On universal infinite-dimensional spaces

by

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Abstract. In this paper we construct a universal compact metric space with given transfinite D-dimension. A similar result is proved for separable metric spaces. Since $D(X) = \dim X$ for finite-dimensional spaces, these results are extensions of well-known theorems for finite-dimensional spaces. Also we prove that every separable metric space X is contained in a compact metric space $R \in \mathrm{AR}$ such that $D(R) \leqslant D(X)+1$.

§ 1. Definitions and notation. In this paper we consider the transfinite D-dimension introduced in [1], Henderson. Some of our results we announced in [2], Luxemburg, without proof. All spaces in this paper are assumed to be metric and all mappings continuous. For every ordinal number β the equality $\beta = \alpha + n$ holds where α is a limit number or 0 and $n = 0, 1, 2, \ldots$ (1). Then we put $K(\beta) = n$, $J(\beta) = \alpha$.

1.1. Definition. We put $D(\emptyset) = -1$. If $X \neq \emptyset$, then $D(X)$ is the smallest ordinal number β such that there exists a collection of sets $\{A_\xi\colon 0 \leqslant \xi \leqslant \gamma\}$, where γ is an ordinal number, satisfying the following conditions:

(a) $X = \bigcup \{A_\xi\colon 0 \leqslant \xi \leqslant \gamma\}$.

(b) Every set A_ξ is closed and finite-dimensional.

(c) For any $\delta \leqslant \gamma$, the set $\bigcup \{A_\alpha\colon \delta \leqslant \alpha \leqslant \gamma\}$ is closed in X.

(d) $J(\beta) = \gamma$, $\dim A_\gamma \leqslant K(\beta)$.

(e) For any point $x \in X$, there exists the greatest number $\delta \leqslant \gamma$ such that $x \in A_\delta$.

If there is no such number β, we put $D(X) = \Delta$ where Δ is an abstract symbol such that $\Delta > \beta$ for any ordinal number β. If conditions (a)–(e) hold, then equality (a) is called a *β-D-representation of a space X.*

It is evident that

(1) if $X \subset Y$, then $D(X) \leqslant D(Y)$.

Moreover, $D(X) = \dim X = \operatorname{Ind} X$ for finite-dimensional spaces. For any space X of weight $\leqslant \tau$, we have $|D(X)| \leqslant \tau$ (see [1], Henderson, Theorem 10); consequently, for any separable space (in particular, a compact space) X, we have $D(X) < \omega_1$ or $D(X) = \Delta$.

(1) We always consider $\beta + 0 = 0 + \beta = \beta$.

1.2. THEOREM. *There is a universal element in the class of all compact spaces X such that $D(X)\leqslant\beta$ $(\beta<\omega_1)$.*

1.3. THEOREM. *There is a universal element in the class of all separable spaces X such that $D(X)\leqslant\beta$ $(\beta<\omega_1)$.*

We note that for every $\beta<\omega_1$ such that $\beta\geqslant\omega_0$, there exists a separable space X_β satisfying the following condition:

$$D(X_\beta)=\beta \text{ and for any compact space } Y\supset X \text{ we have } D(Y)>D(X),$$

(see [3], Luxemburg, Theorem 8.2). Consequently, universal elements in Theorems 1.2 and 1.3 are different for any $\beta\geqslant\omega_0$. These theorems are extensions of well-known results (see [4], Nöbeling) for finite-dimensional spaces. We will also prove the following theorem:

1.4. THEOREM. *For every separable space X there exists a compact space $Y\in$ AR and a homeomorphism $f\colon X\to Y$ such that $D(Y)\leqslant D(X)+1$.*

This theorem is an extension of a similar theorem for finite-dimensional spaces see [5], Bothe). To prove this theorem we need some preliminary constructions.

§ 2. The main constructions.

2.1. DEFINITION. Let X be a compactum and

$$(1)\qquad \varphi_X\colon X\times I\to CX$$

the identification mapping of the product $X\times I$, where I denotes the unit segment $[0,1]$, onto the cone CX. (We obtain the cone by identifying all points of the set $X\times\{0\}\subset X\times I$. The point $\varphi(X\times 0)=a\in CX$ is called the *apex of the cone* CX.)

2.2. CONSTRUCTION. Let $\sum X_i$ be a discrete union of spaces X_i, $i=1,2,\ldots$ Suppose in any X_i there are two closed sets A_i and B_i, $A_i\cap B_i=\emptyset$, and for any i there exists a homeomorphism $g_i\colon B_i\to A_{i+1}$. We identify every point $x\in B_i$ in a space $\sum X_i$ with a point $g_i(x)$ for all i. Then we get a factor mapping:

$$(2)\qquad \mu\colon \textstyle\sum X_i\to\Phi$$

onto the factor space Φ. We shall consider a set $F\subset\Phi$ to be closed if and only if the set $\mu^{-1}(\Phi)$ is closed. It is evident that for each i we have an embedding

$$(3)\qquad f_i\colon X_i\to\Phi$$

and

$$\bigcup_{i=1}^{\infty} f_i(X_i)=\Phi$$

where f_i is a restriction of μ to X_i. We put

$$(4)\qquad \Phi=\Phi(X_i,A_i,B_i,g_i);\qquad \overline{X}_i=f_i(X_i).\ \blacksquare$$

2.3. DEFINITION. Let $\mathscr{F}=\{F_i\colon i=1,2,\ldots\}$ be a countable family of sets in a space X and let the set $U\subset X$ be open. Then the family $\mathscr{F}$ is called *simple with respect to* U if

$$(5)\qquad U=\bigcup_{i=1}^{\infty}F_i,$$

$$(6)\qquad F_i\cap F_j=\emptyset \text{ for } |i-j|>1,$$

(7) the family $\mathscr{F}$ is locally finite on U and sets F_i are closed in X.

2.4. LEMMA. *Let a space Φ be defined by equality* (4); *then the family of sets $\{\overline{X}_i\}$ is simple with respect to Φ. Moreover, if the spaces X_i are compact, then Φ is separable and locally compact.*

The lemma is evident.

2.5. CONSTRUCTION. Let $\{X_i\colon i=1,2,\ldots\}$ be a family of disjoint compact spaces, and for each i, there exists a homeomorphism $h_i\colon X_i\to X_{i+1}$. We put:

$$(8)\qquad B(X_i)=X_i\times I\times CX_i$$

where CX_i is the cone with the apex a_i. Let

$$(9)\qquad A_i=X_i\times\{0\}\times\{a_i\}\subset B(X_i),\qquad B_i=X_i\times\{1\}\times\{a_i\}\subset B(X_i).$$

Since A_i and B_i are homeomorphic to X_i, there exist homeomorphisms $g_i\colon B_i\to A_{i+1}$, $i=1,2,\ldots$ We put

$$\Phi=\Phi\big(B(X_i),A_i,B_i,g_i\big).$$

Since all spaces X_i are compact, all spaces $B(X_i)$ are also compact and, by virtue of Lemma 2.4, Φ is a locally compact, separable space. We put

$$\mathscr{K}=\mathscr{K}(X_i,h_i)=\{\omega\}\cup\Phi,$$

where $\mathscr{K}$ is a compactification of Φ with an extra point ω. Consequently,

$$(10)\qquad \mathscr{K}=\{\omega\}\cup\bigcup_{i=1}^{\infty}B(X_i)$$

and

(11) the set $B(X_{i+1})\cap B(X_i)$ is homeomorphic to X_i. $\blacksquare$

2.6. LEMMA. *The family of sets $\{B(X_i)\}$ is simple with respect to $\mathscr{K}\setminus\omega$.*

This theorem follows from Lemma 2.4.

2.7. LEMMA. *If all compacta $X_i\in$ AR, then $\mathscr{K}=\mathscr{K}(X_i,h_i)\in$ AR.*

Proof. Let $f\colon F\to\mathscr{K}$ be a mapping of a closed subset F of a space X into $\mathscr{K}$. We shall extend f to X. We can easily find a function $\Psi\colon\mathscr{K}\to[0,1]$ such that

$$(12)\qquad \Psi^{-1}(0)=\omega,\quad \Psi^{-1}\big(1/(i+1)\big)=S_i,\quad \Psi^{-1}[1/(i+1),1/i]=B(X_i),$$

where $S_i=B(X_i)\cap B(X_{i+1})$. Then we have a mapping $\Psi\circ f\colon F\to[0,1]$. Let $\mu\colon X\to[0,1]$ be an extension of $\Psi\circ f$. We put

$$(13)\qquad C_i=\mu^{-1}\left(\frac{1}{i+1}\right),\quad B_i=\mu^{-1}\left(\left[\frac{1}{i+1},\frac{1}{i}\right]\right),\quad W=\mu^{-1}(0),$$
$$C_0=\emptyset\qquad (i=1,2,\ldots).$$

Then

$$X = W \cup \bigcup_{i=1}^{\infty} B_i, \quad C_i = B_i \cap B_{i+1}.$$

By virtue of (12) and (13), $f(F \cap C_i) \subset S_i$. Since $X_i \in \mathrm{AR}$, by virtue of (11), the set $S_i = B(X_i) \cap B(X_{i+1})$ is also an AR-space. Consequently, for any i, there exists an extension $g_i\colon C_i \to S_i$ of the restriction of f to $F \cap C_i$. We put

$$g(x) = f(x) \text{ for } x \in F, \quad g(x) = g_i(x) \text{ for } x \in C_i, \quad g(W) = \omega.$$

Then we have a continuous mapping $g\colon R = F \cup W \cup \bigcup_{i=1}^{\infty} C_i \to \mathscr{K}$, and clearly

$$g(R \cap B_i) \subset B(X_i) \subset \mathscr{K}, \quad g(B_i \cap B_{i+1} \cap R) \subset S_i.$$

Since $X_i \in \mathrm{AR}$, the cone $CX_i \in \mathrm{AR}$ and consequently $B(X_i) = X_i \times I \times CX_i \in \mathrm{AR}$. Therefore, for each i, there is an extension $k_i\colon B_i \to B(X_i)$ of the mapping $r_i\colon B_i \cap R \to B(X_i) \subset \mathscr{K}$, where r_i is a restriction of g to $R \cap B_i$. We put

$$k(x) = k_i(x) \text{ for } x \in B_i, \quad k(x) = \omega \text{ for } x \in W.$$

Then clearly $k\colon X \to \mathscr{K}$ is a continuous extension of f. ■

2.8. Lemma. *Let a family of sets $\mathscr{F} = \{F_i\}$ be simple with respect to $U \subset X$. Suppose the family of spaces $\{X_i\}$ $(i = 1, 2, \ldots)$ satisfies the conditions of construction 2.2, and, for each i, there exists a homeomorphism $\varphi_i\colon F_i \to X_i$ such that*

$$g_i \circ \varphi_i(x) = \varphi_{i+1}(x) \quad \textit{for} \quad x \in F_i \cap F_{i+1}.$$

Then the mapping $\varphi\colon U \to \Phi = \Phi(X_i, A_i, B_i, g_i)$, defined by the equality

$$\varphi(x) = f_i \circ \varphi_i(x),$$

where f_i is a homeomorphism (3), is a homeomorphism and

$$\varphi(F_i) \subset \overline{X}_i = f_i(X_i). \tag{14}$$

The lemma follows directly from Construction 2.2. ■

2.9. Lemma. *Let Y be a compactum, CX the cone over $X \in \mathrm{AR}$ with the apex a and*

$$B(X) = X \times I \times CX, \quad A_i = X \times \{i\} \times \{a\} \subset B(X), \quad i = 0, 1 \in [0, 1].$$

If there exists a homeomorphism

$$f\colon Y \to X,$$

then for any disjoint closed subsets $F, G \subset Y$ and any homeomorphisms

$$f_0\colon F \to A_0, \quad f_1\colon G \to A_1,$$

there exists a homeomorphism $g\colon Y \to B(X)$, which extends f_0 and f_1.

Proof. Let $\pi_1\colon B(X) \to X \times I$ be a projection. Since $X \in \mathrm{AR}$ and $I \in \mathrm{AR}$, we have $X \times I \in \mathrm{AR}$. Therefore, there exists a mapping $k\colon Y \to X \times I$, which extends $\pi_1 \circ f_0$ and $\pi_1 \circ f_1$ to Y. Then, clearly,

(15) k is injective on $F \cup G$.

Let $\mu\colon Y \to [0, 1]$ be a continuous function such that $\mu^{-1}(0) = F \cup G$. Let a mapping $h\colon Y \to X \times I$ be defined by the equality

$$h(y) = \{f(y), \mu(y)\}.$$

We put $l(y) = \varphi_X \circ h$, where φ_X is a mapping (1). Then, clearly,

(16) l is injective on $Y \setminus (F \cup G)$.

We put $g(y) = \big(k(y), l(y)\big)$, then by virtue of (15) and (16) $g\colon Y \to B(X) = X \times \times I \times CX$ is injective on Y. Since Y is compact, g is a homeomorphism. ■

2.10. Lemma. *Let X be a compactum and the equality*

$$X = \bigcup \{A_\gamma\colon \gamma \leqslant J(\beta)\} \tag{17}$$

be a β-D-representation such that $A_{J(\beta)}$ consists of exactly one point. Let there exists an increasing sequence $\{\gamma_i\}$ of ordinal numbers such that

$$\sup \gamma_i = \alpha = J(\beta) \geqslant \omega_0$$

and a sequence of absolute retracts $\{X_i\}$ satisfying the following conditions:

(18) *Every compactum with D-dimension $\leqslant \gamma_i$ has an embedding in X_i.*

(19) *There exists a homeomorphism $h_i\colon X_i \to X_{i+1}$ for $i = 1, 2, \ldots$*

Then there exists a homeomorphism

$$h\colon X \to \mathscr{K}(X_i, h_i) \tag{20}$$

such that

$$h^{-1}(\omega) = A_{J(\beta)} \tag{21}$$

where ω is the compactification point in $\mathscr{K} = \mathscr{K}(X_i, h_i)$. (See Construction 2.5.)

Proof. By virtue of Lemma 8.2 in [3], Luxemburg, there exists a family of sets $\mathscr{F} = \{F_i\}$, simple with respect to $X \setminus A_{J(\beta)}$, such that

$$D(F_i) \leqslant \gamma_i. \tag{22}$$

By Definition 2.3, the sets F_i are closed in X and are consequently compact. Let the sets $B(X_i), A_i, B_i$ be defined by conditions (8) and (9); $g_i\colon B_i \to A_{i+1}$ are homeomorphisms from Construction 2.5. Since A_i and B_i are homeomorphic to X_i by virtue of (18) and (22), there exist homeomorphisms

$$f_i\colon F_i \to X_i, \quad k_i\colon F_i \cap F_{i+1} \to B_i, \quad r_{i+1}\colon F_i \cap F_{i+1} \to A_{i+1}$$

where

$$r_{i+1} = g_i \circ k_i. \tag{23}$$

By virtue of Lemma 2.9 there exists a homeomorphism $\varphi_i\colon F_i \to B(X_i)$ which is an extension of k_i and r_i. Consequently, from (23), it follows that

$$g_i \circ \varphi_i(x) = \varphi_{i+1}(x) \quad \text{for} \quad x \in F_i \cap F_{i+1}.$$

By Lemma 2.8 there exists a homeomorphism $\varphi\colon X \setminus A_{J(\beta)} = \Phi = \mathscr{K} \setminus \omega$. We put $h(x) = \varphi(x)$ for $x \notin A_{J(\beta)}$ and $h(A_{J(\beta)}) = \omega$. Then, clearly, h is a desired homeomorphism. ■

§ 3. Natural sums and β-D-representations of compacta. In Sections 3 and 4 the symbol β is intended to denote infinite ordinal numbers. In what follows we need some information about the natural sum of ordinal numbers; see [6], Toulmin, and [7], Hausdorff (²).

Every ordinal number β has a unique representation:

(1) $$\beta = \alpha_1 + \ldots + \alpha_{k+1}$$

where $\alpha_{k+1} = 0, 1, 2, \ldots$ and α_i $(i \leqslant k)$ are indecomposable transfinite numbers such that $\alpha_{i+1} \leqslant \alpha_i$. (A transfinite number ξ is called indecomposable if ξ is not the sum of a finite number of ordinal numbers less than ξ.) The representation (1) is called canonic. It is evident that $K(\beta) = \alpha_{k+1}$.

3.1. Definition. Let (1) be a canonic representation of β and

(2) $$\gamma = \delta_1 + \ldots + \delta_{p+1}$$

be a canonic representation of γ. Let $\xi_1, \ldots, \xi_{p+k+2}$ be elements of the set $\alpha_1, \ldots, \alpha_{k+1}, \beta_1, \ldots, \beta_{p+1}$ with decreasing order $(\xi_i \geqslant \xi_{i+1})$. Then the natural sum $\gamma \oplus \beta$ is defined by the equality $\gamma \oplus \beta = \xi_1 + \ldots + \xi_{p+k+2}$. ■

If $n = 0, 1, 2, \ldots$, then by definition $\gamma \oplus n = \gamma + n = n \oplus \gamma$. It is evident that $\beta \oplus \gamma = \gamma \oplus \beta$.

In [1], Henderson, Theorem (8), it was proved that for any spaces X and Y

(3) $$D(X \times Y) \leqslant D(X) \oplus D(Y).$$

3.2. Lemma. *Let* (1) *and* (2) *be representations of β, γ and $\alpha_{k+1} = 0$. Let $\gamma < \beta$ and l be the first integer such that $\delta_l \neq \alpha_l$. Then $\delta_l < \alpha_l$.*

Proof. If $\delta_l > \alpha_l$, then, since the sequence $\{\alpha_i\}$ is decreasing, $\delta_l > \alpha_j$ for $j \geqslant l$. Besides that, clearly $l < p+1$. Therefore, δ_l is an indecomposable transfinite number. Consequently, $\delta_l > \alpha_l + \ldots + \alpha_{k+1}$ and $\gamma \geqslant \delta_1 + \ldots + \delta_l > \alpha_1 + \ldots + (\alpha_l + \ldots + \alpha_{k+1}) = \beta$. This contradiction proves the lemma. ■

3.3. Lemma. *Let CX be the cone over a compact space X. Then*

$$D(CX) \leqslant D(X) + 1.$$

Proof. Let a be the apex of CX; clearly, $CX \setminus \{a\}$ can be embedded into $X \times I$. Consequently (see [1], Henderson, Theorem 8),

$$D(CX \setminus \{a\}) \leqslant D(X \times I) \leqslant D(X) + 1.$$

(²) In [6] this sum is called "lower".

Moreover, it is evident that the adding of a point to a nonempty set whose D-dimension is defined does not change its D-dimension. ■

3.4. Lemma. *If $D(X) < \alpha$ and α is indecomposable transfinite, then $D(B(X)) < \alpha$ where $B(X) = CX \times X \times I$.*

The lemma follows from Lemma 3.3 and (3). ■

3.5. Lemma. *Let L be a compactum such that*

$$L = \{l\} \cup \Phi, \quad l \notin \Phi,$$

for some point $l \in L$, and $\{H_i\}$ be a family, simple with respect to Φ, such that $D(H_i) \leqslant \gamma_i < \alpha$. Then there exists an α-D-representation of L

$$L = \bigcup \{A_\gamma\colon \gamma \leqslant J(\alpha)\}$$

such that $A_{J(\alpha)} = \{l\}$.

Proof. For each i, since $D(H_i) \leqslant \gamma_i$, by Lemma 1 in [1], Henderson, there exists a γ_i-D-representation of the space H_i

$$H_i = \bigcup \{A^i_\mu\colon \mu \leqslant J(\gamma_i)\}$$

such that

$$\dim A^i_\mu \leqslant K(\mu).$$

We put $A^i_\mu = \emptyset$ for $\mu > J(\gamma_i)$ and

$$A_\delta = \{l\} \cup \bigcup \{A^i_\delta\colon i = 1, 2, \ldots\}.$$

From the sum theorem it follows that

$$\dim A_\delta \leqslant K(\delta).$$

Moreover, $A_{J(\alpha)} = \{l\}$. It is easy to see that the equality $L = \bigcup \{A_\delta\colon \delta \leqslant J(\alpha)\}$ is an α-D-representation of L. ■

3.6. Lemma. *Let α be an indecomposable ordinal number and, for $i = 1, 2, \ldots$, X_i a compact space such that $D(X_i) < \alpha$. There exists an α-D-representation of $\mathscr{K}_i = \mathscr{K}(X_i, h_i)$*

$$\mathscr{K} = \{A_\gamma\colon \gamma \leqslant J(\alpha)\}$$

such that $A_{J(\alpha)} = \{\omega\}$. Consequently, $D(\mathscr{K}) \leqslant \alpha$.

Our lemma follows from Lemmas 2.6, 3.4, and 3.5. ■

3.7. Lemma. *If X is compact, $D(X) = \beta$, and*

(4) $$X = \bigcup \{A_\gamma\colon \gamma \leqslant J(\beta)\}$$

is a β-D-representation of X, then we have:

$$C_\gamma = \bigcup \{A_\delta\colon \gamma \leqslant \delta \leqslant J(\beta)\} \neq \emptyset$$

and C_γ is compact for each $\gamma \leqslant J(\beta)$. In particular, $C_{J(\beta)} = A_{J(\beta)} \neq \emptyset$.

Proof. First we will show that $C_\gamma \neq \emptyset$ for $\gamma < J(\beta)$. Indeed, if $C_\gamma = \emptyset$, then

$$X = U_\gamma = X \setminus C_\gamma .$$

By Lemma 8.3 in [8], Luxemburg, $D(U_\gamma) < J(\beta) \leqslant \beta$. This contradicts the condition of the Lemma. Therefore, $C_\gamma \neq \emptyset$ for $\gamma < J(\beta)$. From condition (c) of Definition 1.1 the set C_γ is closed in X and consequently is compact. Since $C_\gamma \subset C_{\gamma'}$ for $\gamma > \gamma'$, we have

$$\bigcap \{C_\gamma \colon \gamma < J(\beta)\} \neq \emptyset .$$

But from condition (e) of Definition 1.1 it follows that $\bigcap\{C_\gamma\colon \gamma<J(\beta)\} = A_{J(\beta)} \neq \emptyset$. ■

3.8. Lemma. *Let* (4) *be a* β-D-*representation of* X *and*

$$Y = \bigcup \{B_\gamma \colon \gamma \leqslant J(\delta)\}$$

be a δ-D-*representation of* Y. *Then the equality*

(5) $$X \times Y = \bigcup \{D_\mu \colon D_\mu = \bigcup \{A_{\gamma_1} \times B_{\gamma_2} \colon \gamma_1 \oplus \gamma_2 = \mu\},\ \mu \leqslant J(\delta) \oplus J(\beta)\}$$

is a $(\beta \oplus \delta)$-D-*representation of* $X \times Y$ *and* $D_{J(\beta\oplus\delta)} = A_{J(\beta)} \times B_{J(\delta)}$.

Proof. The equality (5) is a $(\beta\oplus\delta)$-D-representation of $X \times Y$ by virtue of [1], Henderson. From the definition of the natural sum it follows that $J(\beta\oplus\delta) = J(\beta) \oplus J(\delta)$. If $\gamma \leqslant J(\beta)$, $\gamma' < J(\delta)$ or $\gamma < J(\beta)$, $\gamma' \leqslant J(\delta)$, we have $\gamma \oplus \gamma' < J(\beta\oplus\delta)$. Consequently, $D_{J(\beta\oplus\delta)} = A_{J(\beta)} \times B_{J(\delta)}$. ■

3.9. Corollary. *Let* (4) *be a* β-D-*representation of* X *and let* K *be an arbitrary space with* $\dim K \leqslant m$; *then the equality*

$$X \times K = \bigcup \{B_\gamma = A_\gamma \times K \colon \gamma \leqslant J(\beta) = J(\beta+m)\}$$

is a $(\beta+m)$-D-*representation of* $X \times K$. ■

§ 4. On compacta $Z(\alpha_i)$.

4.1. Definition. Let X be a compactum and let

(1) $$\beta = \alpha_1 + \ldots + \alpha_{k+1}, \quad \alpha_{k+1} = n = 0, 1, 2, \ldots, \quad \alpha_0 = 0$$

be a canonic representation of ordinal number $\beta \geqslant \omega_0$ and let

(2) $$X = \bigcup \{A_\gamma \colon \gamma \leqslant J(\beta)\}$$

be a β-D-representation of X. We put

(3) $$Y(\alpha_i) = \bigcup \{A_\gamma \colon \alpha_1 + \ldots + \alpha_i \leqslant \gamma \leqslant J(\beta)\} \subset X, \quad 1 \leqslant i \leqslant k, \quad Y(\alpha_0) = X .$$

We note that by Lemma 3.7, $Y(\alpha_i)$ is compact and $Y(\alpha_i) \neq \emptyset$. Moreover,

(4) $$Y(\alpha_k) = A_{J(\beta)} .$$

Let

(5) $$\varrho_i \colon Y(\alpha_{i-1}) \to Z(\alpha_i), \quad i = 1, \ldots, k$$

be a mapping on a compactum $Z(\alpha_i)$ obtained by identification of all points of the compactum $Y(\alpha_i) \subset Y(\alpha_{i-1})$. Let $\{b_i\} = \varrho_i(Y(\alpha_i))$. We also put

(6) $$Z(\alpha_{k+1}) = A_{J(\beta)} = Y(\alpha_k), \quad \varrho_{k+1} = \mathrm{id} \colon Y(\alpha_k) \to Z(\alpha_{k+1}) .$$

Thus, the compacta $Y(\alpha_i)$ are completely defined for $i = 0, \ldots, k$,

$$X = Y(\alpha_0) \supset \ldots \supset Y(\alpha_k) = A_{J(\beta)}$$

and the compacta $Z(\alpha_i)$ are defined as quotient spaces for $i = 1, \ldots, k+1$,

$$Z(\alpha_i) = Y(\alpha_{i-1})/Y(\alpha_i) \quad \text{for} \quad i \leqslant k$$

and

$$Z(\alpha_{k+1}) = Y(\alpha_k) = A_{J(\beta)} . \ ■$$

We introduce one more notation. Let $\gamma \leqslant \alpha_1$, then we put $r_1(\gamma) = \gamma$. If $\alpha_1 + \ldots + \alpha_i \leqslant \gamma \leqslant \alpha_1 + \ldots + \alpha_{i+1}$, then the ordinal number $r_{i+1}(\gamma)$ is defined by the equality $\alpha_1 + \ldots + \alpha_i + r_{i+1}(\gamma) = \gamma$ for all $\gamma \leqslant \beta$.

4.2. Lemma. *We put*

(7) $$B_{r_i(\gamma)} = \varrho_i(A_\gamma), \quad \alpha_1 + \ldots + \alpha_{i-1} \leqslant \gamma \leqslant \alpha_1 + \ldots + \alpha_i .$$

Then the equality

(8) $$Z(\alpha_i) = \bigcup \{B_\mu \colon \mu \leqslant \alpha_i\}$$

is an α_i-D-*representation of the compactum* $Z(\alpha_i)$ *and*

(9) $$B_{\alpha_i} = B_{J(\alpha_i)} = \{b_i\} .$$

Proof. The equality in (8) follows from (3), (7), and the construction of ϱ_i. Since $Y(\alpha_{i-1})$ is compact, the mapping ϱ_i is closed. The mapping ϱ_i clearly does not raise the dimension of closed finite-dimensional sets. Consequently, from condition (7), it follows that the sets β_μ are closed and finite-dimensional. Thus properties (a) and (b) of Definition 1.1 are proved. Property (c) follows from the closedness of ϱ_i. Condition (9) is evident and (e) follows from (9). Condition (d) is true because ϱ_i is a homeomorphism on $\varrho_i^{-1}(Z(\alpha_i) \setminus b_i)$. ■

In the following two lemmas we adopt the notation of Definition 4.1.

4.3. Lemma. *For any two distinct points* x *and* y *in the compactum* X, *there exists a number* $i = 0, \ldots, k$ *such that* $x \in Y(\alpha_i)$, $y \in Y(\alpha_i)$ *and* $\varrho_{i+1}(x) \neq \varrho_{i+1}(y)$.

Proof. For any point $z \in X$, let $\mu(z)$ be the greatest number i such that $z \in Y(\alpha_i)$. Let $\mu(x) = \mu(y) = p$. Then: either $p < k$ or $p = k$.

In the second case $\varrho_{k+1}(x) = x \neq y = \varrho_{k+1}(y)$. Since ϱ_{p+1} is clearly injective on $Y(\alpha_p) \setminus Y(\alpha_{p+1})$, we also have $\varrho_{p+1}(x) \neq \varrho_{p+1}(y)$ in the first case. If $\mu(x) \neq \mu(y)$, for example, $\mu(x) > \mu(y) = p$, then $x \in Y(\alpha_{p+1})$, $y \in Y(\alpha_p) \setminus Y(\alpha_{p+1})$. Therefore, $\varrho_{p+1}(x) = b_{p+1} \neq \varrho_{p+1}(y)$. ■

4.4. Lemma. *Let* X *be a compactum and, for* $i = 1, 2, \ldots, k+1$, *there exists a homeomorphism*

$$h_i \colon Z(\alpha_i) \to P_i \in \mathrm{AR}$$

in a space $P_i \in \mathrm{AR}$. Then there exists a homeomorphism

$$h\colon X \to \prod_{i=1}^{k+1} P_i$$

of the space X in the product of the spaces P_i. Moreover,

(10) *If the set $A_{J(\beta)} = Z(\alpha_{k+1})$ in a β-D-representation (2) consists of exactly one point b_{k+1}, then $h(A_{J(\beta)})$ is a point whose i-th-coordinate in the product $\prod_{i=1}^{k+1} P_i$ is a point $h_i(b_i)$ $(b_i \in Z(\alpha_i))$.*

Proof. Since $P_i \in \mathrm{AR}$ there exists an extension

$$g_i\colon X \to P_i$$

of the mapping $h_i \circ \varrho_i\colon Y(\alpha_{i-1}) \to P_i$. Let

$$h\colon X \to \prod_{i=1}^{h+1} P_i$$

be a mapping whose i-coordinate is g_i. Let x, y be a pair of distinct points in X. Then, by Lemma 4.3, $\varrho_{i+1}(x) \neq \varrho_{i+1}(y)$ for some i. Since h_{i+1} is a homeomorphism,

$$g_{i+1}(x) = h_{i+1} \circ \varrho_{i+1}(x) \neq h_{i+1} \circ \varrho_{i+1}(y) = g_{i+1}(y).$$

Consequently, $h(x) \neq h(y)$. Therefore, h is injective. Since X is compact, h is a homeomorphism. Condition (10) is evident. ■

§ 5. On compacta P_β.

5.1. Construction. For each ordinal $\beta < \omega_1$, we will define a compactum P_β and a fixed point $q_\beta \in P_\beta$. For each pair of compacta P_γ and P_β $(\gamma \leqslant \beta)$, we will define a homeomorphism

(1) $$h_{\gamma\beta}\colon P_\gamma \to P_\beta.$$

We put

$$P_n = I^n \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

where I^n is an n-dimensional cube. Points q_n we select in an arbitrary way. Then, clearly, for $\gamma \leqslant \beta < \omega_0$, there exist homeomorphisms (1). Suppose, for $\beta < \beta_0$, compacta P_β and homeomorphisms (1) have been constructed. If β_0 is indecomposable transfinite, then there exists a sequence of ordinal numbers $\{\gamma(\beta_0, i)\}$ such that

(2) $$\sup\{\gamma(\beta_0, i)\colon i = 1, 2, \ldots\} = \beta_0 \quad \text{and} \quad \gamma(\beta_0, i) < \gamma(\beta_0, i+1)$$

and thus we put

(3) $$P_\beta = \mathscr{K}(P_{\gamma(\beta,i)}, h_{\gamma(\beta,i)\gamma(\beta,i+1)}).$$

We define q_β as a compactification point in $\mathscr{K} = P_\beta$ (see Construction 2.5). Let $\gamma < \beta$. Then, by virtue of (2), $\gamma \leqslant \gamma(\beta, i)$ for some i. By inductive assumption there

exists a homeomorphism $h_{\gamma\gamma(\beta,i)}\colon P_\gamma \to P_{\gamma(\beta,i)}$. Consequently, there exists a homeomorphism (1) because $P_{\gamma(\beta,i)}$ is homeomorphic to a subset of P_β. (See (10) and (11) in § 2.)

If α is a decomposable transfinite number and the equality

(4) $$\beta = \alpha_1 + \ldots + \alpha_{k+1}$$

is a canonic representation of β, then we put

(5) $$P_\beta = \prod_{i=1}^{k+1} P_{\alpha_i}.$$

Let $\pi_i\colon P_\beta \to P_{\alpha_i}$ be a projection on a factor. Then the point q_β is defined by the equalities

$$\pi_i(q_\beta) = q_{\alpha_i}, \quad i = 1, \ldots, k+1.$$

Let us define the homeomorphism $h_{\gamma\beta}$. Let $\gamma < \beta$ and let the equality

$$\gamma = \delta_1 + \ldots + \delta_{p+1}, \quad p = 0, 1, 2, \ldots$$

be a canonic representation of γ. Let $l \leqslant p+1$ be the first number such that $\delta_l \neq \alpha_l$. Then by Lemma 3.2 $\delta_l < \alpha_l$. If $l = k+1$, then clearly

$$P_\beta = P_\gamma \times I^n$$

where $n = \alpha_{k+1} - \delta_{k+1}$, and the homeomorphism $h_{\gamma\beta}$ exists. Let $l \leqslant k$; then α_l is an indecomposable number. Since $\delta_s \leqslant \delta_l < \alpha_l$ for $s \geqslant l$, we have

$$\xi = \delta_l + \ldots + \delta_{p+1} < \alpha_l.$$

By inductive assumption there exists a homeomorphism

(6) $$h_{\xi\alpha_l}\colon P_\xi \to P_{\alpha_l}.$$

Moreover, by our construction

(7) $$P_\gamma = \prod_{i=1}^{p+1} P_{\gamma_i} = \prod_{i=1}^{l-1} P_{\alpha_i} \times P_\xi.$$

By virtue of (5), (6), and (7) there exists a homeomorphism (1). ■

5.2. Lemma. $D(P_\beta) \leqslant \beta$.

Proof. We will prove this lemma by induction on β. If $\beta < \omega_0$, then clearly $D(P_\beta) = \beta$. Let β be a decomposable transfinite number and let (4) be its canonic representation. Consequently, by virtue of (3), § 3, and (5), along with Definition 3.1 and inductive assumption,

$$D(P_\beta) \leqslant D(P_{\alpha_1}) \oplus \ldots \oplus D(P_{\alpha_{k+1}}) \leqslant \alpha_1 \oplus \ldots \oplus \alpha_{k+1} = \beta.$$

If β is indecomposable transfinite, then our inequality follows from Lemma 3.6 and the inductive assumption. ■

5.3. LEMMA. *If α is a limit number, then there exists an α-D-representation of the compactum P_α*

(8) $$P_\alpha = \{A_\gamma\colon \gamma \leqslant \alpha = J(\alpha)\}$$

such that the set $A_{J(\alpha)}$ consists of exactly one point q_α.

Proof. We will prove this lemma by induction. If α is an indecomposable transfinite number, our lemma follows from Lemmas 3.6 and 5.2 and Construction 5.1. Consequently, our lemma is true for the first limit number $\alpha = \omega_0$. Let $\alpha = \alpha_1 + \ldots + \alpha_k$ be a canonic representation of the decomposable limit number α. Then $\alpha = \gamma + \alpha_k$ for some limit number γ; and from Construction 5.1 it follows that

$$P_\alpha = P_\gamma \times P_{\alpha_k}.$$

Therefore, by the inductive assumption there exists a γ-D-representation of P_γ and a α_k-D-representation of P_{α_k}, $P_\gamma = \{B_\delta\colon \delta \leqslant \gamma = J(\gamma)\}$, $P_{\alpha_k} = \{C_\delta\colon \delta \leqslant \alpha_k = J(\alpha_k)\}$ such that $B_\gamma = \{q_\gamma\}$, $C_{\alpha_k} = \{q_{\alpha_k}\}$. Since, by Construction 5.1, $\{q_\alpha\} = \{q_\gamma \times q_{\alpha_k}\} = B_\gamma \times C_{\alpha_k}$, our lemma follows from Lemma 3.8. ■

5.4. LEMMA. *For each $\beta < \omega_1$, $P_\beta \in \mathrm{AR}$.*

We will prove our lemma by induction. If $\beta < \omega_0$ then P_β is a Euclidean cube and our lemma is true. If β is an indecomposable transfinite number then our lemma follows from Lemma 2.7 and inductive assumption. Let β be a decomposable transfinite number and (4) its canonic representation. Then (5) holds and $P_\beta \in \mathrm{AR}$ as a product of AR-spaces. ■

§ 6. The main theorems on embeddings of compacta.

6.1. THEOREM. (i) *Let X be a compactum and $D(X) = \beta = \alpha + n$, $\alpha = J(\beta)$, $K(\beta) = n$. There exists an embedding*:

$$h\colon X \to P_{\alpha+2n+1} = P_{\beta+n+1}.$$

(ii) *Let the equality*

(1) $$X = \bigcup \{A_\gamma\colon \gamma \leqslant \alpha = J(\beta)\}$$

be a β-D-representation of X. We suppose that the set $A_{J(\beta)}$ consists of exactly one point. There exists an embedding

(2) $$h\colon X \to P_\alpha$$

such that

(3) $$h^{-1}(q_\alpha) = A_{J(\beta)}.$$

Proof. We note first that since P_{2n+1} is $(2n+1)$-dimensional cube and, for $\alpha = 0$, $D(X) = \dim X = n$, our theorem extends the well-known result of Nöbeling [4], on embedding of n-dimensional sets in Euclidean space.

We now prove this theorem by induction on β. If $\beta < \omega_0$, then as we have just said, the theorem is true. Let β be a decomposable ordinal number $\geqslant \omega_0$ and let the equality

(4) $$\beta = \alpha_1 + \ldots + \alpha_{k+1}, \quad \alpha_{k+1} = n = 0, 1, 2, \ldots$$

be its canonic representation. Then, by inductive assumption and by Lemma 4.2, there exist embeddings

$$h_i\colon Z(\alpha_i) \to P_{\alpha_i}, \quad i \leqslant k, \quad h_{k+1}\colon Z(\alpha_{k+1}) = A_{J(\beta)} \to P_{2n+1}$$

such that

(5) $$h_i^{-1}(q_{\alpha_i}) = \{b_i\} \quad (i \leqslant k),$$

where $\{b_i\} = B_{\alpha_i}$ (see Lemma 4.2, conditions (8) and (9) of § 4). Since, by Lemma 5.4, $P_\gamma \in \mathrm{AR}$, by Lemma 4.4 there exists a homeomorphism

$$h\colon X \to \prod_{i=1}^{k} P_{\alpha_i} \times P_{2n+1} = P_{\alpha+2n+1}.$$

If the set $A_{J(\beta)} = Z(\alpha_{k+1})$ consists of exactly one point, then we consider a mapping $h_{k+1}\colon Z(\alpha_{k+1}) \to P_0$. By virtue of Lemma 4.4 there exists a homeomorphism (2) such that $h(A_{J(\beta)})$ is a point whose i-coordinate is $h_i(b_i)$. By property (5) and by Definition 5.1 this point is q_α. Therefore, condition (3) holds. Let β be an indecomposable ordinal number. Then there exists a canonic representation of β

$$\beta = \alpha_1 + \alpha_2, \quad \alpha_2 = 0.$$

By inductive assumption, Construction 5.1, Lemma 2.10, and Definition 4.1, there exists a homeomorphism

$$g\colon Z(\alpha_1) \to \mathscr{K}(P_{\gamma(\beta,i)}, h_{\gamma(\beta,i)}) = P_\beta = P_{\alpha_1}$$

such that $g^{-1}(q_\alpha) = b_1$. If $A_{J(\beta)}$ consists of exactly one point, then the mapping $\varrho_1\colon X \to Z(\alpha_1)$ is a homeomorphism and $\varrho_1(A_{J(\beta)}) = \{b_1\}$. Therefore $h = g \circ \varrho_1$ is a homeomorphism and condition (3) holds. Clearly, the case when $A_{J(\beta)}$ is 0-dimensional but not of cardinality 1 can be settled as above.

6.2. THEOREM. *If $f\colon X \to Y$ is a closed mapping of a space X onto a space Y, then*:

(a) *If* $\sup\{\dim f^{-1}(y)\colon y \in Y\} \leqslant k$, $k = 0, 1, 2, \ldots$, *then*

(6) $$D(X) \leqslant D(Y) + k.$$

(b) *If $f^{-1}(y)$ consists of no more than $(k+1)$ points for each $y \in Y$, then*

(7) $$D(Y) \leqslant D(X) + k.$$

This theorem extends Hurewicz's formulas for finite-dimensional spaces.

Proof. (a) Let $D(Y) = \beta$ and

(8) $$Y = \bigcup \{B_\gamma\colon \gamma \leqslant J(\beta)\}$$

be a β-D-representation of Y. We put

(9) $$A_\gamma = f^{-1}(B_\gamma).$$

Then

(10) $$X = \bigcup \{A_\gamma\colon \gamma \leqslant J(\beta) = J(\beta+k)\}.$$

We will prove that (10) is a $(\beta+k)$-D-representation. By Hurewicz's formula for finite-dimensional spaces (see [9]),

$$\dim A_\gamma \leqslant \dim B_\gamma + k .$$

In particular, for $J(\beta) = J(\beta+k)$

$$\dim A_{J(\beta+k)} \leqslant \dim B_{J(\beta)+k} \leqslant K(\beta)+k = K(\beta+k) .$$

Therefore, conditions (a) and (b) of Definition 1.1 hold. Conditions (c) and (e) follow from (9). Hence, (10) is a $(\beta+k)$-D-representation of X and inequality (6) holds.

(b) Let $D(X) = \beta$ and let the equality

$$X = \{B_\gamma\colon \gamma \leqslant J(\beta)\}$$

be a β-D-representation of X. We put

(11) $$A_\gamma = f(B_\gamma) .$$

Then

(12) $$f(X) = Y = \bigcup \{A_\gamma\colon \gamma \leqslant J(\beta) = J(\beta+k)\} .$$

By virtue of Hurewicz's formula for finite-dimensional spaces (see [10])

(13) $$\dim A_\gamma \leqslant \dim B_\gamma + k , \quad \dim B_{J(\beta+k)} \leqslant \dim A_{J(\beta)} + k \leqslant K(\beta)+k = K(\beta+k) .$$

Moreover, the sets A_γ are closed because f is a closed mapping and the sets B_γ are closed. Therefore, conditions (a), (b), (d) of Definition 1.1 hold. Condition (c) holds because f is a closed mapping. Condition (e) holds because the set $f^{-1}(y)$ is finite for every $y \in Y$. Hence equality (12) is a $(\beta+k)$-D-representation of Y and (7) holds. ■

6.3. THEOREM. *Let X be a compactum, then $D(X) \leqslant \beta$ if and only if there exists a zero-dimensional mapping $f\colon X \to P_\beta$.*

Proof. We will use the following two assertions:

(14) (See [8], Luxemburg, Lemma 8.7.)
Let X be a compactum and (1) be its β-D-representation. We define a mapping

(15) $$\pi\colon X \to X_\#$$

as the identification of all points of the set $A_{J(\beta)}$. We put

$$p = \pi(A_{J(\beta)}) .$$

Then the equality

$$X_\# = \bigcup \{B_\gamma = \pi(A_\gamma)\colon \gamma \leqslant J(\beta)\}$$

is a $J(\beta)$-D-representation of $X_\#$ and the set $B_{J(\beta)}$ is a point p. Furthermore, π is injective on $X \setminus A_{J(\beta)}$.

(16) (See [8], Luxemburg, Lemma 8.8.)
Let U be an open set in X, $A = X \setminus U$. If $f\colon X \to K$ and $g\colon X \to T$ are mappings such that

$$\dim(f^{-1}(x) \cap U) \leqslant 0 , \quad \dim(g^{-1}(y) \cap A) \leqslant 0 \quad \text{for} \quad y \in T,\ x \in K,$$

then the mapping:

$$F\colon X \to K \times T$$

defined by the equality

$$F(x) = (f(x), g(x))$$

is zero-dimensional.

Let (1) be a β-D-representation of X and let π be the mapping (15). Then, by virtue of assertion (14) and Theorem 6.1, there exists an embedding

$$h\colon X_\# \to P_\alpha \quad (\alpha = J(\beta)) .$$

Therefore, by virtue of (14), the mapping

$$q = h \circ \pi\colon X \to P_\alpha$$

is injective and consequently, zero-dimensional on $U = X \setminus A_{J(\beta)}$. Since $\dim A_{J(\beta)} \leqslant K(\beta) = n$ (condition (d) of Definition 1.1), there exists a zero-dimensional mapping $r\colon A_{J(\beta)} \to I^n$ in n-dimensional cube I^n (see [11], Hurewicz). Let $g\colon X \to I^n$ be any extension of r. Then by virtue of (16), there exists a zero-dimensional mapping

$$f\colon X \to P_\alpha \times I^n = P_{\alpha+n} = P_\beta$$

which is defined by the equality: $f(x) = (q(x), g(x))$.

On the other hand, let $f\colon X \to P_\beta$ be a zero-dimensional mapping, then by Theorem 6.2 and Lemma 5.2 $D(X) \leqslant D(P_\beta) \leqslant \beta$. ■

Proof of Theorem 1.2. Let R be a compactum and $Z(R)$ be the class of all compacta X having a zero-dimensional mapping $f\colon X \to R$. By virtue of [12], Pasynkov, Theorem 8.8, there is a universal element in the class $Z(R)$. Let D_β be a universal element in the class $Z(P_\beta)$. Then our theorem follows from Theorem 6.3.

6.4. COROLLARY. $D(P_\beta) = D(D_\beta) = \beta$.

Proof. Since for any β there exists a compact space X with $D(X) = \beta$ (see [1], Henderson), $D(D_\beta) \geqslant D(X) = \beta$. By the definition of D_β there exists a zero-dimensional mapping $f\colon D_\beta \to P_\beta$. Therefore, by Theorem 6.2 and Lemma 5.2

$$D(D_\beta) \leqslant D(P_\beta) \leqslant \beta . \ ■$$

§ 7. Universal spaces for noncompact separable spaces. As mentioned in § 1, the universal element in the class of compact spaces X with $D(X) \leqslant \beta$ does not coincide with the one in the class of separable spaces with D-dimension $\leqslant \beta$ for $\beta \geqslant \omega_0$. To prove Theorem 1.3 we need some preliminary lemmas.

7.1. LEMMA. *Let the equality*

$$(1)\qquad X = \bigcup \{A_\gamma\colon \gamma \leqslant J(\beta)\}, \quad J(\beta) = \alpha$$

be a β-D-representation of a space X and let $M \subset A_{J(\beta)}$ be an arbitrary set of dimension $\operatorname{Ind} M \leqslant s$. *Then the equality*

$$(X \setminus A_{J(\beta)}) \cup M = \bigcup \{B_\gamma\colon \gamma \leqslant J(\beta)\}$$

where $B_\gamma = (A_\gamma \cap (X \setminus A_{J(\beta)})) \cup M$, is an $(\alpha+s)$-D-representation of $(X \setminus A_{J(\beta)}) \cup M$.

The lemma is evident. ■

7.2. DEFINITION. Let

$$(2)\qquad \beta = \alpha + n, \quad \alpha = J(\beta), \quad n = K(\beta)$$

and A_n be a universal n-dimensional compact space. Then (see [5], Bothe), there exists $(n+1)$-dimensional compactum $R_{n+1} \supset A_n$ such that $R_{n+1} \in \mathrm{AR}$. Let $q_\beta \in P_\beta$ be a fixed point (see Construction 5.1). Let

$$\pi_1\colon P_\alpha \times R_{n+1} \to P_\alpha, \quad \pi_2\colon P_\alpha \times R_{n+1} \to R_{n+1}$$

be projections. Then we put

$$S_\beta = P_\alpha \times R_{n+1} \setminus \{x\colon \pi_1(x) = q_\alpha,\ \pi_2(x) \in (R_{n+1} \setminus A_n),\ x \in P_\alpha \times R_{n+1}\}. \ ■$$

7.3. LEMMA. $D(S_\beta) \leqslant \beta$.

Proof. Let (8) (§ 5) be an α-D-representation of P_α satisfying the conditions of Lemma 5.3. Then, by Corollary 3.9, the equality

$$(3)\qquad P_\alpha \times R_{n+1} = \bigcup \{B_\gamma = A_\gamma \times R_{n+1}\colon \gamma \leqslant \alpha = J(\alpha+n+1)\}$$

is an $(\alpha+n+1)$-D-representation of the space $P_\alpha \times R_{n+1}$. We put

$$M = \{x\colon x \in P_\alpha \times R_{n+1},\ \pi_2(x) \in A_n,\ \pi_1(x) = q_\alpha\}.$$

Then M is homeomorphic to A_n and consequently $\dim M = n$. By Lemma 7.1 the equality

$$S_\beta = (P_\alpha \times R_{n+1} \setminus (A_\alpha \times R_{n+1})) \cap M = \bigcup \{C_\gamma\colon \gamma \leqslant \alpha = J(\alpha+n)\} \qquad (A_\alpha = q_\alpha)$$

where $C_\gamma = ((P_\alpha \times R_{n+1} \setminus A_\alpha \times R_{n+1}) \cap B_\gamma) \cup M$, is an $(\alpha+n)$-D-representation of S_β. Therefore $D(S_\beta) \leqslant \alpha + n = \beta$. ■

7.4. LEMMA. *If $D(X) \leqslant \beta$ and X is separable, then there exists an embedding $f\colon X \to S_\beta$.*

Proof. Suppose condition (2) holds. Then, by Lemma 8.9 in [3], Luxemburg, there exists a compactum $K \supset X$ such that

$$(4)\qquad K = R_{n+1} \cup H, \quad H \cap R_{n+1} = \varnothing\ (^3),$$

$$(5)\qquad X \subset H \cup A_n \quad (A_n \subset R_{n+1}),$$

(³) We use here the notation of Definition 7.2.

(6) H is an open set such that there exists a family of compact sets $\{H_i\}$, simple with respect to H, such that $D(H_i) < \alpha$.

Let $\pi\colon K \to L$ be a mapping onto the quotient compactum L which we obtain by identification of all points of $R_{n+1} \subset K$; we let $\pi(R_{n+1}) = \{l\} \subset L$. Then by condition (6), Lemma 3.5, and Theorem 6.1, there exists a homeomorphism $g\colon L \to P_\alpha$ such that

$$(7)\qquad g(l) = q_\alpha \in P_\alpha.$$

Since $R_{n+1} \in \mathrm{AR}$, there exists a retraction:

$$r\colon K \to R_{n+1}.$$

We define a mapping $F\colon K \to P_\alpha \times R_{n+1}$ by the equality:

$$(8)\qquad F = (g \circ \pi, r).$$

From conditions (5) and (7) it follows that

$$(9)\qquad F(X) \subset S_\beta \subset P_\alpha \times R_{n+1}.$$

Since g is a homeomorphism and π is injective on H, the composition $g \circ \pi$ is injective on H. Further, r is injective on R_{n+1}. Therefore, by virtue of (4) and (8), F is injective on K. Since K is compact, F is a homeomorphism. Let f be a restriction of F to X, then f is also a homeomorphism and by virtue of (9), $f(X) \subset S_\beta$. ■

Proof of Theorem 1.3. Since S_β is contained in the compactum $P_\alpha \times R_{n+1}$ it is a separable space. Our theorem now follows from Lemmas 7.3 and 7.4. ■

§ 8. On compactifications. For any separable space X there exists a compactification $K \supset X$ such that

$$D(K) \leqslant D(X) + 1$$

(see [13], Kozlovsky, and [3], Luxemburg, for the proof). Moreover, for any separable finite-dimensional space X there exists a compactification $K \in \mathrm{AR}$ such that

$$\dim K \leqslant \dim X + 1$$

(see [5], Bothe). The following theorem is an extension of both these results.

8.1. THEOREM. *For each $\beta < \omega_1$ there exists a compactum $Q_\beta \in \mathrm{AR}$ such that*

$$D(Q_\beta) \leqslant \beta + 1$$

and Q_β contains a homeomorphic image of any separable space Y with $D(Y) \leqslant \beta$.

Proof. We put

$$Q_\beta = P_\alpha \times R_{n+1}, \quad \alpha = J(\beta), \quad n = K(\beta)$$

(see Construction 5.1 and Definition 7.2). We have proved in § 7 (see (3) § 7) that $D(Q_\beta) \leqslant \alpha + n + 1 = \beta + 1$. Since $Q_\beta \supset S_\beta$, it follows from Lemma 7.4 that Q_β contains a homeomorphic image of each separable space Y with $D(Y) \leqslant \beta$. Moreover, $Q_\beta \in \mathrm{AR}$ because $P_\alpha \in \mathrm{AR}$ and $R_{n+1} \in \mathrm{AR}$. ■

It is also easy to prove that $D(Q_\beta) = \beta+1$.

8.2. Definition (see [14], Zarelua). A mapping $f\colon X \to Y$ is called *scattering* if for each point $x \in X$ and for each neighborhood $V \ni x$ there exists a neighborhood $U \ni f(x)$, $U \subset Y$, such that there exist open sets P and W satisfying the conditions:

$$f^{-1}(U) = P \cup W, \quad P \cap W = \emptyset, \quad x \in P \subset V.$$

It is easy to prove that for compact spaces the class of all zero-dimensional mappings coincides with the class of all scattering mappings. Therefore, by Theorem 6.2

(1) If $f\colon X \to Y$ is a scattering mapping and X and Y are compact, then $D(X) \leqslant D(Y)$.

As we mentioned above a separable space need not have a compactification with the same D-dimension (see [1]).

The following theorem gives a necessary and sufficient condition for the existence of such a compactification.

8.3. Theorem. *Let X be a separable space with $D(X) = \beta$. Then the existence of a compactum $K \supset X$ such that $D(K) = D(X)$ is equivalent to the existence of a scattering mapping $f\colon X \to P_\beta$.*

Proof. Let $f\colon X \to P_\beta$ be a scattering mapping. Then by virtue of [14], Corollary 5, Zarelua, there exists a compactification $K \supset X$ and a scattering extension $F\colon K \to P_\beta$ of the mapping f. Therefore, by virtue of (1), $D(K) \leqslant D(P_\beta) = \beta$. Moreover, $D(K) \geqslant D(X) = \beta$ because $K \supset X$. We have thus proved that our condition is sufficient. Let K be a compactification of X with $D(K) = D(X) = \beta$. Then, by Theorem 6.3, there exists a zero-dimensional and, consequently, scattering, mapping $f\colon K \to P_\beta$. Let $g\colon X \to P_\beta$ be a restriction of f to X. Then clearly g is also scattering. ■

8.4. Problem. Let $\mathscr{S}$ be a class of all separable spaces and let $\mathscr{K}$ be a class of all compact spaces. In each of these classes consider two subclasses for $\omega_0 \leqslant \alpha < \omega_1$:

(1) Spaces X with $\operatorname{ind} X \leqslant \alpha$.

(2) Spaces X with $\operatorname{Ind} X \leqslant \alpha$.

So we get four classes of spaces for each α. Do there exist universal elements in these classes?

8.5. Remark. According to [2], § 8, Luxemburg, there exist compact spaces X having $\operatorname{Ind} X = \alpha$ and an arbitrarily large $D(X) < \omega_1$. Therefore, there are no universal elements in the class of compact spaces X having $D(X) < \Delta$ and $\operatorname{Ind} X \leqslant \alpha$. Analogous results could be obtained for the other three classes in Problem 8.4.

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Received 30 November 1981;
in revised form 25 June 1982