is a lifting of $h \circ k_j$ then $\psi_j([0, \frac{1}{2}]) \subset S(\xi_j(I), \varepsilon)$ or $\xi_j([\frac{1}{2}, 1]) \subset S(\psi_j(I), \varepsilon)$ implies $\varepsilon > 1$.

Now, $h_j)_j$ and $k_j)_j$ are sequences of mappings of I to Q such that $\lim\limits_j h_j(I) = X = \lim\limits_j k_j(I)$. If $a\colon I \twoheadrightarrow I$ is a mapping then $\psi_j \circ a$ is a lifting of $h \circ h_j \circ a$ to P. Furthermore, every lifting of $h \circ h_j \circ a$ to P is $\psi_j \circ a$ followed by a deck transformation of P. Let $\delta > 0$ be such that $x, y \in U$ with $d(x, y) < \delta$ then $d(h(x), h(y)) < \frac{1}{2}$. For large j h_j does not δ-uniformize with a piece of k_j and k_j does not δ-uniformize with a piece of h_j.

5. On span of weakly chainable continua. Much of the work in this paper has been motivated by the following:

Question 4 [A. Lelek, problem 81–82, University of Houston Problem Book]. Is $(\sigma_0^*(X) = 0) \Leftrightarrow X$ is chainable?

Question 5. Among atriodic tree-like continua are the following two notions equivalent?

1) $\sigma_0^* = 0$, 2) weak chainability.

It is obvious that chainable continua are weakly chainable. Lelek [14] proved that chainable continua have $\sigma_0^* = 0$.

In Corollary 6 we proved that 1) ⇒ 2). In the next theorem we extend the main result in [23] to show that under certain additional conditions 2) ⇒ 1).

The proof is a simplified version of the proof of 2.1 in [23].

Theorem 15. *If X is a weakly chainable continuum such that $X \in$ class W and $\sigma_0(Y) = 0$ for each proper subcontinuum Y of X then $\sigma_0(X) = 0$.*

Proof. Let $X = \varprojlim(X_p, f_p^m)$ and let the pseudo-arc $P = \varprojlim(I_p, g_p^m)$ where the X_p's are polyhedra, the I_p's are arcs and f_p^m and g_p^m are onto mappings. Suppose that the map $\varphi\colon P \twoheadrightarrow X$ is weakly induced (see Mioduszewski [19]) by the sequence $\varphi_p)$ of mappings $\varphi_p\colon I_p \to X_p$ with respect to the sequence $\varepsilon_p)$ of positive numbers such that $\lim\limits_p \varepsilon_p = 0$. We may assume no proper subcontinuum of P maps onto X under φ.

Just suppose $\sigma_0^*(X) > 0$. There exist $\eta > 0$, a continuum C and mappings $h, k\colon C \to X$ such that $h(C) = X$ and $d(h(c), k(c)) > \eta$ for each $c \in C$. We may assume that $k(K) \neq X$ for each proper subcontinuum K of C. Moreover, we may assume that $d(f_1 \circ h(c), f_1 \circ k(c)) > \eta$ for each $c \in C$ where $f_i\colon X \to X_i$ denotes the ith coordinate projection. Choose n a positive integer so that $10\varepsilon_n < \eta$.

Let $P^* = \varprojlim(I_p^*, g_p^m|I_m^*)$ be a proper subcontinuum of P such that $g_n(P^*) = I_n$ ($g_p\colon P \to I_p$ is the pth coordinate projection). We may assume the maps g_p^m) were chosen so that such P^* always exists. Set $Y_1 = \varphi(P^*)$. Then Y_1 is a proper subcontinuum of X. Since X is in class W there exists a proper subcontinuum C^* of C such that $h(C^*) = Y_1$. Set $k(C^*) = Y_2$. Then Y_2 is also a proper subcontinuum of X. Note that

$$Y_i = \varprojlim(f_p(Y_i), f_p^m|f_m(Y_i))$$

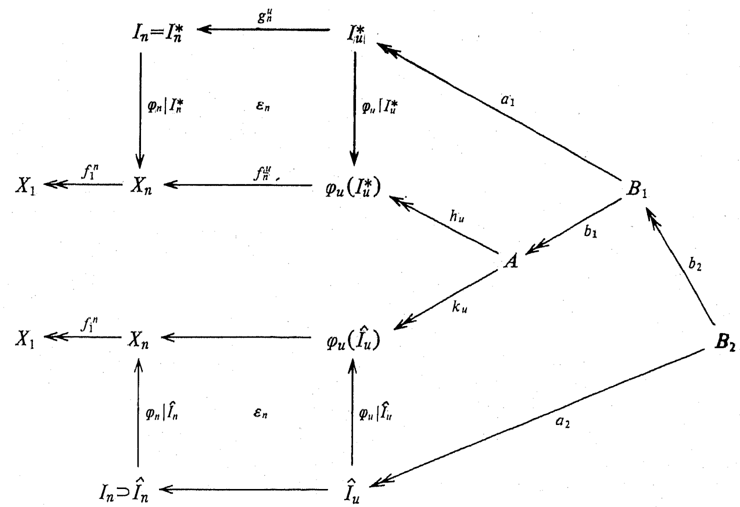
for $i = 1, 2$. Since $X \in$ class W there exists a continuum $\hat{P} = \varprojlim(\hat{I}_p, g_p^m|\hat{I}_m) \subset P$ such that $\varphi(\hat{P}) = Y_2$.

Since the map φ is almost induced and $\sigma(Y_i) = 0$ for $i = 1, 2$ there exists an integer $u > n$, an arc A (see [22], 2.4) approximating C^* and mappings $h_u\colon A \twoheadrightarrow f_u(Y_1)$ and $k_u\colon A \twoheadrightarrow f_u(Y_2)$ (approximating $f_u \circ h$ and $f_u \circ k$ respectively) such that

$$(*) \qquad d(f_1^u \circ h_u(t), f_1^u \circ k_u(t)) > \eta$$

for each $t \in A$ and

$$f_1^n \circ \varphi_n \circ g_n^u \underset{\varepsilon_n}{=} f_1^u \circ \varphi_n.$$



Let $\delta > 0$ such that $d(x, y) < \delta$ in X_u implies $d(f_1^u(x), f_1^u(y)) < \varepsilon_n$. Since $\sigma_0(Y_1) = 0$ and $\lim \varphi_u(I_u^*) = Y_1 = \lim h_u(A)$, for sufficiently large u the mappings $\varphi_u|I_u^*\colon I_u^* \to \varphi_u(I_u^*)$ and $h_u\colon A \to f_u(Y_1)$ may be δ-uniformized by Corollary 2.

Hence, there exists an arc B_1 and mappings $a_1\colon B_1 \twoheadrightarrow I_u^*$ and $b_1\colon B_1 \twoheadrightarrow A$ such that

$$(**)\qquad \varphi_u \circ a_1 \underset{\delta}{=} h_u \circ b_1 .$$

Similarly, for sufficiently large u there exists an arc B_2 and mappings $a_2\colon B_2 \twoheadrightarrow \hat{I}_u$ and $b_2\colon B_2 \twoheadrightarrow B_1$ such that

$$\varphi_u \circ a_2 \underset{\delta}{=} k_u \circ b_1 \circ b_2 .$$

By $(**)$

$$\varphi_u \circ a_1 \circ b_2 \underset{\delta}{=} h_u \circ b_1 \circ b_2 .$$

Hence

$$f_1^u \circ \varphi_u \circ a_1 \circ b_2 \underset{\varepsilon_n}{=} f_1^u \circ h_u \circ b_1 \circ b_2$$

and

$$f_1^u \circ \varphi_u \circ a_2 \underset{\varepsilon_n}{=} f_1^u \circ k_u \circ b_1 \circ b_2 .$$

Also

$$f_1^u \circ \varphi_u \underset{\varepsilon_n}{=} f_1^n \circ \varphi_n \circ g_n^u$$

and hence

$$f_1^n \circ \varphi_n \circ g_n^u \circ a_1 \circ b_2 \underset{2\varepsilon_n}{=} f_1^u \circ h_u \circ b_1 \circ b_2$$

and

$$(\overset{**}{*})\qquad f_1^n \circ \varphi_n \circ g_n^u \circ a_2 \underset{2\varepsilon_n}{=} f_1^u \circ k_u \circ b_1 \circ b_2 .$$

Since the map $g_n^u \circ a_1 \circ b_2\colon B_2 \to I_n^* = I_n$ is onto and $g_n^u \circ a_2\colon B_2 \to \hat{I}_n \subset I_n$ there exists $t_0 \in B_2$ such that $g_n^u \circ a_1 \circ b_2(t_0) = g_n^u \circ a_2(t_0)$. Hence by $(\overset{**}{*})$

$$d\big(f_1^u \circ h_u \circ b_1 \circ b_2(t_0), f_1^u \circ k_u \circ b_1 \circ b_2(t_0)\big) < 4\varepsilon_n < \eta .$$

This contradicts $(*)$ and the proof is complete.

Notice that Theorem 15 gives another proof that the atriodic tree-like continuum described by W. T. Ingram (Fund. Math. 77 (1972), pp. 99–107) is not weakly chainable since it has positive span and its proper subcontinua are arcs.

We may restate Theorem 15 as follows:

Corollary 16. *If X is a weakly chainable tree-like continuum and $\sigma_0(Y) = 0$ for each proper subcontinuum Y of X then $\sigma_0(X) = 0$.*

Proof. The condition $\sigma_0(Y) = 0$ for each proper subcontinuum of X implies by Theorem 9 that X is atriodic for if X were a triod it would contain a proper subcontinuum which is also a triod. By [9] $X \in$ class W so Theorem 15 applies.

The condition that X be tree-like in Corollary 16 is necessary as may be seen by considering the simple closed curve.

Theorem 15 may be strengthened in the following way:

Theorem 17. *If X is a weakly chainable continuum in class W such that there exist continua Y_i in X with $\lim_i Y_i = X$ and $\sigma_0(Y_i) = 0$ for each i then $\sigma_0(X) = \sup\{\sigma_0(Z) \mid Z \in C(X) \setminus \{X\}\}$.*

Proof. In the proof of Theorem 15 we may choose P^* and Y_1 with the additional requirement that $\sigma_0(Y_1) = 0$. The rest of the proof is now a relatively straightforward modification of the proof of Theorem 15.

Note. Theorem 15 may be compared with the following result [see 22, proof of Theorem 3.6]: If X is a homogeneous hereditarily indecomposable continuum such that the semi-span of all proper subcontinua is zero, then $\sigma_0(X) = 0$.

6. Decomposition and union theorems. Propositions 20–22 in this section are stated for not necessarily metric continua.

Let Y be a (not necessarily metric) continuum (i.e. Y is a compact connected Hausdorff space). Then the semi-span $\sigma_0(Y)$ of Y is zero if for every continuum $Z \subset Y \times Y$ such that $\pi_1(Z) \supset \pi_2(Z)$, $Z \cap \Delta Y \neq \emptyset$.

A mapping $f\colon X \twoheadrightarrow Y$ of (not necessarily metric) continua is *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$.

We are interested in obtaining some converses to the following well-known theorem:

Theorem. *If $f\colon X \twoheadrightarrow Y$ is a monotone mapping of (not necessarily metric) continua and $\sigma^*(X) = 0$ (resp. $\sigma_0^*(X) = 0$) then $\sigma^*(Y) = 0$ (resp. $\sigma_0^*(Y) = 0$).*

Proof. Continua in $Y \times Y \setminus \Delta Y$ pull back under $f \times f$ to continua in $X \times X \setminus \Delta X$.

We say a continuum X is a *strongly terminal subcontinuum* of a (not necessarily metric) continuum Y if $X \subset Y$ and if B, C continua in Y such that $B \neq B \setminus X \neq \emptyset \neq C \setminus X \neq C$ then $B \subset C$ or $C \subset B$. For example, if Y is hereditarily indecomposable every subcontinuum of Y is strongly terminal.

The proofs of the next two propositions are similar to that of Proposition 20 and are omitted.

Proposition 18. *Let $X \subset Y$ be metric continua and let $\varepsilon > 0$ be such that X is a strongly terminal continuum in Y, $\sigma_0(X) \leqslant \varepsilon$ and $\sigma_0(Y/X) = 0$. Then $\sigma_0(Y) \leqslant \varepsilon$.*

Proposition 19. *Let $X \subset Y$ be metric continua and let $\varepsilon > 0$ be such that X is strongly terminal in Y, $\sigma_0^*(X) \leqslant \varepsilon$ and $\sigma_0(Y/X) = 0$. Then $\sigma_0^*(Y) \leqslant \varepsilon$.*

Proposition 20. *Let $X_1, \ldots, X_n$ be pairwise disjoint strongly terminal subcontinua of a (not necessarily metric) continuum Y such that $\sigma_0(Y/\{X_i\}_{i=1}^n) = 0$ and $\sigma_0(X_i) = 0$ for each $i = 1, \ldots, n$. Then $\sigma_0(Y) = 0$.*

Proof. The proof is by induction on n. We will first consider the case $n = 1$. Put $X_1 = X$. Suppose Z is a subcontinuum of $Y \times Y$ such that $\pi_1(Z) \supset \pi_2(Z)$ and $Z \cap \Delta Y = \emptyset$. Then $Z \not\subset X \times X$ and $Z \cap (X \times X) \neq \emptyset$ since $\sigma_0(X) = \sigma_0(Y/X) = 0$. We may suppose without loss of generality that $\pi_1(Z) = Y$.

Let $U_\alpha)_{\alpha \in A}$ be a directed family of closed neighbourhoods of X such that $X = \bigcap_{\alpha \in A} U_\alpha$. Hence for every $\alpha, \beta \in A$ there exists a $\gamma \in A$ such that $X \subset U_\gamma \subset U_\alpha \cap U_\beta$. Let $(x_0, y_0) \in Z \cap (X \times X)$. For each $\alpha \in A$ let Z_α denote the component of (x_0, y_0) in $Z \cap \big(\mathrm{Cl}(U_\alpha) \times \mathrm{Cl}(U_\alpha)\big)$. By the Boundary Bumping Theorem either $\pi_1(Z_\alpha) \cap \mathrm{Bd}(U_\alpha) \neq \emptyset$ or $\pi_2(Z_\alpha) \cap \mathrm{Bd}(U_\alpha) \neq \emptyset$. Hence there exists a cofinal subset A' of A such that either $\pi_1(Z_\alpha) \cap \mathrm{Bd}(U_\alpha) \neq \emptyset$ or $\pi_2(Z_\alpha) \cap \mathrm{Bd}(U_\alpha) \neq \emptyset$ for all $\alpha \in A'$.

Without loss of generality we may suppose $\pi_1(Z_\alpha) \cap \mathrm{Bd}(U_\alpha) \neq \emptyset$ for all $\alpha \in A$. For each $\alpha, \beta \in A$ such that $U_\beta \subset U_\alpha$ let $j_\alpha^\beta\colon U_\beta \to U_\alpha$ denote the inclusion. Let $Z_0 = \varprojlim\{Z_\alpha, j_\alpha^\beta|Z_\beta\}$, then Z_0 is a continuum and $Z_0 \approx \bigcap_{\alpha \in A} Z_\alpha \subset X \times X$. Since $\mathrm{Bd}(U_\alpha) \cap \pi_1(Z_\alpha) \neq \emptyset$ and X is strongly terminal in Y, $\pi_1(Z_\alpha) \supset X$ for each $\alpha \in A$. Hence $\pi_1(Z_0) = X \supset \pi_2(Z_0)$ and $\sigma_0(X) \neq 0$. This is a contradiction.

The proposition now follows easily from induction on n.

LEMMA 21. *Suppose $Y = \varprojlim\{X_\alpha, f_\alpha^\beta\}$ is an inverse system of continua and $\sigma(X_\alpha) = 0$ for each α. Then $\sigma(Y) = 0$.*

Proof. We may suppose that each f_α^β is an onto map. Suppose $Z \subset Y \times Y$ is a continuum such that $\pi_1(Z) \supset \pi_2(Z)$ and $Z \cap \Delta Y = \emptyset$. Now,

$$Y \times Y = \varprojlim\{X_\alpha \times X_\alpha, f_\alpha^\beta \times f_\alpha^\beta\}.$$

Since $Y \times Y$ is compact and Z and ΔY are disjoint closed subsets of $Y \times Y$ there exists an α such that

$$\Delta X_\alpha \cap f_\alpha \times f_\alpha(Z) = f_\alpha \times f_\alpha(\Delta Y) \cap f_\alpha \times f_\alpha(Z) = \emptyset$$

where $f_\alpha\colon Y \to X_\alpha$ is the coordinate projection. This contradicts the assumption that $\sigma(X_\alpha) = 0$.

PROPOSITION 22. *Suppose $f\colon X \twoheadrightarrow Y$ is a monotone map of continua such that for each open cover $\mathscr{U}$ of X there exist only finitely many $y \in Y$ with $f^{-1}(y) \not\subset U$ for any $U \in \mathscr{U}$. Suppose also that if $y \in Y$ such that $f^{-1}(y)$ is non-degenerate then $f^{-1}(y)$ is a strongly terminal subcontinuum of X with $\sigma(f^{-1}(y)) = 0$. If $\sigma(Y) = 0$ then $\sigma(X) = 0$.*

Proof. Let $A = \{y \in Y | f^{-1}(y)$ is non-degenerate$\}$. Let $\mathscr{F}$ denote the family of finite subsets of A partially ordered under inclusion. For each $F \in \mathscr{F}$ let $X_F = X/\{f^{-1}(y)\}_{y \in Y \setminus F}$. If $F, G \in \mathscr{F}$ with $F \subset G$ there exists a natural projection $\varphi_F^G\colon X_G \twoheadrightarrow X_F$. By Proposition 20 $\sigma(X_F) = 0$ for each $F \in \mathscr{F}$ and it is easy to see that

$$X \approx \varprojlim\{X_F, \varphi_F^G\}.$$

Hence, the proposition follows by Lemma 21.

QUESTION 6. *If $f\colon X \twoheadrightarrow Y$ is a monotone map of atriodic continua such that $\sigma(Y) = \sigma(f^{-1}(y)) = 0$ for each $y \in Y$, then is $\sigma(X) = 0$?*

Question 6 is related to Problem 105 (due to H. Cook and J. B. Fugate) in the University of Houston Problem Book, where in each case span zero was replaced by chainability.

Recall now the following result of Duda and Kell [5]:

THEOREM (Duda and Kell [5]). *Let $X = A \cup B$ be an atriodic continuum. If A and B are continua such that $A \cap B$ is connected and $\sigma_0(A) = \sigma_0(B) = 0$ then $\sigma_0(X) = 0$.*

PROPOSITION 23. *Let X be a subcontinuum of the atriodic continuum Y. If $\sigma_0(Y/X) = \sigma_0(X) = 0$ and $Y = A \cup B$ where A and B are continua with $X = A \cap B$ then $\sigma_0(Y) = 0$.*

Proof. Let $A^* = \mathrm{Cl}(A \setminus X)$ and $X^* = A^* \cap X$. Now $\sigma_0(A^*/X^*) = 0$ since A^*/X^* is the monotone image of Y/X. By Proposition 18 $\sigma_0(A^*) = 0$. By the Duda and Kell result $\sigma_0(A) = \sigma_0(A^* \cup X) = 0$. Similarly, $\sigma_0(B) = 0$. Applying the Duda and Kell result again we get $\sigma_0(Y) = 0$.

It is unclear whether the proof of Duda and Kell can be extended to the case $\sigma_0^*(A) = 0 = \sigma_0^*(B)$. It is sometimes useful to know that a continuum X with $\sigma_0^*(X) = 0$ can be embedded as a proper subcontinuum of a continuum Y with $\sigma_0^*(Y) = 0$.

PROPOSITION 24. *If X is a continuum with $\sigma^*(X) = 0$* (resp. $\sigma_0^*(X) = 0$) *and Y is a compactification of a half line by X such that $Y \setminus X$ is dense in Y then $\sigma^*(Y) = 0$* (resp. $\sigma_0^*(Y) = 0$).

We do the proof for the case $\sigma^*(X) = 0$. The other case follows from Proposition 19.

Proof. Identify $Y \setminus X$ with $[0, \infty)$. For $x \in [0, \infty)$ let $Y_x = Y \setminus [0, x)$.

Suppose L is a continuum in $Y \times Y \setminus \Delta Y$ such that $\pi_i(L) = Y$ where ΔY denotes the diagonal of Y and $\pi_i\colon Y \times Y \to Y$ is the ith coordinate projection. There is a component $Z_0 = L$ of $(Y_0 \times Y_0) \cap L$ which meets both $\{0\} \times Y_0$ and $Y_0 \times \{0\}$.

By induction there exists for each positive integer n a component Z_n of $(Y_n \times Y_n) \cap Z_{n-1}$ such that $\pi_i(Z_n) = Y_n$ for $i = 1, 2$. Then $Z = \bigcap_{n=0}^{\infty} Z_n$ is a continuum in $L \cap X \times X$ such that $\pi_i(Z) = X$ for $i = 1, 2$ and Z misses the diagonal of X. This contradicts the assumption that $\sigma^*(X) = 0$.

PROPOSITION 25. *Let X be a continuum with $\sigma_0^*(X) = 0$. Let A and B be compactifications of half lines by X such that $A \setminus X$ and $B \setminus X$ are dense in A and B respectively and $A \cap B = X$. Then $\sigma_0^*(A \cup B) = 0$.*

The proof uses Proposition 24 and is similar to that given by Duda and Kell [5], 3.2.

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On d-paracompactness and related properties

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Abstract. We give a simple definition of d-paracompact spaces and use it in order to prove that the following classes of spaces are preserved by perfect mappings: the class of perfect preimages of metacompact (metalindelöf) Moore spaces and the class of spaces which, for an arbitrary open cover $\mathcal{U}$, admit a perfect $\mathcal{U}$-mapping onto a Moore space. These results were announced in [Ch6] and it was shown there that the class of perfect preimages of Moore spaces is not preserved by perfect mappings.

A method of constructing mappings onto developable spaces has been introduced in [P]. This method uses the concept of d-paracompact spaces. The definition of d-paracompactness in [P] is very technical. A simpler but still technical concept of a kernel-normal sequence of open covers has been introduced in [Br].

In the first section of this paper we give a simple definition of d-paracompactness which is analogous to certain characterizations of paracompactness and subparacompactness. We also define related concepts of d-regularity, d-normality and collectionwise d-normality. All these properties are weaker than developability (= d-metrizability). We prove that our definition of d-paracompactness is equivalent to the one given in [P] and show some simple facts about related concepts which are useful in further investigations of d-paracompactness.

In the second section we prove that, in the class of metacompact (metalindelöf) spaces, d-paracompactness shows some analogies with the weaker concept of subparacompactness. In particular, it is preserved (in both directions) by perfect mappings and, consequently, the class of perfect preimages of metacompact (metalindelöf) Moore spaces is preserved by perfect mappings (this solves Problem 3.1 of [Ch5]).

The third section is devoted to the investigation of the preservation of d-paracompactness by perfect mappings. We give a method of constructing perfect mappings from spaces which are not d-normal (d-normal but not d-paracompact) onto Moore spaces. According to the results of the second section, such Moore spaces cannot be metalindelöf. We prove that perfect images of d-paracompact p-spaces are d-paracompact p-spaces. This shows that the property of having, for an arbitrary open cover $\mathcal{U}$, a perfect $\mathcal{U}$-mapping onto a Moore space is an invariant of perfect mappings.