On the $\diamondsuit$ principle

by

Adam Piołunowicz (Warszawa)

Abstract. We shall consider the principle $\diamondsuit(E)$ and some of its versions which seem to be weaker, and prove that under some additional simple assumption they are equivalent. Then, in particular cases, we obtain as corollaries two versions of the principle $\diamondsuit$.

DEFINITIONS. Let λ be an infinite cardinal number, $E \subseteq \lambda^+$.

The principle $\diamondsuit_{\lambda^+}(E)$ holds iff there is a sequence $\langle S_\alpha \colon \alpha \in E \rangle$ such that:

1) $\forall_{\alpha \in E}\, S_\alpha \subseteq \alpha$,

2) $\forall_{X \subseteq \lambda^+} \{\alpha \in E \colon X \cap \alpha = S_\alpha\}$ is stationary in λ^+.

The principle $\diamondsuit^*_{\lambda^+}(E)$ holds iff there is a sequence $\langle S_\alpha \colon \alpha \in E \rangle$ such that:

1) $\forall_{\alpha \in E}\, S_\alpha \subseteq \alpha$,

2) $\forall_{X \subseteq \lambda^+} \exists_{\alpha \in E}\, X \cap \alpha = S_\alpha$.

The principle $\diamondsuit^-_{\lambda^+}(E)$ holds iff there is a sequence $\langle S_\alpha \colon \alpha \in E \rangle$ such that:

1) $\forall_{\alpha \in E}\, S_\alpha \subseteq P(\alpha)$,

2) $\forall_{\alpha \in E} |S_\alpha| \leqslant \lambda$,

3) $\forall_{X \subseteq \lambda^+} \exists_{\alpha \in E}\, X \cap \alpha \in S_\alpha$.

$\diamondsuit_{\lambda^+}$ means the principle $\diamondsuit_{\lambda^+}(\lambda^+)$.

We see that the principle $\diamondsuit_{\lambda^+}(E)$ implies $\diamondsuit^*_{\lambda^+}(E)$ and that $\diamondsuit^*_{\lambda^+}(E)$ implies $\diamondsuit^-_{\lambda^+}(E)$. Our main task is to answer the question under what assumption the converse implications hold.

DEFINITION. We call a tree T *continuous* iff, for all $\alpha \in \mathrm{Lim}$, $\alpha > 0$, and, for all $x, y \in T^{(\alpha)}$, $x \neq y \Rightarrow \{z \in T \colon z \prec x\} \neq \{z \in T \colon z \prec y\}$.

We shall use the following two facts:

THEOREM 1. *Let λ be an infinite cardinal number and E a stationary subset of λ^+. Then there is a continuous tree T such that:*

a) $|T| \leqslant 2^\lambda$,

b) *every element of T is of the form $\langle f, g, \alpha \rangle$ where: f, g are functions*; $\mathrm{rg}(f)$, $\mathrm{rg}(g) \subseteq \{0, 1\}$; $\mathrm{dom}(g)$ *and α are ordinal numbers less than λ^+*; $\mathrm{dom}(f) = \mathrm{dom}(g) \times \mathrm{dom}(g)$;

c) *if* $X \subseteq \lambda^+ \times \lambda^+$, *and* C *is a closed unbounded set in* λ^+, *then there is a branch* G *such that every element of* G *is of the form* $\langle \chi_{X|\beta\times\beta}, \chi_{C|\beta}, \alpha \rangle$ *where* $\beta \in C$ *and* $\chi_{X|\beta\times\beta}, \chi_{C|\beta}$ *denote restrictions of characteristic functions of* X *and* C, *and the length of* G *is* λ^+,

d) *if* $\langle f, g, \alpha \rangle$ *belongs to* $T^{(\beta)}$ *then* $\operatorname{dom}(g) \geqslant \beta$; *moreover, if* $\beta \in E$ *then also* $\operatorname{dom}(g) \in E$.

We shall show a method of the construction of the required tree T later in the paper.

THEOREM 2. *Let* λ *be an infinite cardinal number,*

$$K = \{\alpha \in \lambda^+ \colon \exists_{\beta\in\lambda^+}[\beta \neq 0 \,\&\, \alpha = \lambda\cdot\beta]\},$$

$E \subseteq K$, *and assume that principle* $\diamondsuit^-_{\lambda^+}(E)$ *holds. Then, for every continuous tree* T *such that* $|T| \leqslant 2^\lambda$, *there is a set* S, $S \subseteq T$, *such that*:

a) $\forall_{\alpha\in\lambda^+} |S \cap T^{(\alpha)}| \leqslant \lambda$,

b) $\forall_{\alpha\in\lambda^+\setminus E}\, S \cap T^{(\alpha)} = \emptyset$,

c) *for every branch* G *of the tree* T, *if the length of the branch* G *is* $\geqslant \lambda^+$ *then* $G \cap S \neq \emptyset$.

Proof. Let $T_1 = \{f\colon f$ is a function, $\operatorname{dom}(f) \in K$ and $\operatorname{rg}(f) \subseteq \{0,1\}\}$. The partial ordering $\langle T_1, \subseteq \rangle$ is a continuous tree, T_1 has 2^λ minimal elements, every element of T_1 has 2^λ immediate successors the height of T_1 is λ^+.

Let $\langle S_\alpha\colon \alpha \in E\rangle$ be a sequence satisfying $\diamondsuit^-_{\lambda^+}(E)$. The characteristic functions of elements S_α (in the space α) belong to T_1. Let S_1 be the set of all these functions. Obviously, S_1, T_1 satisfy points a) and c) of the theorem.

Let $\langle T, \underset{T}{\leqslant} \rangle$ be a continuous tree, $|T| \leqslant 2^\lambda$. Then for $T_0 = \bigcup_{\alpha\in K} T^{(\alpha)}$, $\langle T_0, \underset{T}{\leqslant}|_{T_0} \rangle$ is also a continuous tree, $|T_0| \leqslant 2^\lambda$.

We may assume that T_0 is isomorphic to a subtree of T_1 with the following property (after suitable identification): if $x \in T_0^{(\alpha)}$ (for some α) then $x \in T_1^{(\alpha)}$. (The levels are preserved). We put $S = S_1 \cap T_0$.

Obviously S, T satisfy a) and c). We must show that b) holds as well. It is sufficient to prove that if $s \in S_\alpha$ and f is the characteristic function of s (in the space α) then f (as an element of T_0) belongs to $T^{(\alpha)}$. This, however, is immediate. ■

THEOREM 3. *Let* λ *be an infinite cardinal number,*

$$K = \{\alpha \in \lambda^+ \colon \exists_{\beta\in\lambda^+}[\beta \neq 0 \,\&\, \alpha = \lambda\cdot\beta]\}$$

and $E \subseteq K$. *Then the following are equivalent*:

a) $\diamondsuit_{\lambda^+}(E)$,

b $\diamondsuit^*_{\lambda^+}(E)$,

c) $\diamondsuit^-_{\lambda^+}(E)$,

d) *there is a continuous tree* T *such that*: $T \neq \emptyset$, *every element of* T *has* 2^λ *immediate successors, the length of every branch of* T *belongs to* E *and for every* α *less than* λ^+ *there are at most* λ *branches of length* α.

Proof. Obviously a) $\Rightarrow$ b) $\Rightarrow$ c). The proof of c) $\Leftrightarrow$ d) strongly resembles the proof of Theorem 2. Thus we must show c) $\Rightarrow$ a). Assume $\diamondsuit^-_{\lambda^+}(E)$.

LEMMA 1. *E is a stationary set.*

Proof. Suppose that E is not stationary. Since K is closed unbounded set, we can find a closed unbounded set C such that $C \subseteq K$ and $C \cap E = \emptyset$. Let $\langle c_\xi\colon \xi < \lambda^+\rangle$ be an increasing enumeration of C. By induction on $\xi < \lambda^+$ we construct a family $\langle X_\xi\colon \xi < \lambda^+ \rangle$ such that:

1) $X_\xi \subseteq c_\xi$,

2) for $\xi_1 < \xi_2$ $X_{\xi_1} = X_{\xi_2} \cap c_{\xi_1}$,

3) $\forall_{\alpha < c_\xi} X_\xi \cap \alpha \notin S_\alpha$.

Let X_ξ be given. There are 2^λ sets Y such that $Y \cap c_\xi = X_\xi$ and $Y \subseteq c_\xi + \lambda$. We have $c_\xi \in K \setminus E$ and $E \subseteq K$; hence if $c_\xi \leqslant \alpha < c_\xi + \lambda$ then $\alpha \notin E$. We note also that $|\bigcup_{c_\xi \leqslant \alpha \leqslant c_{\xi+1}} S_\alpha| \leqslant \lambda$. Hence there is a set Y such that $Y \subseteq c_{\xi+1}$, $Y \cap c_\xi = X_\xi$ and $\forall_\alpha[c_\xi \leqslant \alpha \leqslant c_{\xi+1} \Rightarrow Y \cap \alpha \notin S_\alpha]$. We put $X_{\xi+1} = Y$. By the induction assumption and the construction, $X_{\xi+1}$ satisfies 1), 2), 3). The construction for the case $\xi = 0$ is the same. For $\xi \in \mathrm{Lim}$, $\xi > 0$ we put $X_\xi = \bigcup_{\mu<\xi} X_\mu$. We define $X = \bigcup_{\xi<\lambda^+} X_\xi$. Then $\forall_{\alpha\in E} X \cap \alpha \notin S_\alpha$. However this contradicts $\diamondsuit^-_{\lambda^+}(E)$. ■

LEMMA 2. *There is a sequence* $\langle S'_\alpha\colon \alpha \in E\rangle$ *such that*:

1) $\forall_{\alpha\in E} S'_\alpha \subseteq P(\alpha\times\alpha)$,

2) $\forall_{\alpha\in E} |S'_\alpha| \leqslant \lambda$,

3) $\forall_{X \subseteq \lambda^+\times\lambda^+} \{\alpha \in \lambda^+\colon X \cap (\alpha\times\alpha) \in S'_\alpha\}$ *is stationary.*

Proof. We already know that E is a stationary set. So, let T be a tree given by Theorem 1 and let S be a set given by Theorem 2. Then S satisfies the following:

a) $\forall_{\alpha\in\lambda^+} |S \cap T^{(\alpha)}| \leqslant \lambda$,

b) $\forall_{\alpha\in\lambda^+\setminus E} S \cap T^{(\alpha)} = \emptyset$,

c) if G is a branch of length λ^+ then $G \cap S \neq \emptyset$.

For $\alpha \in \lambda^+$ we put:

$$S'_\alpha = \{s\colon \exists_{f,g,\beta}[\langle f, g, \beta\rangle \in S \,\&\, \operatorname{dom}(g) = \alpha \,\&\, s = f^{-1}(\{0\})]\}.$$

We shall show that S'_α's satisfy 1), 2), 3). Obviously $S'_\alpha \subseteq P(\alpha\times\alpha)$.

First we note that by point b) of this lemma elements of S exist only on the levels from E. Hence, by point d) of Theorem 1, for every $\langle f, g, \beta\rangle \in S$, $\operatorname{dom}(g) \in E$. This implies that, for $\alpha \in \lambda^+ \setminus E$, $S'_\alpha = \emptyset$. Hence we must consider only a sequence $\langle S'_\alpha\colon \alpha \in E\rangle$.

By d) of Theorem 1 we infer that if $\langle f, g, \beta\rangle \in T^{(\gamma)}$ then $\operatorname{dom}(g) \geqslant \gamma$. Now we can rewrite the definition of S'_α in the following fashion:

$$S'_\alpha = \{s\colon \exists_{f,g,\beta}[\langle f, g, \beta\rangle \in S \cap \bigcup_{\gamma\leqslant\alpha} T^{(\gamma)} \,\&\, \operatorname{dom}(g) = \alpha \,\&\, s = f^{-1}(\{0\})]\}.$$

But

$$|S \cap \bigcup_{\gamma \leqslant \alpha} T^{(\gamma)}| = |\bigcup_{\gamma \leqslant \alpha} (S \cap T^{(\gamma)})| = \sum_{\gamma \leqslant \alpha} |S \cap T^{(\gamma)}| \leqslant |\alpha| \cdot \lambda \leqslant \lambda .$$

Hence $|S'_\alpha| \leqslant \lambda$.

Now we show 3). Let us fix a set $X \subseteq \lambda^+ \times \lambda^+$ and a closed unbounded set $C \subseteq \lambda^+$. It is sufficient to prove that $\{\alpha \in \lambda^+ \colon X \cap (\alpha \times \alpha) \in S'_\alpha\} \cap C \neq \emptyset$. Let G be a branch in the tree T such that every element of G is of the form $\langle \chi_{X|\beta\times\beta}, \chi_{C|\beta}, \alpha \rangle$ where $\beta \in C$ (see c) of Theorem 1). Let $y_0 \in G \cap S$ then $y_0 = \langle \chi_{X|\beta_0\times\beta_0}, \chi_{C|\beta_0}, \alpha \rangle$ and $\beta_0 \in C$. We see $X \cap (\beta_0 \times \beta_0) \in S'_{\beta_0}$ and $\beta_0 \in C$. ■

Lemma 3 (Kunen [4]). *Let* $\langle S'_\alpha \colon \alpha \in E \rangle$ *be a sequence from Lemma* 2, *and let* S'^{β}_α *be an enumeration* S'_α *such that* $S'_\alpha = \{S'^{\beta}_\alpha \colon \beta < |S'_\alpha|\}$. *For* $X \subseteq \lambda^+ \times \lambda^+$ *and* $\beta \in \lambda^+$ *we put* $R_\beta(X) \overset{\text{df}}{=} \{\alpha \in \lambda^+ \colon \langle \alpha, \beta \rangle \in X\}$. *Then there is* $\beta_0 \in \lambda$ *such that* $\forall_{X \subseteq \lambda^+} \exists_{Y \subseteq \lambda^+ \times \lambda^+} R_{\beta_0}(Y) = X \,\&\, \{\alpha \in E \colon Y \cap (\alpha \times \alpha) = S'^{\beta_0}_\alpha\}$ *is stationary.*

Proof. Suppose the lemma is not true. Then for every $\beta < \lambda$ we can pick $X_\beta \subseteq \lambda^+$ such that

$$\forall_{Y \subseteq \lambda^+ \times \lambda^+} [R_\beta(Y) = X_\beta \Rightarrow \{\alpha \in E \colon Y \cap (\alpha \times \alpha) = S'^{\beta}_\alpha\} \text{ is not stationary}] .$$

We put $Y_0 = \{\langle \alpha, \beta \rangle \colon \alpha \in X_\beta\}$. We note that $\forall_{\beta < \lambda} R_\beta(Y_0) = X_\beta$. Hence $\forall_{\beta < \lambda} Z_\beta \overset{\text{df}}{=} \{\alpha \in E \colon Y_0 \cap (\alpha \times \alpha) = S'^{\beta}_\alpha\}$ is not stationary.

But then also $\{\alpha \in E \colon Y_0 \cap (\alpha \times \alpha) \in S'_\alpha\} = \bigcup_{\beta < \lambda} Z_\beta$ is not stationary. This contradicts point 3) of Lemma 3. ■

Proof of Theorem 3. We follow the argument of Kunen [4]. Let $\{S'^{\beta}_\alpha \colon \beta < |S'_\alpha|, \alpha \in E\}$, $\beta_0 \in \lambda$ be as in Lemma 3. For $\alpha \in E$ we put $S''_\alpha = R_{\beta_0}(S'^{\beta_0}_\alpha)$. We must show that:

1) $\forall_{\alpha \in E} S''_\alpha \subseteq \alpha$,

2) $\forall_{X \subseteq \lambda^+} \{\alpha \in E \colon X \cap \alpha = S''_\alpha\}$ is stationary.

1) follows from the construction of S''_α.

We now show 2). Let a set $X \subseteq \lambda^+$ be given. Then, by Lemma 3, there is a set $Y \subseteq \lambda^+ \times \lambda^+$ such that $R_{\beta_0}(Y) = X$ and $\{\alpha \in E \colon Y \cap (\alpha \times \alpha) = S'^{\beta_0}_\alpha\}$ is stationary. Hence $\{\alpha \in E \colon Y \cap (\alpha \times \alpha) = S'^{\beta_0}_\alpha\} \subseteq \{\alpha \in E \colon R_{\beta_0}(Y \cap (\alpha \times \alpha)) = R_{\beta_0}(S'^{\beta_0}_\alpha)\} = \{\alpha \in E \colon X \cap \alpha = S''_\alpha\}$ is also stationary. ■

We derive as corollaries the following facts.

Theorem 4. *Suppose* λ *is an infinite cardinal number. Then the following are equivalent*:

a) $\diamondsuit_{\lambda^+}$, b) $\diamondsuit_{\lambda^+}(\lambda^+ \setminus \lambda)$.

Theorem 5. *The following are equivalent*:

a) $\diamondsuit \omega_1$,

b) *there is a sequence* $\langle S_\alpha \colon \alpha < \omega_1 \rangle$ *such that*

1) $\forall_{\alpha \in \omega_1} S_\alpha \subseteq \alpha$,

2) $\forall_{X \subseteq \omega_1} |\{\alpha \in \omega_1 \colon X \cap \alpha = S_\alpha\}| \geqslant 3$.

Proof. a) $\Rightarrow$ b) is obvious.

b) $\Rightarrow$ a) Let $\langle S_\alpha \colon \alpha < \omega_1 \rangle$ be a sequence as in b). We put

$$A = \{X \subseteq \omega \colon |\{n \in \omega \colon X \cap n = S_n\}| \leqslant 2\} .$$

Lemma. $|A| = 2^\omega$.

Proof. It is enough to prove that $|A| \geqslant 2^\omega$. Let

$$B = \{n \in \omega \colon \exists_{k \in \omega} [0 < k < n \,\&\, S_k = S_n \cap k]\} .$$

I. Suppose $|B| < \omega$. Let $n_0 \in \omega$ be such that $\forall_{n \in B}\, n < n_0$. We put

$$C = \{X \subseteq \omega \colon \forall_{m < n_0} [m \in X \Leftrightarrow m \notin S_{m+1}]\} .$$

It is sufficient to prove that:

i) $C \subseteq A$, ii) $|C| \geqslant 2^\omega$.

i) By the definition of C, if $0 < n < n_0$ and $X \in C$ then $X \cap n \neq S_n$. But $|\{n \geqslant n_0 \colon X \cap n = S_n\}| \leqslant 1$ by the definition of B. We see that for every $X \in C$ $|\{n \in \omega \colon X \cap n = S_n\}| \leqslant 2$, so $C \subseteq A$.

ii) follows from the fact that for every element of $P(\omega \setminus n_0)$ we can pick a set y belonging to C which contains that element. We obtain a function f, $f \colon P(\omega \setminus n_0) \to C$, and the function f is 1-1.

II. Suppose $|B| = \omega$. In this case we put

$$C = \{X \subseteq \omega \colon \forall_{m \in \omega} [m+1 \notin B \Rightarrow (m \in X \Leftrightarrow m \notin S_{m+1})]\} .$$

We need to show that:

i) $C \subseteq A$, ii) $|C| \geqslant 2^\omega$.

The proof follows the argument of case I. Thus in any case $|A| = 2^\omega$. ■

Let f be a function such that $f \colon A \xrightarrow[\text{onto}]{1-1} P(\omega)$. For $\omega \leqslant \alpha < \omega_1$ we put

$$S'_\alpha = \begin{cases} \{(S_\alpha \setminus \omega) \cup f(S_\alpha \cap \omega)\} & \text{if } S_\alpha \cap \omega \in A , \\ \{\emptyset\} & \text{otherwise} . \end{cases}$$

We prove that

i) $\forall_{\omega \leqslant \alpha < \omega_1} S'_\alpha \subseteq P(\alpha)$,

ii) $\forall_{\omega \leqslant \alpha < \omega_1} |S'_\alpha| = 1 \leqslant \omega$,

iii) $\forall_{X \subseteq \omega_1} \exists_{\alpha \geqslant \omega} X \cap \alpha \in S'_\alpha$.

i) and ii) are obvious.

iii) Let $X\subseteq\omega_1$. We put $Y=(X\setminus\omega)\cup f^{-1}(X\cap\omega)$. We note $Y\cap\omega\in A$. Hence, there is $\alpha\geqslant\omega$ such that $Y\cap\alpha=S_\alpha$. But then

$$X\cap\alpha=((X\setminus\omega)\cap\alpha)\cup((X\cap\omega)\cap\alpha)=((Y\setminus\omega)\cap\alpha)\cup(X\cap\omega)$$
$$=(S_\alpha\setminus\omega)\cup f(S_\alpha\cap\omega)\in S'_\alpha .$$

We see that the principle $\diamondsuit^-_{\omega_1}(\omega_1\setminus\omega)$ holds. By Theorem 4 we have $\diamondsuit_{\omega_1}$. ■

We now give the proof of Theorem 1.

I. The construction of the set T. For $\alpha\in E$ we define:

$A_\alpha=\{f\colon f$ is a function, $\mathrm{dom}(f)=\alpha\times\alpha$, $\mathrm{rg}(f)\subseteq\{0,1\}\}$,
$B_\alpha=\{g\colon g$ is a function, $\mathrm{dom}(g)=\alpha$, $\mathrm{rg}(g)\subseteq\{0,1\}$, $g^{-1}(\{0\})$ is a closed set in $\alpha\}$,
$D_\alpha=A_\alpha\times B_\alpha$,
$D=\bigcup_{\alpha\in E}D_\alpha$.

In D we define a partial ordering $\preccurlyeq$ as follows:

$\langle f_1,g_1,\alpha_1\rangle\preccurlyeq\langle f_2,g_2,\alpha_2\rangle$ iff $f_1\subseteq f_2$, $g_1\subseteq g_2$, $\alpha_1<\alpha_2$, $g_2(\alpha_1)=0$ or $\langle f_1,g_1,\alpha_1\rangle=\langle f_2,g_2,\alpha_2\rangle$.

Suppose $L=\{\langle f_z,g_z,\alpha_z\rangle\colon z\in Z\}$ is a chain in D, and the order type of L is μ, $0<\mu<\lambda^+$, $\mu\in\mathrm{Lim}$. Then we define:

$$\sigma(L)=\Big\langle\bigcup_{z\in Z}f_z,\ \bigcup_{z\in Z}g_z,\ \sup_{z\in Z}\alpha_z\Big\rangle .$$

We put

$$T_1=D\cup\{\sigma(L)\colon L\neq\emptyset,\ L \text{ is a chain, the order type of } L \text{ is a limit ordinal less than } \lambda^+\}.$$

For $\langle f,g,\alpha\rangle\in T_1$ we define

$$\delta(\langle f,g,\alpha\rangle)=\begin{cases}\{\langle f,g,\beta\rangle\colon \sup(g^{-1}(\{0\})\cap E)<\beta<\alpha\} & \text{if } g^{-1}(\{0\})\cap E\neq\emptyset,\\ \{\langle f,g,\beta\rangle\colon \beta<\alpha\} & \text{otherwise}.\end{cases}$$

First we note

(1) if $\langle f,g,\alpha\rangle\in T_1\setminus D$ then $\delta(\langle f,g,\alpha\rangle)=\emptyset$.

We can already define

$$T=T_1\cup\bigcup_{x\in T_1}\delta(x).$$

II. Some properties of the set T. From the definition of T it follows that:

(2) every element of T is of the form $\langle f,g,\alpha\rangle$ where f,g are functions; $\mathrm{rg}(f)$, $\mathrm{rg}(g)\subseteq\{0,1\}$; $\mathrm{dom}(g),\alpha$ are ordinal numbers less than λ^+; $\mathrm{dom}(f)=\mathrm{dom}(g)\times\mathrm{dom}(g)$.

Hence

(3) $|T|\leqslant 2^\lambda$.

By the definition we also have:

(4) if $\langle f,g,\alpha\rangle\in T$ then $\mathrm{dom}(g)\geqslant\alpha$,
(5) if $\langle f,g,\alpha\rangle\in T$ then $\mathrm{dom}(g)=\alpha\Leftrightarrow\langle f,g,\alpha\rangle\in T_1$.

Then by (1) and (5) we have:

(6) if $\langle f,g,\alpha\rangle\in T$ and $\mathrm{dom}(g)>\alpha$ then $\mathrm{dom}(g)\in E$.

III. A tree ordering on T. We define $\langle f_1,g_1,\alpha_1\rangle\prec\langle f_2,g_2,\alpha_2\rangle$ iff

a) $f_1\subseteq f_2$, $g_1\subseteq g_2$, $\alpha_1<\alpha_2$,
b) if $\mathrm{dom}(g_1)<\mathrm{dom}(g_2)$ then $g_2(\mathrm{dom}(g_1))=0$.

For $x,y\in T$ we define $x\preccurlyeq y\Leftrightarrow x\prec y\vee x=y$. First we note that $\preccurlyeq$ is a well-founded, partial ordering relation. We must show that, for every $x\in T$, a set $\{z\in T\colon z\preccurlyeq x\}$ is a chain. We first notice the following fact:

LEMMA. *If* $\langle f_1,g_1,\alpha_1\rangle\prec\langle f_3,g_3,\alpha_3\rangle$, $\langle f_2,g_2,\alpha_2\rangle\prec\langle f_3,g_3,\alpha_3\rangle$ *and* $\alpha_1\leqslant\alpha_2$ *then* $\mathrm{dom}(g_1)\subseteq\mathrm{dom}(g_2)$.

Proof. Suppose the lemma is not true. Then $\mathrm{dom}(g_2)<\mathrm{dom}(g_1)$. We note that $g_1,g_2\subseteq g_3$, and since $g_2\subsetneq g_1\subseteq g_3$ we have $g_3(\mathrm{dom}(g_2))=0$; hence also

(i) $g_1(\mathrm{dom}(g_2))=0$.

We now prove that

(ii) $g_1^{-1}(\{0\})\cap E\neq\emptyset$ and $\sup g_1^{-1}(\{0\})\cap E\geqslant\mathrm{dom}(g_2)$.

If $\mathrm{dom}(g_2)\in E$ then (i) implies (ii). So suppose $\mathrm{dom}(g_2)\notin E$. Then $\langle f_2,g_2,\alpha_2\rangle\notin D$ (by the definition of T). We also have $\langle f_2,g_2,\mathrm{dom}(g_2)\rangle\in T_1\setminus D$ and therefore $\langle f_2,g_2,\mathrm{dom}(g_2)\rangle=\sigma(Z)$ for some Z. From the definition of σ we obtain $\sup(g_2^{-1}(\{0\})\cap E)=\mathrm{dom}(g_2)$. This concludes (ii) since $g_2\subseteq g_1$.

From (4) we now get $\alpha_1\leqslant\alpha_2\leqslant\mathrm{dom}(g_2)<\mathrm{dom}(g_1)$, so $\alpha_1<\mathrm{dom}(g_1)$. (5) tells us that $\langle f_1,g_1,\alpha_1\rangle\in\delta(x)$ for some x. By the definition of δ and (ii) we have:

(iii) $\alpha_1>\sup(g_1^{-1}(\{0\})\cap E)$.

But by (ii), (iii) and (4) we have $\alpha_1>\sup(g_1^{-1}(\{0\})\cap E)\geqslant\mathrm{dom}(g_2)\geqslant\alpha_2$. This contradicts $\alpha_1\leqslant\alpha_2$. ■

We now prove that $\langle T,\preccurlyeq\rangle$ is a tree. Suppose $\langle f_1,g_1,\alpha_1\rangle\prec\langle f_3,g_3,\alpha_3\rangle$, $\langle f_2,g_2,\alpha_2\rangle\prec\langle f_3,g_3,\alpha_3\rangle$ and $\alpha_1\leqslant\alpha_2$. Then by the lemma $\mathrm{dom}(g_1)\subseteq\mathrm{dom}(g_2)$ and $\mathrm{dom}(f_1)=\mathrm{dom}(g_1)\times\mathrm{dom}(g_1)\subseteq\mathrm{dom}(g_2)\times\mathrm{dom}(g_2)=\mathrm{dom}(f_2)$. Since $f_1,f_2\subseteq f_3$ and $g_1,g_2\subseteq g_3$ we get $f_1\subseteq f_2$, $g_1\subseteq g_2$, $\alpha_1\leqslant\alpha_2$.

If $\alpha_1=\alpha_2$ then we also have $\alpha_2\leqslant\alpha_1$ and so $f_2\subseteq f_1$ and $g_2\subseteq g_1$; thus $\langle f_1,g_1,\alpha_1\rangle=\langle f_2,g_2,\alpha_2\rangle$.

So assume $\alpha_1<\alpha_2$. We must show $\mathrm{dom}(g_1)<\mathrm{dom}(g_2)\Rightarrow g_2(\mathrm{dom}(g_1))=0$. But since $g_1\subsetneq g_2\subseteq g_3$ by the definition of $\prec$, we have $g_3(\mathrm{dom}(g_1))=0$. Then also $g_2(\mathrm{dom}(g_1))=0$. ■

IV. $\langle f,g,\alpha\rangle$ belongs to $T^{(\alpha)}$. It is sufficient to show that:

(i) for $\langle f,g,\alpha\rangle\in T$ and $\beta<\alpha$ there are f_1,g_1 such that $\langle f_1,g_1,\beta\rangle\prec\langle f,g,\alpha\rangle$.

Indeed, in this fashion we see that $\langle f,g,\alpha\rangle$ has exactly α predecessors. (i) reduces to the following:

(ii) for $\langle f, g, \alpha\rangle \in T_1$ and $\beta<\alpha$ there are f_1, g_1 such that $\langle f_1, g_1, \beta\rangle \prec \langle f, g, \alpha\rangle$.

Indeed, suppose $\langle f, g, \alpha\rangle \in T\setminus T_1$. Then $\langle f, g, \alpha\rangle \in \delta(\langle f, g, \mathrm{dom}(g)\rangle)$ and $\langle f, g, \alpha\rangle \prec \langle f, g, \mathrm{dom}(g)\rangle \in T_1$. If we can find f_1, g_1 such that $\langle f_1, g_1, \beta\rangle \prec \langle f, g, \mathrm{dom}(g)\rangle$, we get, since T is a tree, $\langle f_1, g_1, \beta\rangle \prec \langle f, g, \alpha\rangle$ or $\langle f_1, g_1, \beta\rangle = \langle f, g, \alpha\rangle$ or $\langle f, g, \alpha\rangle \prec \langle f_1, g_1, \beta\rangle$. But $\beta<\alpha$, and so $\langle f_1, g_1, \beta\rangle \prec \langle f, g, \alpha\rangle$.

Again (ii) reduces to:

(iii) if $\langle f, g, \alpha\rangle \in T_1$, $\beta<\alpha$ and $\sup(g^{-1}(\{0\}) \cap E \cap (\beta+1))<\beta$ then there are f_1, g_1 such that $\langle f_1, g_1, \beta\rangle \prec \langle f, g, \alpha\rangle$.

Indeed, if $\beta \in g^{-1}(\{0\}) \cap E$ we get $\langle f_{|\beta\times\beta}, g_{|\beta}, \beta\rangle \in D$ and $\langle f_{|\beta\times\beta}, g_{|\beta}, \beta\rangle \prec \langle f, g, \alpha\rangle$. So suppose $\beta \notin g^{-1}(\{0\}) \cap E$ but $\sup(g^{-1}(\{0\}) \cap E \cap (\beta+1)) = \beta$. We define $L = \{\langle f_{|\gamma\times\gamma}, g_{|\gamma}, \gamma\rangle : \gamma \in g^{-1}(\{0\}) \cap E \cap (\beta+1).\}$

We notice that $L \subseteq D$, L is a chain with order type μ and μ is a limit. We get $\langle f_{|\beta\times\beta}, g_{|\beta}, \beta\rangle = \sigma(L) \in T$ and $\langle f_{|\beta\times\beta}, g_{|\beta}, \beta\rangle \prec \langle f, g, \alpha\rangle$.

Thus we prove (iii). Let $Z = \{\langle f_1, g_1, \gamma\rangle \in T_1 : \langle f_1, g_1, \gamma\rangle \preccurlyeq \langle f, g, \alpha\rangle \ \&\ \beta \leqslant \gamma \leqslant \alpha\}$. Then $Z \neq \emptyset$ since $\langle f, g, \alpha\rangle \in Z$. We can pick a minimal element $\langle f_1, g_1, \gamma_0\rangle$ of Z. Assume $\gamma_0>\beta$.

Then, since $g_1 \subseteq g$ and $\mathrm{dom}(g_1) = \gamma_0 > \beta$, $\sup(g_1^{-1}(\{0\}) \cap E \cap (\beta+1)) = \sup(g^{-1}(\{0\}) \cap E \cap (\beta+1))<\beta$. But $\langle f_1, g_1, \gamma_0\rangle$ is a minimal element, and so $\sup(g_1^{-1}(\{0\}) \cap E) = \sup(g_1^{-1}(\{0\}) \cap E \cap (\beta+1))<\beta$. We get $\langle f_1, g_1, \beta\rangle \in \delta(\langle f_1, g_1, \gamma_0\rangle)$ and $\langle f_1, g_1, \beta\rangle \prec \langle f_1, g_1, \gamma_0\rangle \prec \langle f, g, \alpha\rangle$. ■

The following properties of the tree result:

(7) The height of T is not greater than λ^+.

By (4) and IV we have:

(8) If $\langle f, g, \beta\rangle \in T^{(\alpha)}$ then $\mathrm{dom}(g) \geqslant \alpha$.

By (6) and IV we have

(9) If $\langle f, g, \beta\rangle \in T^{(\alpha)}$ and $\alpha \in E$ then $\mathrm{dom}(g) \in E$.

V. T is a continuous tree. Let α be a limit ordinal number, $\alpha>0$, $\langle f, g, \alpha\rangle \in T^{(\alpha)}$ and let $\{\langle f_\beta, g_\beta, \beta\rangle : \beta<\alpha\}$ be the set of all predecessors of $\langle f, g, \alpha\rangle$. It is sufficient to show that $\bigcup_{\beta<\alpha} f_\beta = f$ and $\bigcup_{\beta<\alpha} g_\beta = g$. We first note that $\bigcup_{\beta<\alpha} f_\beta \subseteq f$ and $\bigcup_{\beta<\alpha} g_\beta \subseteq g$. Hence, we must only prove that $\mathrm{dom}(\bigcup_{\beta<\alpha} f_\beta) = \mathrm{dom}(f)$ and $\mathrm{dom}(\bigcup_{\beta<\alpha} g_\beta) = \mathrm{dom}(g)$. We notice that $\mathrm{dom}(\bigcup_{\beta<\alpha} g_\beta) = \mathrm{dom}(g)$ implies $\mathrm{dom}(\bigcup_{\beta<\alpha} f_\beta) = \mathrm{dom}(f)$ since, for every $\beta<\alpha$, $\mathrm{dom}(f_\beta) = \mathrm{dom}(g_\beta)\times\mathrm{dom}(g_\beta)$ and $\mathrm{dom}(f) = \mathrm{dom}(g)\times\mathrm{dom}(g)$. We have: $\mathrm{dom}(\bigcup_{\beta<\alpha} g_\beta) \subseteq \mathrm{dom}(g)$ and $\mathrm{dom}(\bigcup_{\beta<\alpha} g_\beta)$ is an ordinal number. By (4) we have $\forall_{\beta<\alpha} \mathrm{dom}(g_\beta) \geqslant \beta$. Hence $\mathrm{dom}(\bigcup_{\beta<\alpha} g_\beta) \geqslant \alpha$. Suppose $\mathrm{dom}(g) > \mathrm{dom}(\bigcup_{\beta<\alpha} g_\beta) \geqslant \alpha$. From (5) we obtain: $\langle f, g, \alpha\rangle \in \delta(\langle f, g, \mathrm{dom}(g)\rangle)$ and then $\sup(g^{-1}(\{0\}) \cap E)<\alpha$ or even $g^{-1}(\{0\}) \cap E = \emptyset$. But α is a limit ordinal number, and so there is $\beta_0<\alpha$ such that also $\langle f, g, \beta_0\rangle$ belongs to $\delta(\langle f, g, \mathrm{dom}(g)\rangle)$. But now $g_{\beta_0} = g$ and then $g \subseteq \bigcup_{\beta<\alpha} g_\beta$. ■

VI. Conclusion. We know that T is a continuous tree. Points (2) and (3) of part II are exactly the same as points a) and b) in Theorem 1. (8) and (9) imply d). We must prove c). Let $X \subseteq \lambda^+ \times \lambda^+$, C is a closed unbounded subset of λ^+. We put:

$$G_1 = \{\langle \chi_{X|\alpha\times\alpha}, \chi_{C|\alpha}, \alpha\rangle : \alpha \in C \cap E\},$$
$$G_2 = G_1 \cup \{\sigma(L) : L \subseteq G_1,\ L \neq \emptyset,\ L \text{ is a chain, the order type of } L \text{ is a limit ordinal less than } \lambda^+\},$$
$$G = G_2 \cup \bigcup_{x\in G_2} \delta(x).$$

We infer that G is a chain and every element of G is of the form as in c). We must prove that G is a branch. We first note that the order type of G is λ^+ and by (7) the height of T is not greater than λ^+. So, it is sufficient to prove that $\forall_{\alpha<\lambda^+} \exists_{f,g} \langle f, g, \alpha\rangle \in G$. However, this follows from the argument of part IV. ■

References

[1] K. Kuratowski and A. Mostowski, *Set Theory*, Warszawa 1976.

[2] K. Devlin, *Aspects of Constructibility*, Lecture Notes in Math. 354, Springer 1973.

[3] — *Variations on* ◇, to appear.

[4] R. B. Jensen and K. Kunen, *Some Combinatorial Properties of L and V*, unpublished manuscript.

UNIVERSITY OF WARSAW

Received 15 March 1982