On universal infinite-dimensional spaces

by

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Abstract. In this paper we construct a universal compact metric space with given transfinite $D$-dimension. A similar result is proved for separable metric spaces. Since $D(X) = \dim X$ for finite-dimensional spaces, these results are extensions of well-known theorems for finite-dimensional spaces. Also we prove that every separable metric space $X$ is contained in a compact metric space $K \in AR$ such that $D(K) \leq D(X) + 1$.

§ 1. Definitions and notation. In this paper we consider the transfinite $D$-dimension introduced in [1]. Henderson. Some of our results we announced in [2]. Luxembourg, without proof. All spaces in this paper are assumed to be metric and all mappings continuous. For every ordinal number $\beta$ the equality $\beta = \alpha + n$ holds where $\alpha$ is a limit number or 0 and $n = 0, 1, 2, \ldots \ (\cdot)$. Then we put $K(\beta) = n$, $J(\beta) = \alpha$.

1.1. Definition. We put $D(\emptyset) = -1$. If $X \neq \emptyset$, then $D(X)$ is the smallest ordinal number $\beta$ such that there exists a collection of sets $\{A_\delta: 0 \leq \delta \leq \gamma\}$, where $\gamma$ is an ordinal number, satisfying the following conditions:

(a) $X = \bigcup \{A_\delta: 0 \leq \delta \leq \gamma\}$.
(b) Every set $A_\delta$ is closed and finite-dimensional.
(c) For any $\delta \leq \gamma$, the set $\bigcup \{A_\delta: \delta \leq \delta \leq \gamma\}$ is closed in $X$.
(d) $J(\beta) = \gamma$, $\dim A_\delta \leq K(\beta)$.
(e) For any point $x \in X$, there exists the greatest number $\delta \leq \gamma$ such that $x \in A_\delta$.

If there is no such number $\beta$, we put $D(X) = \Delta$ where $\Delta$ is an abstract symbol such that $\Delta > \beta$ for any ordinal number $\beta$. If conditions (a)-(e) hold, then equality (a) is called a $\beta$-$D$-representation of a space $X$.

It is evident that

1) if $X \subseteq Y$, then $D(X) \leq D(Y)$.

Moreover, $D(X) = \dim X - \text{Ind} X$ for finite-dimensional spaces. For any space $X$ of weight $\aleph_1$, we have $|D(X)| = \aleph_1$ (see [1], Henderson, Theorem 10); consequently, for any separable space (in particular, a compact space) $X$, we have $D(X) \leq \aleph_1$ or $D(X) = \Delta$.

(\cdot) We always consider $\beta + 0 = 0 + \beta = \beta$. 
1.2. Theorem. There is a universal element in the class of all compact spaces $X$ such that $D(X) \leq \beta$ ($\beta < \omega_1$).

1.3. Theorem. There is a universal element in the class of all separable spaces $X$ such that $D(X) \leq \beta$ ($\beta < \omega_1$).

We note that for every $\beta < \omega_1$ such that $\beta \geq \omega_0$, there exists a separable space $X_\beta$ satisfying the following condition:

$$D(X_\beta) = \beta$$

and for any compact space $Y \subseteq X$ we have $D(Y) > D(X)$, (see [3], Luxemburg, Theorem 8.2). Consequently, universal elements in Theorems 1.2 and 1.3 are different for any $\beta \geq \omega_0$. These theorems are extensions of well-known results (see [4], Nöbeling) for finite-dimensional spaces. We will also prove the following theorem:

1.4. Theorem. For every separable space $X$ there exists a compact space $Y \subseteq X$ and a homeomorphism $f: X \rightarrow Y$ such that $D(Y) \leq D(X) + 1$.

This theorem is an extension of a similar theorem for finite-dimensional spaces (see [5], Rothen). To prove this theorem, we need some preliminary constructions.

§ 2. The main constructions.

2.1. Definition. Let $X$ be a compactum and

$$\varphi_X: X \times I \rightarrow CX$$

the identification mapping of the product $X \times I$, where $I$ denotes the unit segment $[0, 1]$, onto the cone $CX$. (We obtain the cone by identifying all points of the set $X \times \{0\} \subseteq X \times I$. The point $\varphi(X \times 0) = a \in CX$ is called the apex of the cone $CX$).

2.2. Construction. Let $\sum X_i$, $i = 1, 2, ...$ be a discrete union of spaces $X_i$, $i = 1, 2, ...$. Suppose in any $X_i$ there are two closed sets $A_i$ and $B_i$, $A_i \cap B_i = \emptyset$, and for any $i$ there exists a homeomorphism $g_i: B_i \rightarrow A_{i+1}$. We identify every point $x \in B_i$ in a space $\sum X_i$ with a point $g_i(x)$ for all $i$. Then we get a factor mapping:

$$\mu: \sum X_i \rightarrow \Phi$$

onto the factor space $\Phi$. We shall consider a set $F \subseteq \Phi$ to be closed if and only if the set $\mu^{-1}(F)$ is closed. It is evident that for each $i$ we have an embedding

$$f_i: X_i \rightarrow \Phi$$

and

$$\bigcup_{i=1}^{m} f_i(X_i) = \Phi$$

where $f_i$ is a restriction of $\mu$ to $X_i$. We put

$$\Phi = \Phi(X_1, A_1, B_1, g_1), \quad X_i = f_i(X_i).$$

2.3. Definition. Let $\mathcal{F} = \{F_i: i = 1, 2, ...\}$ be a countable family of sets in a space $X$ and let the set $U \subseteq X$ be open. Then the family $\mathcal{F}$ is called simple with respect to $U$ if

$$U = \bigcup_{i=1}^{n} F_i$$

and

$$F_i \cap F_j = \emptyset$$

for $i\neq j$.

2.4. Lemma. Let a space $\Phi$ be defined by equality (4); then the family of sets $\{X_i\}_{i=1}^{m}$ is simple with respect to $\Phi$. Moreover, if the spaces $X_i$ are compact, then $\Phi$ is separable and locally compact.

The lemma is evident.

2.5. Construction. Let $\{X_i: i = 1, 2, ...\}$ be a family of disjoint compact spaces, and for each $i$, there exists a homeomorphism $h_i: X_i \rightarrow X_{i+1}$. We put

$$B(X_i) = X_i \times I \times CX_{i+1}$$

where $CX_i$ is the cone with the apex $a_i$. Let

$$A_i = X_i \times \{0\} \times \{a_i\} \subseteq B(X_i), \quad B_i = X_i \times \{1\} \times \{a_i\} \subseteq B(X_i).$$

Since $A_i$ and $B_i$ are homeomorphic to $X_i$, there exist homeomorphisms $g_i: B_i \rightarrow A_{i+1}$, $i = 1, 2, ...$. We put

$$\Phi = \Phi(B(X_i), A_i, B_i, g_i).$$

Since all spaces $X_i$ are compact, all spaces $B(X)$ are also compact and, by virtue of Lemma 2.4, $\Phi$ is a locally compact, separable space. We put

$$\mathcal{S} = \mathcal{S}(X_i, h_i) = \{a_i\} \cup \{f\}$$

where $\mathcal{S}$ is a compactification of $\Phi$ with an extra point $a_i$. Consequently,

$$\mathcal{S} = \{a_i\} \cup \bigcup_{i=1}^{m} B(X_i)$$

and

$$\mu: \mathcal{S} \rightarrow X_i$$

is an extension of $\Phi$. We put

$$\mu^{-1}(0) = a_i, \quad \mu^{-1}(1/(i+1)) = S_i, \quad \mu^{-1}(1/(i+1), 1) = B(X_i).$$

where $S_i = B(X_i) \cap B(X_{i+1})$. Then we have a mapping $\Psi: F \rightarrow [0, 1]$. Let

$$\mu: X \rightarrow [0, 1]$$

be an extension of $\mu$. We put

$$C_i = \mu^{-1}\left(\frac{1}{i+1}\right), \quad B_i = \mu^{-1}\left(\frac{1}{i+1}, \frac{1}{i}\right), \quad W = \mu^{-1}(0), \quad C_0 = \emptyset$$

for $i = 1, 2, ...$. 

(5) $U = \bigcup_{i=1}^{n} F_i$

(6) $F_i \cap F_j = \emptyset$ for $i \neq j$.

(7) The family $\mathcal{F}$ is locally finite on $U$ and sets $F_i$ are closed in $X$.

(8) $B(X_i) = X_i \times I \times CX_{i+1}$

(9) $A_i = X_i \times \{0\} \times \{a_i\} \subseteq B(X_i), \quad B_i = X_i \times \{1\} \times \{a_i\} \subseteq B(X_i)$.

(10) $\mathcal{S} = \{a_i\} \cup \bigcup_{i=1}^{m} B(X_i)$

(11) $\mu: \mathcal{S} \rightarrow X_i$. 

(12) $\Psi^{-1}(0) = a_i, \quad \Psi^{-1}(1/(i+1)) = S_i, \quad \Psi^{-1}(1/(i+1), 1) = B(X_i)$.

(13) $C_i = \mu^{-1}\left(\frac{1}{i+1}\right), \quad B_i = \mu^{-1}\left(\frac{1}{i+1}, \frac{1}{i}\right), \quad W = \mu^{-1}(0), \quad C_0 = \emptyset$ for $i = 1, 2, ...$. 

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Then
\[ X = W \cup \bigcup_{i=1}^{n} B_i, \quad C_i = B_i \cap B_{i+1}. \]

By virtue of (12) and (13), \( f(F \cap C_i) \subset S_i \). Since \( X_i \subset AR \), by virtue of (11), the set \( S_i = B(X_i) \cap B(X_{i+1}) \) is also an AR-space. Consequently, for any \( i \), there exists a mapping \( g_i : C_i \rightarrow S_i \) of the restriction of \( f \) to \( F \cap C_i \). We put
\[ g(x) = f(x) \text{ for } x \in F, \quad g(x) = g_i(x) \text{ for } x \in C_i, \quad g(W) = \omega. \]

Then we have a continuous mapping \( g : R = F \cup W \subset \bigcup_{i=1}^{n} C_i \subset X \), and clearly
\[ g(R \cap B_i) = B(X_i) \subset X, \quad g(B_i \cap B_{i+1} \cap R) \subset S_i. \]

Since \( X_i \subset AR \), the cone \( CX_i \subset AR \) and consequently \( B(X_i) = X_i \times I \times CX_i \subset AR \). Therefore, for each \( i \), there is an extension \( k_i : B_i \rightarrow B(X_i) \) of the mapping \( r_i : B_i \cap R \rightarrow B(X_i) \subset X \), where \( r_i \) is a restriction of \( g \) to \( R \cap B_i \). We put
\[ k(x) = k_i(x) \text{ for } x \in B_i, \quad k(x) = \omega \text{ for } x \in W. \]

Then clearly \( k : X \rightarrow X \) is a continuous extension of \( f \).

2.8. Lemma. Let \( F \) be a family of sets \( F = \{ F_1 \} \) be simple with respect to \( U \subset X \).
Answer: Suppose the family of spaces \( \{ X_i \} \) (\( i = 1, 2, \ldots \)) satisfies the conditions of construction 2.2, and for each \( i \), there exists a homeomorphism \( \Phi_i : F_i \rightarrow X_i \) such that
\[ g_i \circ \Phi_i(x) = \varphi_i(x) \quad \text{for} \quad x \in F_i \cap F_{i+1}. \]

Then the mapping \( \Phi : U \rightarrow \Phi(X) \) is defined by the equality
\[ \varphi(x) = f_i \circ \Phi_i(x), \]
where \( f_i \) is a homeomorphism (3), is a homeomorphism and
\[ \Phi(F_i) = X_i \circ f_i(X_i). \]

The lemma follows directly from Construction 2.2.

2.9. Lemma. Let \( Y \) be a compactum, \( CX \) the cone over \( X \in AR \) with the apex \( a \) and
\[ B(X) = X \times I \times CX, \quad A_i = X \times \{ 0 \} \times \{ 0 \} \subset B(X), \quad i = 0, 1 \in [0, 1]. \]

If there exists a homeomorphism
\[ f : Y \rightarrow X, \]
then for any disjoint closed subsets \( F, G \subset Y \) and any homeomorphisms
\[ f_0 : F \rightarrow A_0, \quad f_1 : G \rightarrow A_1, \]
there exists a homeomorphism \( g : Y \rightarrow B(X) \), which extends \( f_0 \) and \( f_1 \).

Proof. Let \( \pi_1 : B(X) \rightarrow X \times I \) be a projection. Since \( X \in AR \) and \( I \in AR \), we have \( X \times I \in AR \). Therefore, there exists a mapping \( k : Y \rightarrow X \times I \), which extends \( \pi_1 \circ f_0 \) and \( \pi_1 \circ f_1 \) to \( Y \). Then,
\[ (15) \quad k \text{ is injective on } F \cup G. \]

Let \( \mu : Y \rightarrow [0, 1] \) be a continuous function such that \( \mu^{-1}(0) = F \cup G \). Let a mapping \( h : Y \rightarrow X \times I \) be defined by the equality
\[ h(y) = (f(y), \mu(y)). \]

We put \( l(y) = \varphi_x \circ h \), where \( \varphi_x \) is a mapping (1). Then, clearly,
\[ (16) \quad l \text{ is injective on } Y \setminus (F \cup G). \]

We put \( g(y) = (k(y), l(y)) \), then by virtue of (15) and (16) \( g : Y \rightarrow B(X) \subset X \times I \times CX \) is injective on \( Y \). Since \( Y \) is compact, \( g \) is a homeomorphism.

2.10. Lemma. Let \( X \) be a compactum and the equality
\[ (17) \quad X = \bigcup \{ A_\beta \mid \beta \in J(\beta) \} \]
be a \( \beta \)-D-representation such that \( A_{\beta(x)} \) consists of exactly one point. Let there exists an increasing sequence \( \{ \gamma_\alpha \} \) of ordinal numbers such that
\[ \sup \gamma_\alpha = x = J(\beta) \supseteq \omega_0 \]
and a sequence of absolute retracts \( \{ X_i \} \) satisfying the following conditions:
\[ (18) \quad \text{Every compactum with } D \text{-dimension } \leq \gamma_\alpha \text{ has an embedding in } X_i. \]

\[ (19) \quad \text{There exists a homeomorphism } h_i : X_i \rightarrow X_{i+1}, \text{ for } i = 1, 2, ... \]

Then there exists a homeomorphism
\[ h : X \rightarrow X(X_i, h_i) \]
such that
\[ (20) \quad h^{-1}(\omega) = A_{\beta(x)} \]
where \( \omega \) is the compactification point in \( X = X(X_i, h_i) \).

Proof. By virtue of Lemma 8.2 in [3], Luxembourg, there exists a family of sets \( F_i \), simple with respect to \( X \setminus A_{\beta(x)} \), such that
\[ D(F_i) \leq \gamma_\alpha. \]

By Definition 2.3, the sets \( F_i \) are closed in \( X \) and are consequently compact. Let the sets \( B(X_i), A_i, B_i \) be defined by conditions (8) and (9); \( g_i : B_i \rightarrow A_{i+1} \) are homeomorphisms from Construction 2.5. Since \( A_i \) and \( B_i \) are homeomorphic to \( X_i \) by virtue of (18) and (22), there exist homeomorphisms
\[ (21) \quad f_i : F_i \rightarrow X_i, \quad k_i : F_i \cap F_{i+1} \rightarrow B_i, \quad r_{i+1} : F_i \cap F_{i+1} \rightarrow A_{i+1} \]

then for any disjoint closed subsets \( F, G \subset Y \) and any homeomorphisms
\[ f_0 : F \rightarrow A_0, \quad f_1 : G \rightarrow A_1, \]
there exists a homeomorphism \( g : Y \rightarrow B(X) \), which extends \( f_0 \) and \( f_1 \).
By virtue of Lemma 2.9 there exists a homeomorphism \( g_1 : F_1 \to B(X) \) which is an extension of \( h_1 \) and \( r_1 \). Consequently, from (23), it follows that
\[
g_1 \circ \varphi(x) = \varphi_1 \circ (x) \quad \text{for} \quad x \in F_1 \cap F_1^{+}.\]

By Lemma 2.8 there exists a homeomorphism \( \varphi : \mathcal{N} \setminus A_{M} = \Phi = \mathcal{N} \setminus \Phi_0. \) We put \( h(x) = \varphi(x) \) for \( x \notin A_{M} \) and \( H(A_{M}) = \omega. \) Then, clearly, \( h \) is a desired homeomorphism.

§ 3. Natural sums and \( \beta \)-\( D \)-representations of compacts. In Sections 3 and 4 the symbol \( \beta \) is intended to denote infinite ordinal numbers. In what follows we need some information about the natural sum of ordinal numbers; see [6], Toulmin, and [7], Hausdorff (7).

Every ordinal number \( \beta \) has a unique representation:
\[
\beta = a_1 + \ldots + a_n(1)
\]
where \( a_1, a_2, \ldots, a_n \) are indecomposable transfinite numbers such that \( a_1 < \beta \). (A transfinite number \( \xi \) is called indecomposable if \( \xi \) is not the sum of a finite number of ordinal numbers less than \( \xi \).) The representation (1) is called canonical. It is evident that \( K(\beta) = a_n+1. \)

1. DEFINITION. Let (1) be a canonical representation of \( \beta \) and
\[
\gamma = \delta_1 + \ldots + \delta_\mu(2)
\]
be a canonical representation of \( \gamma \). Let \( \xi_1, \ldots, \xi_{\mu+\beta} \) be elements of the set \( a_1, \ldots, a_\mu, a_{\mu+1}, \ldots, a_{\mu+\beta} \) with decreasing order \( \xi_1 > \xi_2 > \ldots > \xi_{\beta+1} \). Then the natural sum \( \gamma \oplus \beta \) is defined by the equality \( \gamma \oplus \beta = \xi_1 + \ldots + \xi_{\beta+1}. \)

If \( n = 0, 1, 2, \ldots \), then by definition \( \gamma \oplus n = \gamma + n = \gamma \oplus \gamma. \) It is evident that \( \beta \oplus \beta = \gamma \oplus \beta. \)

In [1], Henderson, Theorem (8), it was proved that for any spaces \( X \) and \( Y \)
\[
D(X \times Y) \leq D(X) \oplus D(Y)(3)
\]

2. LEMMA. Let (1) and (2) be representations of \( \beta, \gamma \) and \( a_{n+1} = 0. \) Let \( \gamma < \beta \) and \( l \) be the first integer such that \( \delta_l > a_{l+1}. \) Then \( \delta_l < a_{l+1}. \)

Proof. If \( \delta_l = a_{l+1} \), then, since the sequence \( \{a_i\} \) is decreasing, \( \delta_l > a_{l+1} \) for \( j < l. \) Besides that, clearly \( l < p+1. \) Therefore, \( \delta_l \) is an indecomposable transfinite number.

Consequently, \( \delta_l > a_{l+1} + \ldots + a_{\mu+1} \) and \( \gamma \geq a_1 + \ldots + \delta_l > a_{l+1} + \ldots + a_{\mu+1} + a_{\mu+1} = \beta. \)

This contradiction proves the lemma.

3. LEMMA. Let \( CX \) be the cone over a compact space \( X. \) Then
\[
D(CX) = D(X \times I) \leq D(X) + 1(4).
\]

Proof. Let \( a \) be the apex of \( CX; \) clearly, \( CX \setminus \{a\} \) can be embedded into \( X \times I. \)

Consequently (see [1], Henderson, Theorem (8)),
\[
D(CX \setminus \{a\}) \leq D(X \times I) \leq D(X) + 1.
\]

Moreover, it is evident that the adding of a point to a nonempty set whose \( D \)-dimension is defined does not change its \( D \)-dimension.

3.4. LEMMA. If \( D(X) < a \) and \( a \) is indecomposable transfinite, then \( D(B(X)) < \alpha \) where \( B(X) = CX \times X \times I. \)

The lemma follows from Lemma 3.3 and (3).

3.5. LEMMA. Let \( L \) be a compactum such that
\[
L = \bigcup \{A_\gamma : \gamma \in J(\alpha)\}
\]

for some point \( l \in L, \) and \( \{H_\gamma\} \) be a family, simple with respect to \( \Phi, \) such that \( D(H_\gamma) \leq \gamma \leq \alpha. \) Then there exists a \( \alpha \)-\( D \)-representation of \( L \)
\[
L = \bigcup \{A_\gamma : \gamma \in J(\alpha)\}
\]
such that \( A_{H_\gamma} = \{l\}. \)

Proof. For each \( i, \) since \( D(H_\gamma) \leq \gamma_i, \) by Lemma 1 in [1], Henderson, there exists a \( \gamma_i \)-\( D \)-representation of the space \( H_i \)
\[
H_i = \bigcup \{A_{\gamma_i} : \mu \in J(\gamma_i)\}
\]
such that
\[
dim A_{H_i} = \delta \in \mu(\kappa). \]

We put \( A_{H_i} = \bigcup \{A_{\gamma_i} : \mu > J(\gamma_i)\} \) and \( A_{H_i} = \{l\} \cup \bigcup \{A_{\gamma_i} : i = 1, 2, \ldots\}. \)

From the sum theorem it follows that
\[
dim A_{H_i} = \kappa = \dim(\beta). \]

Moreover, \( A_{H_i} = \{a\}. \) It is easy to see that the equality \( L = \bigcup \{A_\gamma : \delta \in J(\alpha)\} \) is an \( \alpha \)-\( D \)-representation of \( L. \)

3.6. LEMMA. Let \( a \) be an indecomposable ordinal number and, for \( i = 1, 2, \ldots, \)
\( X_i \) a compact space such that \( D(X_i) < a. \) There exists an \( \alpha \)-\( D \)-representation of \( X \),
\[
X = \bigcup \{A_{\gamma} : \gamma \in J(\alpha)\}
\]
such that \( A_{X_i} = \{a\}. \) Consequently, \( D(X) < a. \)

Our lemma follows from Lemmas 2.6, 3.4, and 3.5.

3.7. LEMMA. If \( X \) is compact, \( D(X) = \beta, \) and
\[
X = \bigcup \{A_\gamma : \gamma \in J(\beta)\}
\]
is a \( \beta \)-\( D \)-representation of \( X, \) then we have:
\[
C_\gamma = \bigcup \{A_{\gamma} : \delta \in J(\beta)\} \neq \emptyset
\]
and \( C_\gamma \) is compact for each \( \gamma \in J(\beta). \) In particular, \( C_{H_\gamma} = A_{H_\gamma} \neq \emptyset. \)
Proof. First we shall show that \( \mathcal{C}_\gamma \neq \emptyset \) for \( \gamma < J(\beta) \). Indeed, if \( \mathcal{C}_\gamma = \emptyset \), then 
\[ X = \mathcal{U}_\gamma = X - \mathcal{C}_\gamma. \]
By Lemma 8.3 in [8], Luxembourg, \( D(V) \cup J(\beta) \subseteq \beta \). This contradicts the condition of the Lemma. Therefore, \( \mathcal{C}_\gamma \neq \emptyset \) for \( \gamma < J(\beta) \). From condition (c) of Definition 1.1 the set \( \mathcal{C}_\gamma \) is closed in \( X \) and consequently is compact. Since \( \mathcal{C}_\gamma \subseteq \mathcal{C}_\gamma \) for \( \gamma < \gamma' \), we have 
\[ \bigcap \mathcal{C}_\gamma = \emptyset. \]
But from condition (e) of Definition 1.1 it follows that \( \bigcap \mathcal{C}_\gamma = A_{J(\beta)} \neq \emptyset. \]

3.8. Lemma. Let \( \mathfrak{B} \) be a \( \mu \)-D-representation of \( X \) and 
\[ Y = \bigcup \{ A_{\gamma} : \gamma < J(\beta) \} \]
be a \( \delta'(\beta) \)-representation of \( Y \). Then the equality 
\[ X \times Y = \bigcup \{ A_{\gamma} : \gamma < J(\beta) \} \]
is a \( \delta'(\beta) \)-D-representation of \( X \times Y \) by virtue of [1].
Henderson. From the definition of the natural sum it follows that \( J(\beta \oplus \delta) = J(\beta) \oplus \delta \). If \( \gamma < J(\beta) \) or \( \gamma < J(\beta) \), then \( \gamma \oplus \gamma' = J(\beta \oplus \delta) \). Consequently, 
\[ D(\beta \oplus \delta) = A_{J(\beta)} \times B_\delta. \]

3.9. Corollary. Let \( \mathfrak{B} \) be a \( \mu \)-D-representation of \( X \) and let \( K \) be an arbitrary space with \( \dim_K \leq m \); then the equality 
\[ X \times K = \bigcup \{ A_{\gamma} : \gamma < J(\beta) = J(\beta \oplus \delta) \} \]
is a \( (\mu + m) \)-D-representation of \( X \times K \).

4. On compacta \( Z(a_\gamma) \).

4.1. Definition. Let \( X \) be a compactum and let 
\[ \beta = a_\gamma + \ldots + a_{\gamma + 1}, \quad a_{\gamma + 1} = n_0 = 0, 1, 2, \ldots, \quad a_{\gamma + \gamma} = 0 \]
be a canonical representation of ordinal number \( \beta \geq a_0 \) and let 
\[ X = \bigcup \{ A_{\gamma} : \gamma < J(\beta) \} \]
be a \( \beta \)-D-representation of \( X \). We put 
\[ Y(a_\gamma) = \bigcup \{ A_{\gamma} : a_\gamma < J(\beta) \}. \]
We note that by Lemma 3.7, \( Y(a_\gamma) \) is compact and \( Y(a_\gamma) \neq \emptyset \). Moreover, 
\[ Y(a_\gamma) = A_{J(\beta)} \]
Let 
\[ \gamma_1 : Y(a_{\gamma - 1}) \to Z(a_\gamma), \quad i = 1, \ldots, k \]
be a mapping on a compactum \( Z(a_\gamma) \) obtained by identification of all points of the compactum \( Y(a_\gamma) \). Let \( \beta_i = \gamma_1(Y(a_\gamma)) \). We also put 
\[ Z(a_{\gamma + 1}) = A_{J(\beta)} = Y(a_\gamma), \quad \beta_{\gamma + 1} : i = id : Y(a_\gamma) \to Z(a_{\gamma + 1}). \]
Thus, the compacta \( Y(a_\gamma) \) are completely defined for \( i = 0, \ldots, k \), 
\[ X = Y(a_\gamma) \Rightarrow \ldots \Rightarrow Y(a_\gamma) = A_{J(\beta)} \]
and the compacta \( Z(a_\gamma) \) are defined as quotient spaces for \( i = 1, \ldots, k + 1 \), 
\[ Z(a_\gamma) = Y(a_{\gamma - 1})/Y(a_\gamma) \quad \text{for} \quad i = k \]
and 
\[ Z(a_{\gamma + 1}) = Y(a_\gamma) = A_{J(\beta)} \]

We introduce one more notation. Let \( \gamma < a_\gamma \); then we put \( \gamma_{\min} = \gamma \). If \( a_\gamma + \ldots + a_{\gamma + 1} < \gamma < a_\gamma + \ldots + a_{\gamma + 1} \), then the ordinal number \( r_{\gamma + 1}(\gamma) \) is defined by the equality \( a_\gamma + \ldots + a_{\gamma + 1} + r_{\gamma + 1}(\gamma) = \gamma \) for all \( \gamma \leq \beta \).

4.2. Lemma. We put 
\[ B_{a_\gamma} = a_\gamma(\Delta_\gamma), \quad a_\gamma + \ldots + a_{\gamma - 1} \leq \gamma < a_\gamma + \ldots + a_{\gamma + 1} \]
Then the equality 
\[ Z(a_\gamma) = \bigcup \{ B_\beta : \beta \leq a_\gamma \} \]
is an \( a_\gamma \)-D-representation of the compactum \( Z(a_\gamma) \) and 
\[ B_{a_\gamma} = B_{a_\gamma} = \{ \beta_i \}. \]

Proof. The equality in (8) follows from (3), (7), and the construction of \( q_0 \). Since \( Y(a_{\gamma - 1}) \) is compact, the mapping \( q_0 \) is closed. The mapping \( q_0 \) clearly does not raise the dimension of closed finite-dimensional sets. Consequently, from (7), it follows that the sets \( \beta_\mu \) are closed and finite-dimensional. Thus properties (a) and (b) of Definition 1.1 are proved. Property (c) follows from the closedness of \( q_0 \). Condition (9) is evident and (e) follows from (9). Condition (4) is true because \( q_0 \) is a homeomorphism on \( q_1^*(Z(a_\gamma) \Delta \beta) \).

In the following two lemmas we adopt the notation of Definition 4.1.

4.3. Lemma. For any two distinct points \( x \) and \( y \) in the compactum \( X \), there exists a number \( i = 0, \ldots, k \) such that \( x \in Y(a_\gamma), y \in Y(a_\gamma) \) and \( q_\gamma(x) \neq q_\gamma(y) \).
Proof. For any point \( x \in X \), let \( \mu(x) \) be the greatest number \( i \) such that \( x \in Y(a_\gamma) \).
Let \( \mu(x) = \mu(y) = p \). Then: either \( p < k \) or \( p = k \).
In the second case \( q_\gamma(x) = \gamma \neq \gamma \). Since \( q_{\gamma + 1} \) is clearly injective on \( Y(a_{\gamma + 1}) \setminus Y(a_{\gamma + 1}) \), we also have \( q_{\gamma + 1}(x) \neq q_{\gamma + 1}(y) \) in the first case. If \( \mu(x) \neq \mu(y) \), for example, \( \mu(x) > \mu(y) = p \), then \( x \in Y(a_{\gamma + 1}) \), \( y \in Y(a_{\gamma + 1}) \). Therefore, \( q_{\gamma + 1}(x) = q_{\gamma + 1}(y) \).

4.4. Lemma. Let \( X \) be a compactum and, for \( i = 1, 2, \ldots, k + 1 \), there exists a homeomorphism 
\[ h_i : Z(a_\gamma) \to P_\gamma \in \mathcal{A}_R \]
in a space $P_t \in AR$. Then there exists a homeomorphism

$$h: X \to \prod_{i=1}^{k+1} P_t$$

of the space $X$ in the product of the spaces $P_t$. Moreover,

$$h(x) = h_1(x_1) \times h_2(x_2) \times \cdots \times h_{k+1}(x_{k+1})$$

(10) If the set $A_{k+1} = \{x_i \geq 0\}$ in a $\beta$-D-representation (2) consists of exactly

one point $b_{k+1}$, then $h(A_{k+1})$ is a point whose $i$-th coordinate in the product

$$\prod_{i=1}^{k+1} P_t$$

is a point $h_i(b_i)$ ($h_i \in Z(x_i)$).

Proof. Since $P_t \in AR$ there exists an extension

$$g_i: X \to P_t$$

of the mapping $h_i = g_i$: $Y(x_{i-1}) \to P_t$. Let

$$h: X \to \prod_{i=1}^{k+1} P_t$$

be a mapping whose $i$-coordinate is $g_i$. Let $x, y$ be a pair of distinct points in $X$.

Then, by Lemma 4.3, $g_i(x) \neq g_i(y)$ for some $i$. Since $h_{i+1}$ is a homeomorphism,

$$g_i(x) = h_{i+1} + g_{i-1}(x) \neq h_{i+1} + g_{i-1}(y)$$

Consequently, $h(x) \neq h(y)$. Therefore, $h$ is injective. Since $X$ is compact, $h$

is a homeomorphism. Condition (10) is evident.

§ 5. On compacts $P_t$.

5.1. Construction. For each ordinal $\beta < \omega_1$, we will define a compactum $P_\beta$

and a fixed point $g_\beta \in P_\beta$. For each pair of compacts $P_\gamma$ and $P_\beta$ ($\gamma < \beta$), we will define a homeomorphism

$$h_{\beta\gamma}: P_\gamma \to P_\beta$$

(1)

We put

$$P_n = \mathcal{P}$$

for $n = 0, 1, 2, \ldots$

where $\mathcal{P}$ is an $n$-dimensional cube. Points $g_\beta$, we select in an arbitrary way. Then, clearly, for $\gamma < \beta < \omega_0$, there exist homeomorphisms (1). Suppose, for $\beta < \beta_0$, compacts $P_\beta$ and homeomorphisms (1) have been constructed. If $b_\beta$ is indecomposable

transfinite, then there exists a sequence of ordinal numbers $\gamma(b_\beta, i)$ such that

$$\sup \{\gamma(b_\beta, i): i = 1, 2, \ldots = \beta_0 \}$$

and thus we put

$$P_\beta = \mathcal{P}$$

(2)

We define $g_\beta$ as a compactification point in $\mathcal{P} = P_\beta$ (see Construction 2.5). Let

$\gamma < \beta$. Then, by virtue of (2), $\gamma < \gamma(b_\beta, i)$ for some $i$. By inductive assumption there

exists a homeomorphism $h_{\gamma(b_\beta, i)}: P_\gamma \to P_\beta$. Consequently, there exists a homeo-

morphism (1) because $P_{\gamma(b_\beta, i)}$ is homeomorphic to a subset of $P_\beta$ (See (10) and (11)

in § 2.)

If $\beta$ is a decomposable transfinite number and the equality

$$\beta = \alpha_1 + \cdots + \alpha_{k+1}$$

is a canonical representation of $\beta$, then we put

$$P_\beta = \prod_{i=1}^{k+1} P_{\alpha_i}$$

(5)

Let $\pi_i: P_\beta \to P_{\alpha_i}$ be a projection on a factor. Then the point $g_\beta$ is defined by the equalities

$$\pi_i(g_\beta) = \eta_{\alpha_i}, \quad i = 1, \ldots, k+1.$$

Let $\gamma < \beta$ and let the equality

$$\gamma = \delta_1 + \cdots + \delta_{p+1}, \quad p = 0, 1, 2, \ldots$$

be a canonical representation of $\gamma$. Let $l < p + 1$ be the first number such that $\delta_l \neq \alpha_l$.

Then by Lemma 3.2, $\delta_l < \alpha_l$. If $l = k+1$, then clearly $P_\beta = P_\gamma \times \mathcal{P}^0$

where $n = \alpha_{k+1} - \delta_{k+1}$, and the homeomorphism $h_{\beta\gamma}$ exists. Let $l \leq k$; then $\alpha_l$

is an indecomposable number. Since $\delta_l \leq \delta_i < \alpha_l$ for $s \geq l$, we have

$$\xi = \delta_1 + \cdots + \delta_{p+1} < \alpha_l.$$

By inductive assumption there exists a homeomorphism

$$h_{\beta\gamma}: P_\gamma \to P_\beta$$

(6)

Moreover, by our construction

$$P_\gamma = \prod_{i=1}^{k+1} P_{\alpha_i}$$

(7)

By virtue of (5), (6), and (7) there exists a homeomorphism (1).

5.2. Lemma. $D(P_\beta) \leq \beta$.

Proof. We will prove this lemma by induction on $\beta$. If $\beta < \omega_0$, then clearly

$D(P_\beta) = \beta$. Let $\beta$ be a decomposable transfinite number and let (4) be its canonical

representation. Consequently, by virtue of (3), § 3, and (5), along with Definition 3.1

and inductive assumption,

$$D(P_\beta) \leq D(P_{\alpha_1}) + \cdots + D(P_{\alpha_{k+1}}) \leq \alpha_1 + \cdots + \alpha_{k+1} = \beta.$$
be its canonical representation. Then, by inductive assumption and by Lemma 4.2, there exist embeddings

\[ h_i: Z(q_i) \to P_{a_i}, \quad i \leq k, \quad h_{k+1}: Z(q_{k+1}) = A_{ij(k)} \to P_{2k+1} \]

such that

\[ h_i^{-1}(q_i) = \{b_i\} \quad (i \leq k), \]

where \( \{b_i\} = B_{a_i} \) (see Lemma 4.2, conditions (8) and (9) of § 4). Since, by Lemma 5.4, \( P_{a_i} \in AR \), by Lemma 4.4 there exists a homeomorphism

\[ h: X \to \bigcup_{a_i \leq k} P_{a_i} \times P_{2k+1} = P_{a_k + 2k + 1}. \]

If the set \( A_{ij(k)} = Z(q_{k+1}) \) consists of exactly one point, then we consider a mapping \( h_{k+1}: Z(q_{k+1}) \to P_{2k+1} \). By virtue of Lemma 4.4 there exists a homeomorphism (2) such that \( h(A_{ij(k)}) \) is a point whose \( i \)-coordinate is \( h(b_i) \). By property (5) and by Definition 1 this point is \( q_{a_i} \). Therefore, condition (3) holds. Let \( \beta \) be an indescomposable ordinal number. Then there exists a canonical representation of \( \beta \)

\[ \beta = \alpha_1 + \alpha_2, \quad \alpha_1 = 0. \]

By inductive assumption, Construction 5.1, Lemma 2.10, and Definition 4.1, there exists a homeomorphism

\[ g: X \to Z(P_{a_j(k)}), h(g(\alpha)) = P_0 = P_{a_k} \]

such that \( g^{-1}(q_\alpha) = b_1. \) If \( A_{ij(k)} \) consists of exactly one point, then the mapping \( q_\alpha: X \to Z(q_\alpha) \) is a homeomorphism and \( g(A_{ij(k)}) = \{b_i\}. \) Therefore \( h = g 	imes q_\alpha \) is a homeomorphism and condition (3) holds. Clearly, the case when \( A_{ij(k)} \) is \( 0 \)-dimensional but not of cardinality 1 can be settled as above.

6.2. THEOREM. If \( f: X \to Y \) is a closed mapping of a space \( X \) onto a space \( Y \), then:

(a) If sup \( \{ \text{dim} f^{-1}(y): \gamma \in Y, k = 0, 1, 2, \ldots \} \), then

\[ D(X) \leq D(Y) + k. \]

(b) If \( f^{-1}(y) \) consists of no more than \( (k+1) \) points for each \( y \in Y \), then

\[ D(Y) \leq D(X) + k. \]

This theorem extends Hurewicz's formulas for finite-dimensional spaces.

Proof. (a) Let \( D(Y) = \beta \) and

\[ Y = \bigcup \{ B_\gamma: \gamma \leq J(\beta) \} \]

be a \( \beta \)-representation of \( Y \). We put

\[ A_\gamma = f^{-1}(B_\gamma). \]

Then

\[ X = \bigcup \{ A_\gamma: \gamma \leq J(\beta) = J(\beta + k) \}. \]
We will prove that (10) is a \((\beta+k)-D\)-representation. By Hurewicz's formula for finite-dimensional spaces (see [9]),
\[
\dim A_\beta \leq \dim B_\beta + k.
\]
In particular, for \(J(\beta) = J(\beta+k)\)
\[
\dim A_{J(\beta+k)} \leq \dim B_{J(\beta+k)} + K(\beta+k) = K(\beta+k).
\]
Therefore, conditions (a) and (b) of Definition 1.1 hold. Conditions (c) and (e) follow from (9). Hence, (10) is a \((\beta+k)-D\)-representation of \(X\) and inequality (6) holds.

(b) Let \(D(X) = \beta\) and let the equality
\[
X = \{B_\gamma : \gamma \leq J(\beta)\}
\]
be a \(\beta-D\)-representation of \(X\). We put
\[
A_\gamma = f(B_\gamma).
\]
Then
\[
f(X) = Y = \bigcup \{A_\gamma : \gamma \leq J(\beta) = J(\beta+k)\}.
\]
By virtue of Hurewicz's formula for finite-dimensional spaces (see [10])
\[
\dim A_\gamma \leq \dim B_\gamma + k, \quad \dim B_\gamma \leq \dim A_{J(\beta+k)} + k = K(\beta+k).
\]
Moreover, the sets \(A_\gamma\) are closed because \(f\) is a closed mapping and the sets \(B_\gamma\) are closed. Therefore, conditions (a), (b), (d) of Definition 1.1 hold. Condition (c) holds because \(f\) is a closed mapping. Condition (e) holds because the set \(f^{-1}(Y)\) is finite for every \(y \in Y\). Hence equality (12) is a \((\beta+k)-D\)-representation of \(Y\) and (7) holds.

6.3. Theorem. Let \(X\) be a compactum, then \(D(X) \leq \beta\) if and only if there exists a zero-dimensional mapping \(f : X \to P_\beta\).

Proof. We will use the following two assertions:

14. (See [8], Luxemburg, Lemma 8.7.) Let \(X\) be a compactum and (1) be its \(\beta-D\)-representation. We define a mapping
\[
\pi : X \to X_\beta
\]
as the identification of all points of the set \(A_{J(\beta)}\). We put
\[
p = \pi(A_{J(\beta)}).
\]
Then the equality
\[
X_\beta = \bigcup \{B_\gamma : \gamma \leq J(\beta)\}
\]
is a \(J(\beta)-D\)-representation of \(X_\beta\) and the set \(B_{J(\beta)}\) is a point \(p\). Furthermore, \(\pi\) is injective on \(X \setminus A_{J(\beta)}\).

16. (See [8], Luxemburg, Lemma 8.8.) Let \(U\) be an open set in \(X\), \(A = X \setminus U\). If \(f : X \to K\) and \(g : X \to T\) are mappings such that
\[
\dim (f^{-1}(x) \cap U) \leq 0, \quad \dim (g^{-1}(y) \cap A) \leq 0
\]
for \(y \in T, x \in K\),
then the mapping
\[
F : X \to K \times T
\]
defined by the equality
\[
F(x) = (f(x), g(x))
\]
is zero-dimensional.

Let (1) be a \(\beta-D\)-representation of \(X\) and let \(\pi\) be the mapping (15). Then, by virtue of assertion (14) and Theorem 6.1, there exists an embedding
\[
h : X_\beta \to P_\beta \quad (\pi = J(\beta)).
\]
Therefore, by virtue of (14), the mapping
\[
q = h \circ \pi : X \to P_\beta
\]
is injective and consequently, zero-dimensional on \(U = X \setminus A_{J(\beta)}\). Since \(\dim A_{J(\beta)} \leq K(\beta) = n\) (condition (d) of Definition 1.1), there exists a zero-dimensional mapping \(r : A_{J(\beta)} \to I^n = n\)-dimensional cube \(I^n\) (see [11], Hurewicz). Let \(g : X \to I^n\) be any extension of \(r\). Then by virtue of (16), there exists a zero-dimensional mapping
\[
f : X \to P_\beta \times I^n = P_{\beta+n} = P_\beta
\]
which is defined by the equality:
\[
f(x) = (g(x), g(x)).
\]
On the other hand, let \(f : X \to P_\beta\) be a zero-dimensional mapping, then by Theorem 6.2 and Lemma 5.2.
\[
D(X) \leq D(P_\beta) \leq \beta.
\]
Proof. Of Theorem 1.2. Let \(R\) be a compactum and \(Z(R)\) be the class of all compacta \(X\) having a zero-dimensional mapping \(f : X \to R\). By virtue of [12], Pasynkov, Theorem 8.8, there is a universal element in the class \(Z(R)\). Let \(D_\beta\) be a universal element in the class \(Z(P_\beta)\). Then our theorem follows from Theorem 6.3.

6.4. Corollary. \(D(P_\beta) = D(D_\beta) = \beta\).

Proof. Since for any \(\beta\) there exists a compact space \(X\) with \(D(X) = \beta\) (see [1], Henderson), \(D(D_\beta) \geq D(X) = \beta\). By the definition of \(D_\beta\) there exists a zero-dimensional mapping \(F : X \to D_\beta \to P_\beta\). Therefore, by Theorem 6.2 and Lemma 5.2
\[
D(D_\beta) \leq D(P_\beta) \leq \beta.
\]

§ 7. Universal spaces for noncompact separable spaces. As mentioned in § 1, the universal element in the class of compact spaces \(X\) with \(D(X) \leq \beta\) does not coincide with the one in the class of separable spaces with \(D\)-dimension \(\leq \beta\) for \(\beta \geq \alpha_0\). To prove Theorem 1.3 we need some preliminary lemmas.
7.1. Lemma. Let the equality
\[ X = \bigcup \{ A:\gamma \leq J(\beta) \} , \quad J(\beta) = \alpha \]
be a \( \beta \)-D-representation of a space \( X \) and let \( M \subset A_{J(\beta)} \) be an arbitrary set of dimension \( \text{Ind} \, M = \alpha \). Then the equality
\[ (X \setminus A_{J(\beta)}) \cup M = \bigcup \{ B_\gamma : \gamma \leq J(\beta) \} \]
where \( B_\gamma = (A_\gamma \cap (X \setminus A_{J(\beta)})) \cup M \), is an \( \alpha + n \)-D-representation of \( (X \setminus A_{J(\beta)}) \cup M \).

The lemma is evident. \( \blacksquare \)

7.2. Definition. Let \( \beta = \alpha + n \), \( \alpha = J(\beta) \), \( n = K(\beta) \)
and \( A_\beta \) be a universal \( n \)-dimensional compact space. Then (see [5], Bothe), there exists an \( \alpha + n \)-dimensional compact 
\( \mathcal{R} \subset A_\beta \) such that \( \mathcal{R} \subset A_\beta \subset A \). Let \( g_\delta \in \mathcal{R}_\delta \) be a fixed point (see Construction 5.1). Let \( \pi_\gamma : P_\delta \times \mathcal{R} \to P_\delta \), \( \pi_\alpha : P_\alpha \times \mathcal{R} \to P_\alpha \) be projections. Then we put
\[ S_\delta = P_\delta \times \mathcal{R} \quad \pi_\gamma (x) = g_\delta, \quad \pi_\alpha (x) = (x \in P_\alpha, x \in P_\delta \times \mathcal{R}) \quad \text{in (5)} \]

7.3. Lemma. \( D(S_\delta) \leq \beta \).

Proof. Let (8) ([5]) be an \( \alpha \)-D-representation of \( P_\alpha \) satisfying the conditions of Lemma 5.3. Then, by Corollary 3.9, the equality
\[ P_\alpha \times \mathcal{R} = \bigcup \{ B_\gamma : A_\gamma \times \mathcal{R} : \gamma \leq J(\alpha + \beta) \} \]
is an \( \alpha + \beta \)-D-representation of the space \( P_\alpha \times \mathcal{R} \). We put
\[ M = \{ x : x \in P_\alpha \times \mathcal{R}, \pi_\gamma (x) \in A_\alpha, \pi_\alpha (x) = g_\delta \} \]
Then \( M \) is homeomorphic to \( A_\alpha \) and consequently \( \text{dim} \, M = \alpha \). By Lemma 7.1 the equality
\[ S_\delta = (P_\delta \times \mathcal{R} \setminus (A_\delta \times \mathcal{R})) \cup M = \bigcup \{ C_\gamma : \gamma \leq J(\alpha + n) \} \quad \text{(4)} \]
where \( C_\gamma = (P_\delta \times \mathcal{R} \setminus (A_\delta \times \mathcal{R})) \cup B_\gamma \cup M \), is an \( \alpha + n \)-D-representation of \( S_\delta \). Therefore \( D(S_\delta) \leq \alpha + n = \beta \). \( \blacksquare \)

7.4. Lemma. If \( D(X) \leq \beta \) and \( X \) is separable, then there exists an embedding \( f : X \to \mathcal{S}_\delta \).

Proof. Suppose condition (2) holds. Then, by Lemma 8.9 in [3], Luxemburg, there exists a compactum \( \mathcal{K} \supset X \) such that
\[ \mathcal{K} = \mathcal{R} \cup \mathcal{B}, \quad \mathcal{B} \cap \mathcal{R} = \emptyset \quad \text{(2)} \]
\[ X \supset \mathcal{B} \cup A_\alpha \quad (A_\alpha \subset \mathcal{R}) \quad \text{(3)} \]
We use here the notation of Definition 7.2.

(6) \( H \) is an open set such that there exists a family of compact sets \( \{ H_\beta \} \), simple with respect to \( H \), such that \( \text{D}(H_\beta) < \alpha \).

Let \( \pi : K \to L \) be a mapping onto the quotient compactum \( L \) which we obtain by identification of all points of \( \mathcal{R} \subset K \); we let \( (\mathcal{R}, \mathcal{L}) = \{ l \} \subset L \). Then by condition (6), Lemma 3.5, and Theorem 6.1, there exists a homeomorphism \( g : L \to P_\alpha \) such that
\[ g(l) = g_\delta \subset P_\delta \]
Since \( \mathcal{R} \subset \mathcal{S}_\delta \), there exists a retraction:
\[ r : K \to R_{\mathcal{R}} \]
We define mapping \( F : \mathcal{K} \to \mathcal{P}_\alpha \times \mathcal{R} \) by the equality:
\[ F = (g \circ \pi, r) \]
From conditions (5) and (7) it follows that
\[ \mathcal{F}(\mathcal{X}) = \mathcal{S}_\delta \subset \mathcal{P}_\delta \times \mathcal{R} \]
Since \( g \) is a homeomorphism and \( \pi \) is injective on \( H \), the composition \( g \circ \pi \) is injective on \( H \). Further, \( r \) is injective on \( \mathcal{R} \). Therefore, by virtue of (4), (6), \( F \) is injective on \( K \). Since \( K \) is compact, \( F \) is a homeomorphism. Let \( f \) be a restriction of \( F \) to \( X \), then \( f \) is also a homeomorphism and by virtue of (9), \( f(\mathcal{X}) = \mathcal{S}_\delta \).

Proof of Theorem 1.3. Since \( \mathcal{S}_\delta \) is contained in the compactum \( \mathcal{P}_\delta \times \mathcal{R} \), it is a separable space. Our theorem now follows from Lemmas 7.3 and 7.4.

§ 8. On compactifications. For any separable space \( X \) there exists a compactification \( \mathcal{K} \supset X \) such that
\[ D(\mathcal{K}) \leq D(\mathcal{X}) + 1 \]
(see [13], Kozlovsky, and [3], Luxemburg, for the proof). Moreover, for any separable finite-dimensional space \( X \) there exists a compactification \( \mathcal{K} \supset X \) such that
\[ \text{dim} \, \mathcal{K} \leq \text{dim} \, \mathcal{X} + 1 \]
(see [5], Bothe). The following theorem is an extension of both these results.

8.1. Theorem. For each \( \beta < \omega_1 \) there exists a compactum \( \mathcal{Q}_\beta \subset \mathcal{K} \) such that
\[ D(\mathcal{Q}_\beta) \leq \beta + 1 \]
and \( \mathcal{Q}_\beta \) contains a homeomorphic image of any separable space \( Y \) with \( D(Y) \leq \beta \).

Proof. We put
\[ \mathcal{Q}_\beta = \mathcal{P}_\alpha \times \mathcal{R} \quad (A \subset \mathcal{R}) \quad \text{(3)} \]
(see Construction 5.1 and Definition 7.2). We have proved in § 7 (see (3)) that
\[ D(\mathcal{Q}_\beta) \leq \alpha + n + \beta = \beta + 1 \]
Since \( \mathcal{Q}_\beta = \mathcal{S}_\delta \), it follows from Lemma 7.4 that \( \mathcal{Q}_\beta \) contains a homeomorphic image of each separable space \( Y \) with \( D(Y) \leq \beta \). Moreover, \( \mathcal{Q}_\beta \subset \mathcal{K} \) because \( \mathcal{P}_\alpha \subset \mathcal{K} \) and \( \mathcal{R} \subset \mathcal{S}_\delta \). \( \blacksquare \)
It is also easy to prove that $D(Q) = \beta + 1$

8.2. Definition (see [14], Zarelua). A mapping $f \colon X \to Y$ is called scattering if for each point $x \in X$ and for each neighborhood $V$ of $x$ there exists a neighborhood $U$ of $f(x)$, $U \subseteq Y$, such that there exist open sets $P$ and $W$ satisfying the conditions:

$$f^{-1}(U) = P \cup W, \quad P \cap W = \emptyset, \quad x \in P \subseteq V.$$

It is easy to prove that for compact spaces the class of all zero-dimensional mappings coincides with the class of all scattering mappings. Therefore, by Theorem 6.2

(1) If $f \colon X \to Y$ is a scattering mapping and $X$ and $Y$ are compact, then $D(X) \leq D(Y)$.

As we mentioned above a separable space need not have a compactification with the same $D$-dimension (see [1]).

The following theorem gives a necessary and sufficient condition for the existence of such a compactification.

8.3. Theorem. Let $X$ be a separable space with $D(X) = \beta$. Then the existence of a compactum $K \subseteq X$ such that $D(K) = D(X)$ is equivalent to the existence of a scattering mapping $f \colon X \to P_\beta$.

Proof. Let $f \colon X \to P_\beta$ be a scattering mapping. Then by virtue of [14], Corollary 5, Zarelua, there exists a compactification $K \subseteq X$ and a scattering extension $f \colon K \to P_\beta$ of the mapping $f$. Therefore, by virtue of (1), $D(K) \leq D(P_\beta) = \beta$. Moreover, $D(K) = D(X) = \beta$ because $K \subseteq X$. We have thus proved that our condition is sufficient. Let $K$ be a compactification of $X$ with $D(K) = D(X) = \beta$. Then, by Theorem 6.3, there exists a zero-dimensional and, consequently, scattering, mapping $f \colon K \to P_\beta$. Let $g \colon X \to P_\beta$ be a restriction of $f$ to $X$. Then clearly $g$ is also scattering.

8.4. Problem. Let $\mathcal{S}$ be a class of all separable spaces and let $\mathcal{X}$ be a class of all compact spaces. In each of these classes consider two subclasses for $\omega \leq \alpha \leq \omega_1$.

1. Spaces $X$ with $\text{ind} X \leq \alpha$.

2. Spaces $X$ with $\text{Ind} X \leq \alpha$.

So we get four classes of spaces for each $\alpha$. Do there exist universal elements in these classes?

8.5. Remark. According to [2], § 8, Luxemburg, there exist compact spaces $X$ having $\text{Ind} X = \alpha$ and an arbitrarily large $D(X) < \omega_1$. Therefore, there are no universal elements in the class of compact spaces $X$ having $D(X) < \alpha$ and $\text{Ind} X \leq \alpha$. Analogous results could be obtained for the other three classes in Problem 8.4.

References


