Dimension of convex hyperspaces

by

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Abstract. It is shown that the rank of a convexity on a separable metric space and the dimension of its convex hyperspace are equal under natural conditions. The structure of the convex hyperspace is investigated with the aid of an induced convexity.

0. Introduction. It was shown in [NQS] that the hyperspace of convex sets of a compact linearly convex set C is homeomorphic to the Hilbert cube provided that C is locally convex, metrizable, and of dimension >1. Sharply contrasting is the fact that the convex hyperspace of a square equipped with a "convex structure" of subsquares is homeomorphic to the 4-cube. These and other examples lead to the question which numerical invariant of an abstract topological convexity is responsible for the dimension of the corresponding convex hyperspace.

In regard of the philosophy that dimension is some sort of freedom degree, it is natural to concentrate on the *degree of variation* that convex sets are allowed to have. There is a simple way to measure the latter: the *rank* of a convexity, which is defined to be the supremum of all n such that there exists a free set with n points. Recall [V_6] that a set is free if no one of its points is in the convex hull of the other ones. The notion of rank comes from [J_3]. The idea goes back to a notion of "closure-independency" in universal algebra, [M]. Rank has been investigated at length in [V_6]. The main result of the present paper is that for separable metrizable S_4-convexities with connected convex sets and with compact polytopes, *rank equals the dimension of the convex hyperspace.*

For the obtaining of a proof it was helpful to equip the convex hyperspace with a convexity which is naturally induced by the original one. Section 1 below is largely concerned with a proof that this "lifting" operation preserves the above mentioned class of convexities. Inside this class, the dimension of the underlying space equals the "convex dimension" of the structure, as was shown in [V_5]. The latter dimension function is used for a simple proof that the dimension of the convex hyperspace does not exceed the rank of the original convexity. The opposite inequality comes from considerations involving a theorem of Hurewicz on the dimension-lowering of maps. See Section 2.

As a particular application we obtain Duda's theorem (initiated by Kelley) on the dimension of the hyperspace of continua of a tree-like space, [D], [K]. We also use convex hyperspaces to determine the *Radon number* of binary convexities of minimal rank. For other applications of convex hyperspaces, see [V_5] and [V_7].

The notation and terminology of our previous paper [V_6] will be continued. We are now concerned with uniformizable or metrizable convexities as designed in [V_5]. Let μ be a (diagonal) uniformity on X and let $\mathscr{C}$ be a topological convexity on X. We say that μ is *compatible* with $\mathscr{C}$ (or: μ is a *uniformity* for $(X, \mathscr{C})$) if for each $U \in \mu$ there is an "associated" $V \in \mu$ with $hV[C] \subset U(C)$ for each $C \in \mathscr{C}$. Recall that h denotes the *convex hull operator of* $(X, \mathscr{C})$ and that for a relation $R \subset X \times X$, $R[A]$ denotes the set

$$\{x|\ \exists a \in A\colon (a, x) \in R\} \qquad (A \subset X).$$

If there exists a (metrizable) uniformity on X compatible with the convexity $\mathscr{C}$, then $(X, \mathscr{C})$ is called a *uniformizable* (*metrizable*) convex structure. See [V_5] for an alternative description in terms of covering uniformities.

1. The convex hyperspace. Recall that the *hyperspace* $H(X)$ of a topological space X is the set of all nonempty closed subsets of X equipped with the topology, generated by the closed subbase of all sets of type ($C \subset X$ closed)

$$\langle C\rangle = \{A \in H(X)|\ A \subset C\};$$
$$\langle C, X\rangle = \{A \in H(X)|\ A \cap C \neq \emptyset\}.$$

If X is not compact, then one usually considers the subspace $H_c(X)$ of $H(X)$, consisting of all compact $A \in H(X)$.

If $(X, \mathscr{C})$ is a topological convex structure, then $\mathscr{C}_c^* \subset \mathscr{C}^*$ will denote the set of all *compact* C in $\mathscr{C}^*$. Note that $\mathscr{C}^* \subset H(X)$, $\mathscr{C}_c^* \subset H_c(X)$. The set $\mathscr{C}_c^*$, equipped with the relative topology, will be called the *hyperspace of compact convex sets*, or simply, the *convex hyperspace*.

1.1. Induced convexity on convex hyperspaces. In studying the dimension of $\mathscr{C}_c^*$ it is helpful to consider a natural convexity $\mathscr{H}(\mathscr{C})$ on $\mathscr{C}_c^*$ which is said to be *induced by* $\mathscr{C}$: $\mathscr{H}(\mathscr{C})$ is the (topological) convexity generated by the collection of all (closed) sets of type

$$\langle C\rangle \cap \mathscr{C}_c^* \quad \text{or} \quad \langle C, X\rangle \cap \mathscr{C}_c^*, \qquad C \in \mathscr{C}^*.$$

Such a convexity was already considered (in more restrictive circumstances) in [vMV_1], [vMV_2] and [V_1].

The convex hull operator of $\mathscr{H}(\mathscr{C})$ will again be denoted by h (the argument of h will make it clear which space is involved). As $\mathscr{H}(\mathscr{C})$ is described in terms of subbase sets only, it is somewhat diffcult to give a direct formula for the hull of a subset of $\mathscr{C}_c^*$. However, by [J_1, p. 8, 9] every nonempty polytope is the intersection of a family of subbasic sets. Hence, if

$$\mathscr{A} = \{A_1, \ldots, A_n\} \subset \mathscr{C}_c^*,$$

then obviously

$$h(\mathscr{A}) = \langle h^*(\textstyle\bigcup\mathscr{A})\rangle \cap \bigcap\{\langle C, X\rangle|\ \mathscr{A} \subset \langle C, X\rangle,\ C \in \mathscr{C}^*\} \cap \mathscr{C}_c^*,$$

where h^* stands for the *convex closure operator* of $(X, \mathscr{C})$ (cf. [V_2, 1.2]).

There are two ways to simplify this formula. First note that $\mathscr{A} \subset \langle C, X\rangle$ means exactly that C meets all members of $\mathscr{A}$, i.e. C is a *transversal set* of $\mathscr{A}$ (terminology of [vMV_2]). Let $\perp(\mathscr{B})$ denote the collection of all closed convex sets transversal to $\mathscr{B}$, and let

$$\perp_c(\mathscr{B}) = \perp(\mathscr{B}) \cap \mathscr{C}_c^*.$$

Then obviously,

$$(1) \qquad h(\mathscr{A}) = \langle h^*(\textstyle\bigcup\mathscr{A})\rangle \cap \perp_c\perp(\mathscr{A}) \qquad (\mathscr{A} \text{ finite}).$$

A generalization of this formula to non-finite $\mathscr{A}$ is obtained in 1.6 below.

Next, note that if $\mathscr{A}$ is finite and if $C \in \perp(\mathscr{A})$ then there exists a point $x(A)$ in $C \cap A$ for each $A \in \mathscr{A}$, and the polytope $h\{x(A)|\ A \in \mathscr{A}\}$ is a transversal set of $\mathscr{A}$ included in C. Hence if $\mathscr{A} = \{A_1, \ldots, A_n\}$ then

$$(2) \qquad h(\mathscr{A}) = \{A \in \mathscr{C}_c^*|\ A \subset h^*(\textstyle\bigcup\mathscr{A});\ A \cap h\{a_1, \ldots, a_n\} \neq \emptyset \text{ if } a_1 \in A_1, \ldots, a_n \in A_n\}.$$

In many cases these formulas facilitate to decide on the membership of $h(\mathscr{A})$. This will be done without further reference to (1) or (2).

1.2. THEOREM. *Let $(X, \mathscr{C})$ be a closure-stable convex structure such that*
(1) *every two distinct points in X can be screened with convex closed sets;*
(2) *if C_1, C_2 are compact convex, then so is $h^*(C_1 \cup C_2)$.*

If $\mathscr{A} \subset \mathscr{C}_c^*$ is a closed half-space, then there exists a closed half-space C of X such that

$$\mathscr{A} = \langle C\rangle \cap \mathscr{C}_c^* \quad \text{or} \quad \mathscr{A} = \langle C, X\rangle \cap \mathscr{C}_c^*.$$

Proof of 1.2. We first derive some intermediate results.

(I) *If $A \in \mathscr{A}$ is minimal with respect to inclusion, then A consists of only one point.* Indeed, let $x_1 \neq x_2$ be in A. By (1) above there exist $C_1, C_2 \in \mathscr{C}^*$ such that

$$x_1 \in C_1 \setminus C_2, \quad x_2 \in C_2 \setminus C_1, \quad C_1 \cup C_2 = X.$$

Then $A \cap C_1$, $A \cap C_2$ are in $\mathscr{C}_c^*$ and both are proper subsets of A. Hence by minimality

$$A \cap C_1 \notin \mathscr{A}, \quad A \cap C_2 \notin \mathscr{A}.$$

As $\mathscr{C}_c^* \setminus \mathscr{A}$ is convex, we find that

$$A \in h\{A \cap C_1, A \cap C_2\} \subset \mathscr{C}_c^* \setminus \mathscr{A},$$

a contradiction.

Let A_0 be the set of all $x \in X$ with $\{x\} \in \mathscr{A}$. Note that A_0 is closed in X since $\mathscr{A}$ is closed in $\mathscr{C}_c^*$.

(II) A_0 *is a closed half-space of* X. Indeed, let $x_1, \ldots, x_n \in \mathscr{A}_0$. Then

$$\langle h\{x_1, \ldots, x_n\}\rangle \cap \mathscr{C}_c^* = h\{\{x_1\}, \ldots, \{x_n\}\} \subset \mathscr{A}.$$

Hence if $x \in h\{x_1, \ldots, x_n\}$, then $\{x\} \in \mathscr{A}$ and consequently $x \in A_0$. This proves convexity of A_0, and convexity of $X \setminus A_0$ is proved in a similar way.

(III) $\langle A_0\rangle \cap \mathscr{C}_c^* \subset \mathscr{A} \subset \langle A_0, x\rangle \cap \mathscr{C}_c^*$. Let $B \in \langle A_0\rangle \cap \mathscr{C}_c^*$. If $F \subset B$ then the polytope $h(F) \subset B$ is compact and

$$\langle h(F)\rangle \cap \mathscr{C}_c^* = h\{\{x\} \mid x \in F\} \subset \mathscr{A}.$$

In particular, $h(F) \in \mathscr{A}$. As B is the limit of the (upward filtered) net $h(F)$, $F \subset B$ finite, and as $\mathscr{A}$ is closed, we find that $B \in \mathscr{A}$. Let B be in $\mathscr{A}$ now. As B is compact and as $\mathscr{A}$ is closed, there is a minimal $B' \subset B$ with $B' \in \mathscr{A}$. By (I), $B' = \{x\}$ for some $x \in X$, and hence B meets A_0 in x.

We now show that one of the inclusions in (III) must be an equality. We distinguish between two possibilities.

First case: $\bigcup \mathscr{A} = X$. Let $D \in \langle A_0, X\rangle \cap \mathscr{C}_c^*$. Then D meets A_0, and consequently the family

$$\mathscr{D} = \{A \mid A \subset D,\ A \in \mathscr{A}\}$$

is nonempty. Let $\mathscr{D}' \subset \mathscr{D}$ be a chain with respect to inclusion. Then $\bigcup \mathscr{D}$ is convex, and hence $\mathrm{Cl}(\bigcup \mathscr{D}')$ is compact and convex. This set is also the limit of the net $\mathscr{D}' \subset \mathscr{A}$, and hence $\mathrm{Cl}(\bigcup \mathscr{D}') \in \mathscr{A}$. This shows that $\mathscr{D}$ is inductively ordered upwards by inclusion, and by Zorn's lemma there is a maximal element $D_0 \in \mathscr{D}$. If $D_0 \neq D$ then fix a point $x \in D \setminus D_0$. As $\bigcup \mathscr{A} = X$, there is an $A \in \mathscr{A}$ with $x \in A$. Then

$$B = h^*(A \cup D_0) \in h\{A, D_0\} \subset \mathscr{A},$$

and as

$$D_0 \subset B \cap D \subset B,$$

we may conclude that

$$B \cap D \in h\{D_0, B\} \subset \mathscr{A}.$$

However, $x \in B \cap D$, and hence $B \cap D$ properly includes D_0, a contradiction. Applying the second inclusion of (III) then completes the proof that

$$\mathscr{A} = \langle A_0, X\rangle \cap \mathscr{C}_c^*.$$

Second case: $\bigcup \mathscr{A} \neq X$. Fix a point $x \in X \setminus \bigcup \mathscr{A}$. Note that the case $\mathscr{A} = \varnothing$ is a trivial one, and it will be excluded by now. Then $A_0 \neq \varnothing$: just minimize some (compact) member of $\mathscr{A}$. Fix a point $y \in A_0$. We now show that $\bigcup \mathscr{A} \subset A_0$. Suppose to the contrary that there exists a point $z \in \bigcup \mathscr{A} \setminus A_0$. We then have

$$\{z\} \notin \mathscr{A} \qquad (\text{since } z \notin A_0);$$
$$h\{x, y\} \notin \mathscr{A} \qquad (\text{since } h\{x, y\} \not\subset \textstyle\bigcup \mathscr{A}),$$

and polytopes in X are compact by (2). Hence

$$(*) \qquad h\{y, z\} \in h\{\{z\}, h\{x, y\}\} \subset \mathscr{C}^* \setminus \mathscr{A}.$$

On the other hand there is an $A \in \mathscr{A}$ with $z \in A$. Then

$$\{y\} \subset h\{y, z\} \subset h^*(A \cup \{y\}),$$

where $\{y\} \in \mathscr{A}$ and

$$h^*(A \cup \{y\}) \in h(A, \{y\}) \subset \mathscr{A}.$$

Hence $h\{y, z\} \in \mathscr{A}$, in contradiction with $(*)$.

Having shown that $\bigcup \mathscr{A} \subset A_0$, it follows that $\mathscr{A} \subset \langle A_0\rangle \cap \mathscr{C}_c^*$ and equality follows from (III). ■

We note that if $C \subset X$ is a closed half-space, then $\langle C\rangle \cap \mathscr{C}_c^*$ is a half-space of $\mathscr{C}_c^*$ as one can easily verify. However, $\langle C, X\rangle$ is a half-space of $\mathscr{C}_c^*$ if and only if it fullfils the following (rather mild) extra condition: if A_1, A_2 are compact convex sets disjoint with C, then $h^*(A_1 \cup A_2)$ is also disjoint with C.

The latter condition is fulfilled if, for instance, C is of type $f^{-1}[0, t]$, where f is a convexity-preserving (C.P.) map $X \to [0, 1]$. In case $(X, \mathscr{C})$ is at least semi-regular, there is an abundancy of such sets, and Theorem 1.2 then gives a fairly adequate picture of the closed half-spaces of a convex hyperspace.

1.3. THEOREM. *Let* $(X, \mathscr{C})$ *be a closure-stable convex structure such that*

(1) *if* $C \in \mathscr{C}_c^*$ *and* $x \in X \setminus C$, *then there is a* $C' \in \mathscr{C}^*$ *with* $C \subset \mathrm{int}\, C'$, $x \notin C'$;

(2) *if* $C_1, C_2 \in \mathscr{C}_c^*$, *then also* $h^*(C_1 \cup C_2) \in \mathscr{C}_c^*$.

If all convex sets in X *are connected, then the same is true in* $\mathscr{C}_c^*$.

Proof. Let $\mathscr{A} \subset \mathscr{C}_c^*$ be nonempty convex, and let $A, B \in \mathscr{A}$. Then by (2),

$$C = h^*(A \cup B) \in h\{A, B\} \subset \mathscr{A}.$$

Let $\mathscr{D} \subset \mathscr{C}_c^*$ be maximal with the following properties:

(3) $\mathscr{D}$ is totally ordered under inclusion;

(4) $\forall D \in \mathscr{D}$: $A \subset D \subset C$.

Then $\mathscr{D}$ is densely ordered under inclusion: let $D_1 \subset D_2$ be distinct members of $\mathscr{D}$, and fix $x \in D_2 \setminus D_1$. By (1) there is a $D \in \mathscr{C}^*$ with $D_1 \subset \mathrm{int}\, D$; $x \notin D$. As D_2 is connected, we find that D_1 is properly included in $D_2 \cap D$. By construction, $D_2 \cap D$ is properly included in D_2. Therefore, $\mathscr{D}$ cannot have neighbours.

We next show that $\mathscr{D}$ is also *maximal in* $H_c(X)$ with the above properties (3) and (4): let $Y \in H_c(X)$ be such that $A \subset Y \subset C$, and $\mathscr{D} \cup \{Y\}$ is totally ordered under inclusion. Then

$$\varnothing \neq \mathscr{D}_1 = \{D \in \mathscr{D} \mid D \subset Y\};$$
$$\varnothing \neq \mathscr{D}_2 = \{D \in \mathscr{D} \mid Y \subset D\}.$$

Then $\bigcup \mathscr{D}_1$ is a (nonempty) convex set, and hence $\mathrm{Cl}(\bigcup \mathscr{D}_1) \subset Y$ is a compact convex set. Also, $\bigcap \mathscr{D}_2 \in \mathscr{C}_c^*$, and both $\mathrm{Cl}(\bigcup \mathscr{D}_1)$, $\bigcap \mathscr{D}_2$ are in $\mathscr{D}$ since $\mathscr{D}$ is maximal in $\mathscr{C}_c^*$. As $\mathscr{D}$ has no neighbours, we find that

$$\mathrm{Cl}(\textstyle\bigcup \mathscr{D}_1) = Y = \bigcap \mathscr{D}_2.$$

As $\mathscr{D}$ is closed in $\langle C\rangle \subset H_c(X)$ by maximality, we find that $\mathscr{D}$ is a compact densely ordered space, whence $\mathscr{D}$ is connected. Note that

$$\mathscr{D} \subset h\{A, C\} \subset \mathscr{A}.$$

One similarly finds a subcontinuum in $\mathscr{A}$ joining B with C, showing that $\mathscr{A}$ is connected. ■

See [J_2, theorem 6] for a related result.

We finally concentrate on the separation properties of a hyperspace convexity. If the latter convex structure is semi-regular, then each of its convex closed subsets is the intersection of a family of closed half-spaces, which by Theorem 1.2 are of the "subbase" type. Although the canonical subbase of a hyperspace convexity is a rather rich one, there is no indication that the above statement be true in general.

The most convenient way to get around these (and other) problems is to restrict attention to *uniformizable* convexities: by [V_5, 2 5] a uniformizable S_4-convexity is automatically semi-regular, and even regular if polytopes are compact. The S_4 property is considerably easier to check, in view of results in [J_1] or in [V_2].

Let us first discuss some notation. Let μ be a (diagonal) uniformity on X. The topology of $H_c(X)$ is then induced by a uniformity $[\mu]$ with a base of diagonal entourages of type $[V]$, $V \in \mu$, where

$$[V] = \{(A, B)|\ A \subset V[B], B \subset V[A]\}.$$

If (A, B) is in $[V]$, then we also say that A and B are *V-close*.

In case X is equipped with a topological convexity $\mathscr{C}$ then we will be interested in the space $\mathscr{C}_c^*$ rather than $H_c(X)$. We therefore use the above notation $[\mu]$, $[V]$ relative to the subspace $\mathscr{C}_c^*$ of $H_c(X)$.

We note that $[\mu]$ is a metrizable uniformity if μ is.

1.4. THEOREM. *Let $(X, \mathscr{C})$ be a uniformizable convex structure such that, for each two compact convex sets C_1, C_2, the convex closure of $C_1 \cup C_2$ is compact. Then $(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))$ is also uniformizable. In fact, if μ is a uniformity for $(X, \mathscr{C})$, then $[\mu]$ is a uniformity for $(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))$. In particular, $(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))$ is metrizable if $(X, \mathscr{C})$ is.*

Proof. Let $U \in \mu$, and consider a sequence of symmetric uniform diagonal neighbourhoods

$$W \subset V \subset U' \subset \overline{U}' \subset U$$

where W, resp. V, are associated to V, resp. U'. We will show that $[W]$ is associated to $[U]$, that is, for each convex $\mathscr{D} \subset \mathscr{C}_c^*$,

(1) $$h([W][\mathscr{D}]) \subset [U][\mathscr{D}].$$

Note that $\mathscr{D}$ is the union of the upward filtered collection of all polytopes included in $\mathscr{D}$, and it directly follows that (1) is valid if it can be obtained *for polytopes only*.

To this end, let $C_1, \ldots, C_n \in \mathscr{C}_c^*$, and let

$$D \in h([W][h\{C_1, \ldots, C_n\}]).$$

We put

$$E_1 = h^*(V[D]), \quad E_2 = h^*(\bigcup_{i=1}^{n} C_i),$$
$$E = E_1 \cap E_2.$$

Note that E_2 is compact convex, and hence that E is compact convex. In order to show that

$$D \in [U][h\{C_1, \ldots, C_n\}],$$

we will prove that

(2) D and E are U-close,
(3) $E \in h\{C_1, \ldots, C_n\}$.

Proof of (2). There exist compact convex sets $D_1, \ldots, D_p$ with

(4) $D_1, \ldots, D_p \in [W][h\{C_1, \ldots, C_n\}]$;
(5) $D \in h\{D_1, \ldots, D_p\}$.

By (4) there exist compact convex sets $A_1, \ldots, A_p$ with

(6) $A_1, \ldots, A_p \in h\{C_1, \ldots, C_n\}$;
(7) A_i is W-close to D_i, $i = 1, \ldots, p$.

We then have

$$\bigcup_{j=1}^{p} D_j \subset W[\bigcup_{j=1}^{p} A_j],$$
$$\bigcup_{j=1}^{p} A_j \subset h^*(\bigcup_{i=1}^{n} C_i) = E_2,$$

whence by (5),

$$D \subset h^*(\bigcup_{j=1}^{p} D_j) \subset h^*(W[E_2]) \subset \mathrm{Cl}(V[E_2]).$$

Hence if $x \in D$ then there is a point in E_2 which is $\overline{V}$-close to x. It follows that

$$\emptyset \neq \overline{V}[\{x\}] \cap E_2 = \overline{V}[\{x\}] \cap E,$$

since $\overline{V}[\{x\}] \subset E_1$. Consequently,

$$D \subset \overline{V}[E] \subset U[E].$$

On the other hand,

$$E \subset E_1 = h^*(V[D]) \subset \mathrm{Cl}(U'[D]) \subset U[D],$$

establishing (2).

Proof of (3). We have

$$E \subset E_2 = h^*(\bigcup_{i=1}^{n} C_i),$$

and it remains to be verified that

(8) if $c_1 \in C_1, \ldots, c_n \in C_n$, then $E \cap h\{c_1, \ldots, c_n\} \neq \emptyset$.

By (6) there exist points

$$x_j \in A_j \cap h\{c_1, \ldots, c_n\}, \quad j = 1, \ldots, p.$$

Then

$$h\{x_1, \ldots, x_p\} \subset h\{c_1, \ldots, c_n\} \subset E_2,$$

and it suffices to show that E_1 meets $h\{x_1, \ldots, x_p\}$. By (7) we have for each $j = 1, \ldots, p$ that $A_j \subset W[D_j]$, and hence that there is a point $y_j \in D_j$ which is W-close to x_j. Then

$$\{y_1, \ldots, y_p\} \subset W[\{x_1, \ldots, x_p\}] \subset W[h\{x_1, \ldots, x_p\}],$$

and consequently

$$(9) \qquad h\{y_1, \ldots, y_p\} \subset h(W[h\{x_1, \ldots, x_p\}]) \subset V[h\{x_1, \ldots, x_p\}].$$

By (5), D meets the former set of (9), and hence

$$D \cap V[h\{x_1, \ldots, x_p\}] \neq \emptyset.$$

Then also

$$\emptyset \neq V[D] \cap h\{x_1, \ldots, x_p\} \subset E_1 \cap h\{x_1, \ldots, x_p\},$$

establishing (8).

As the sets of type $[U]$, $U \in \mu$, form a base for $[\mu]$, it follows that the uniformity $[\mu]$ is compatible with $\mathscr{H}(\mathscr{C})$. ■

It was shown in [V_1, theorem 3] that if X is compact and if $\mathscr{C}$ is a normal convexity on X with $\mathscr{C}^*$ closed in $H(X)$, then the induced convexity on $\mathscr{C}^*$ enjoys the same properties. It appears that this result is a particular case of the present one since it was proved in [V_5, 3.3] that a convexity $\mathscr{C}$ on a compact space X is uniformizable iff $\mathscr{C}^*$ is closed in $H(X)$ and for each $C \in \mathscr{C}^*$, $x \in X \setminus C$ there is a $D \in \mathscr{C}^*$ with

$$C \subset \operatorname{int} D; \quad x \notin D.$$

We now obtain the following result on separation:

1.5. Corollary. *Let $(X, \mathscr{C})$ be a uniformizable S_4 convex structure with $h^*(C_1 \cup C_2)$ compact whenever C_1, C_2 are compact convex. Then $\mathscr{H}(\mathscr{C})$ is an S_4-convexity with compact polytopes. In particular, $\mathscr{H}(\mathscr{C})$ is a regular convexity.*

Proof. Let

$$\mathscr{A} = \{A_1, \ldots, A_n\}, \quad \mathscr{B} = \{B_1, \ldots, B_m\}$$

be finite subsets of $\mathscr{C}_c^*$ with $h(\mathscr{A}) \cap h(\mathscr{B}) = \emptyset$. We put

$$A = h^*(\bigcup \mathscr{A}), \quad B = h^*(\bigcup \mathscr{B}).$$

Note that A and B are compact convex, and that

$$h(\mathscr{A}) = \langle A \rangle \cap \perp_c \perp(\mathscr{A}); \quad h(\mathscr{B}) = \langle B \rangle \cap \perp_c \perp(\mathscr{B}).$$

If $A \cap B$ meets all members of $\perp(\mathscr{A})$ and of $\perp(\mathscr{B})$, then $A \cap B$ is a common member of $h\mathscr{A}$ and $h\mathscr{B}$. Hence $A \cap B \cap C = \emptyset$ for some C in, say, $\perp\mathscr{A}$. Note that $A \cap C$ is another transversal set of $\mathscr{A}$, and we may therefore assume that $C \subset A$, $B \cap C = \emptyset$.

Let μ be a uniformity for $(X, \mathscr{C})$. Then the disjoint compact convex sets B, C are non-proximate, and by [V_5, 2.5] there exist convex closed sets B', C' of X with

$$B \subset B' \setminus C', \quad C \subset C' \setminus B', \quad B' \cup C' = X.$$

Clearly,

$$\langle B' \rangle \cup \langle C', X \rangle \supset \mathscr{C}_c^*, \quad \langle B \rangle \subset \langle B' \rangle \setminus \langle C', X \rangle.$$

As C' is also a transversal set of $\mathscr{A}$, we have $\mathscr{A} \subset \langle C', X \rangle$, and as the original set C is a transversal set of $\mathscr{A}$ disjoint with B' we also have $\mathscr{A} \cap \langle B' \rangle = \emptyset$. This shows that $\mathscr{A}$ and $\mathscr{B}$ can be "screened" with convex subsets $\langle C', X \rangle \cap \mathscr{C}_c^*$, $\langle B' \rangle \cap \cap \mathscr{C}_c^*$ of $\mathscr{C}_c^*$, and it follows from [V_2, 2.2] that $\mathscr{H}(\mathscr{C})$ is S_4.

We next show that polytopes in $\mathscr{C}_c^*$ are compact. Let

$$\mathscr{A} = \{A_1, \ldots, A_n\} \subset \mathscr{C}_c^*.$$

The convex closure A of $\bigcup_{i=1}^{n} A_i$ is compact convex again and the trace convexity $\mathscr{C} \upharpoonright A$ of $\mathscr{C}$ on A is obviously compatible with the trace uniformity of μ on A. Also,

$$(\mathscr{C} \upharpoonright A)^* = \langle A \rangle \cap \mathscr{C}_c^*,$$

and by [V_5, 2.4] the convex closure operator (restricted) $h^*\colon \langle A \rangle \to \langle A \rangle \cap \mathscr{C}_c^*$ is continuous. Hence $(\mathscr{C} \upharpoonright A)^*$ is a compact subset of $\mathscr{C}_c^*$, and the polytope $h(\mathscr{A})$ is a closed — and hence compact — subset of the former.

The second part of the theorem now follows from [V_5, 2.5]. ■

We note that, with an argument as in the above proof, it is possible to show that the convex closure in $\mathscr{C}_c^*$ of the union of two compact convex sets is compact too. This property is somewhat stronger than the requirement that polytopes be compact.

As we observed already before 1.4, semi-regularity of the hyperspace convexity has a rather strong consequence concerning the canonical subbase:

1.6. Corollary. *Let $(X, \mathscr{C})$ be a uniformizable S_4-convex structure such that $h^*(C_1 \cup C_2)$ is compact whenever C_1 and C_2 are compact convex. Then every convex closed subset of $\mathscr{C}_c^*$ is the intersection of subbasic sets. In particular, if $\mathscr{A} \subset \mathscr{C}_c^*$ then*

$$h^*(\mathscr{A}) = \langle h^*(\bigcup \mathscr{A}) \rangle \cap \perp_c \perp(\mathscr{A}).$$

Proof. Let $\mathscr{B} \subset \mathscr{C}_c^*$ be convex closed. If $B \in \mathscr{C}_c^* \setminus \mathscr{B}$, then by semi-regularity there exists a closed half-space $\mathscr{B}'$ of $\mathscr{C}_c^*$ with $\mathscr{B} \subset \mathscr{B}'$, $B \notin \mathscr{B}'$. A uniformizable convexity is closure-stable by [V_5, 2.2] and if it is S_4 moreover, then it is also semi-regular. Hence Theorem 1.2 can be applied, showing that $\mathscr{B}'$ is of the subbasic type.

The second part of the theorem easily follows as in 1.1. ■

1.7. Remark. All results in this section prerequire the condition that the convex closure of the union of two compact convex sets be compact again. Here are some cases in which this condition is fulfilled:

(1) $\mathscr{C}$ is a *completely uniformizable* convexity with compact polytopes: indeed, it follows from [V_5, 2.6] that then the convex closure of *any* compact set is compact.

(2) $\mathscr{C}$ is a semi-regular convexity with compact polytopes, and $\mathscr{C}$ has *finite rank*: if C_1, C_2 are compact convex, then both sets are polytopes by [V_6, 4.6]. Hence $h(C_1 \cup C_2)$ is also a polytope, and therefore it is compact again. Note that in this case

$$h(C_1 \cup C_2) = h^*(C_1 \cup C_2).$$

(3) $\mathscr{C}$ is a semi-regular closure-stable convexity with connected convex sets, compact polytopes, of finite dimension, and of *weakly infinite rank*: by [V_6, 4.7] compact convex sets are polytopes, and the above argument applies.

Of course the condition is also fulfilled if the underlying space is merely *compact*.

2. The main theorem. We first compare rank with convex dimension:

2.1. THEOREM. *Let $(X, \mathscr{C})$ be a uniformizable and S_4 convex structure with connected convex sets and with compact polytopes. Then the following are true.*

(1) *The convex dimension of $(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))$ is at most equal to the rank of $(X, \mathscr{C})$.*

(2) *If $(X, \mathscr{C})$ has weakly infinite rank and if $(X, \mathscr{C})$ is either finite-dimensional, or completely uniformizable, then there is no C.P. map from $\mathscr{C}_c^*$ onto the Hilbert cube with its cubical convexity.*

Proof of (1). Let d be the rank of $(X, \mathscr{C})$. We may assume that $d<\infty$. Hence, by a remark in 1.7 the convex closure of the union of two compact convex sets is compact again, and all results in Section 1 are applicable: $(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))$ is semi-regular, closure-stable (being uniformizable), and its convex sets are connected. If $ind(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))>d$, then by [$V_3$, 4.4] there is an *onto* C.P. map $f\colon \mathscr{C}_c^* \to [0,1]^{d+1}$, where the latter cube is equipped with the "subcube" convexity. For each $i = 1, \ldots, d+1$ we put

$$\mathscr{H}_i = f^{-1}\{y \mid \pi_i(y) = 0\}, \quad \mathscr{H}'_i = f^{-1}\{y \mid \pi_i(y) = 1\},$$

where π_i denotes the ith projection. Note that $\mathscr{H}_i$, $\mathscr{H}'_i$ are nonempty closed half-spaces of $\mathscr{C}_c^*$. Hence by Theorem 1.2 there exist nonempty closed half-spaces C_i, C'_i of X with

$$\mathscr{H}_i = \langle C_i\rangle \cap \mathscr{C}_c^* \text{ or } \langle C_i, X\rangle \cap \mathscr{C}_c^*,$$
$$\mathscr{H}'_i = \langle C'_i\rangle \cap \mathscr{C}_c^* \text{ or } \langle C'_i, X\rangle \cap \mathscr{C}_c^*.$$

Note that $\mathscr{H}_i \cap \mathscr{H}'_i = \emptyset$, whereas

$$\langle C_i, X\rangle \cap \langle C'_i, X\rangle \cap \mathscr{C}_c^* \neq \emptyset$$

(it contains a point of type $h\{x, x'\}$, where $x \in C_i$, $x' \in C'_i$). Hence without loss of generality,

$$\mathscr{H}_i = \langle C_i\rangle \cap \mathscr{C}_c^*, \quad i = 1, \ldots, d+1$$

(the sets $\mathscr{H}'_i$ remain undetermined; they will no longer be needed).

Let $i \in \{1, \ldots, d+1\}$. As f is onto, there exists a $D_i \in \mathscr{C}_c^*$ with

$$\pi_j f(D_i) = 0 \quad (j \neq i), \quad \pi_i f(D_i) = 1.$$

Then $D_i \in \bigcap_{j \neq i} \mathscr{H}_j$, $D_i \notin \mathscr{H}_i$, and hence

$$D_i \subset \bigcap_{j \neq i} C_j, \quad D_i \not\subset C_i.$$

Fix a point $x_i \in D_i \setminus C_i$ for each i. This leads us to a collection $\{x_1, \ldots, x_{d+1}\}$ which is free in X since for each i,

$$x_i \notin C_i, \quad x_j \in D_j \subset C_i \quad (j \neq i).$$

This contradicts with our assumption on the rank of $(X, \mathscr{C})$.

Proof of (2). Let $(X, \mathscr{C})$ have weakly infinite rank. As was observed in 1.7, each of the additional conditions on $(X, \mathscr{C})$ implies that the convex closure of the union of two compact convex sets is compact, and hence that all results of Section 1 are applicable. If $f\colon \mathscr{C}_c^* \to [0,1]^\infty$ is a C.P. map onto, then operating as above on the half-spaces

$$f^{-1}\pi_i^{-1}(0), \quad f^{-1}\pi_i^{-1}(1) \quad (i = 1, 2, \ldots)$$

leads us to an infinite free collection in X, contradicting to our assumption. ■

Let us reflect for a moment on the conclusion of (2) above. If $(X, \mathscr{C})$ is a semi-regular closure-stable convex structure with connected convex sets and with compact polytopes, then by [V_3, 4.4] the following statements are equivalent for each $n<\infty$:

(a) $ind(X, \mathscr{C}) \geq n$,

(b) there is a C.P. map $X \to [0,1]^n$ which is onto.

An example given in [V_3, 4.13] shows that these statements are no longer equivalent if $n = \infty$ (though (b) ⇒ (a) remains valid).

Hence the non-existence of a C.P. map onto the Hilbert cube may be interpreted as an expression of "weakly infinite" (or finite) convex dimension. However, we do not propose this as a definition: there are several other — maybe even more natural — definitions possible, and the relationships between such concepts (or their relationships with "topological" weak infinite dimensionality) are still unclear.

A natural question concerning (2) is whether or not the reverse statement is true: if there is no C.P. map from $\mathscr{C}_c^*$ onto the Hilbert cube, does $(X, \mathscr{C})$ then have finite or weakly infinite rank? We have a negative example which, however, does not satisfy any one of the additional conditions in (2):

2.2. EXAMPLE. A separable space X and a metrizable S_4-convexity $\mathscr{C}$ on X with connected convex sets, with compact polytopes, and such that

(1) $(X, \mathscr{C})$ has strongly infinite rank;

(2) there is no C.P. map from $\mathscr{C}_c^*$ onto the Hilbert cube.

Let X be the subspace of $[0,1]^\infty$ consisting of all x such that $\pi_n(x) = 0$ for all but finitely many $n \in N$. Note that X is convex in $[0,1]^\infty$ relative to the "subcube" convexity. Hence, if $\mathscr{C}$ denotes the trace convexity on X, then $\mathscr{C}$ is a metrizable S_4-convexity with connected convex sets and with compact polytopes.

Let $a(n)$ be the point of X of which all coordinates are 0 except for the nth one, which is 1. Then $\{a(n)|\ n\in N\}$ is free in X (establishing (1)), and it is clear that if $C\subset X$ is compact convex, then C is a subcube and $C\subset h\{a(1),\ldots,a(n)\}$ for some $n\in N$. It follows that the convex closure of the union of two compact convex sets is compact again, and that

$$\mathscr{C}_c^* = \bigcup_{n\in N}\mathscr{D}_n, \quad \text{where} \quad \mathscr{D}_n = \langle h\{a(1),\ldots,a(n)\}\rangle \cap \mathscr{C}_c^*.$$

If $f\colon \mathscr{C}_c^* \to [0,1]^\infty$ were a C.P. map onto, then

$$[0,1]^\infty = \bigcup_{n\in N} f(\mathscr{D}_n),$$

where each $f(\mathscr{D}_n)$ is compact. By the Baire theorem, some $f(\mathscr{D}_n)$ has nonempty interior, and hence there is an infinite-dimensional closed subcube $Q\subset f(\mathscr{D}_n)$. Let $p\colon [0,1]^\infty \to Q$ be the "nearest point" map associated to the (normal binary) subcube convexity. Then p is convexity preserving, and hence $p\circ f$ restricts to a C.P. map onto, $\mathscr{D}_n \to Q$. However, $\mathscr{D}_n$ is the convex hyperspace of an n-cube, whence $ind\,\mathscr{D}_n \leqslant 2n$ by 2.1. This leads to a contradiction, since a C.P. map does not raise the convex dimension by [V_3, 4.8]. This establishes (2).

We note that X is not compact, and not finite-dimensional. Also, $(X,\mathscr{C})$ is not completely uniformizable: if it were so, then the convex closure of a compact set were compact again. However, the set

$$Y = \{x|\ \pi_n(x) = 0 \text{ for all but one } n\}$$

is a compact subset of X, and its convex closure (in X) equals X. ■

We now come to our main theorem. $d(X,\mathscr{C})$ denotes the rank of $(X,\mathscr{C})$.

2.3. Theorem. *Let $\mathscr{C}$ be a metrizable S_4-convexity on the separable nontrivial space X, such that convex sets are connected and polytopes are compact. Then*

$$dim\,\mathscr{C}_c^* = d(X,\mathscr{C}).$$

We note that if X is a one-point space, then $dim\,\mathscr{C}_c^* = 0$ and $d(X,\mathscr{C}) = 1$.

Proof of 2.3. We first show that $dim\,\mathscr{C}_c^* \leqslant d(X,\mathscr{C})$ with the aid of Theorem 2.1. For $d(X,\mathscr{C}) = \infty$, the inequality is obvious. If $d(X,\mathscr{C})<\infty$, then all results of Section 1 apply. In particular, $(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C}))$ is a metrizable S_4 convexity with connected convex sets and with compact polytopes. Also, $\mathscr{C}_c^*$ is separable, being a subspace of the separable metric space $H_c(X)$. Hence by [V_5, 5.3],

$$dim\,\mathscr{C}_c^* = ind(\mathscr{C}_c^*, \mathscr{H}(\mathscr{C})),$$

and the desired inequality follows from 2.1.

We next show that $d(X,\mathscr{C}) \leqslant dim\,\mathscr{C}_c^*$. First note that if C is a dense convex subset of X, then C and X have equal rank since (by semi-regularity) if F is a finite free set in X, then its freedom property persists in small neighbourhoods of the members of F. Hence if $A\subset X$ is a countable dense subset, then

(1) $$d(h(A), \mathscr{C}\restriction h(A)) = d(X,\mathscr{C}).$$

Also, $(\mathscr{C}\restriction h(A))_c^*$ is a subspace of $\mathscr{C}_c^*$ whence

(2) $$dim(\mathscr{C}\restriction h(A))_c^* \leqslant dim\,\mathscr{C}_c^*.$$

Put

$$A = \{a_n|\ n\in N\};\quad A_n = \{a_1,\ldots,a_n\};\quad \mathscr{D}_n = \langle h(A_n)\rangle \cap \mathscr{C}_c^*.$$

Then $\bigcup_{n\in N} h(A_n) = h(A)$, $\bigcup_{n\in N}\mathscr{D}_n \subset (\mathscr{C}\restriction h(A))_c^*$, and hence

(3) $$\sup\{dim\,\mathscr{D}_n|\ n\in N\} \leqslant dim(\mathscr{C}\restriction h(A))_c^*,$$

(4) $$d(h(A), \mathscr{C}\restriction h(A)) = \sup\{d(h(A_n), \mathscr{C}\restriction h(A_n))|\ n\in N\}$$

(the latter equality follows from the fact that the sequence $(h(A_n))_{n=1}^\infty$ is increasing). By (1) to (4), it suffices to show that for each $n\in N$,

$$d(h(A_n), \mathscr{C}\restriction h(A_n)) \leqslant dim\,\mathscr{D}_n.$$

Note that $h(A_n)$ is compact, and that $\mathscr{D}_n$ is its convex hyperspace. This reduces the theorem to the case where X is *compact*.

Under the additional condition that X be compact, we now prove that for each $n<\infty$,

(5) $$d(X,\mathscr{C}) \geqslant n \Rightarrow dim\,\mathscr{C}^* \geqslant n.$$

Note that X is a connected and nontrivial subspace of $\mathscr{C}^*$, whence

$$dim\,\mathscr{C}^* \geqslant dim\,X \geqslant 1.$$

Hence (5) is valid for $n = 1$, and we assume that $n>1$. As $(X,\mathscr{C})$ is metrizable, we find by [V_5, 2.4] that the convex closure operator of $(X,\mathscr{C})$ is continuous. This leads us to a map

$$f\colon X^n \to \mathscr{C}^*,$$
$$f(x_1,\ldots,x_n) = h\{x_1,\ldots,x_n\}.$$

If $d(X,\mathscr{C}) \geqslant n$ then there exists a free collection $F = \{z_1,\ldots,z_n\}$ in X with exactly n points. Let O_i, P_i be open half-spaces of X such that

$$F\setminus\{z_i\} \subset P_i, \quad z_i \in O_i, \quad O_i \cap P_i = \emptyset \quad (i = 1,\ldots,n).$$

Then

$$U_i = \bigcap_{j\neq i} P_j \cap O_i$$

is a convex neighbourhood of z_i. Uniformizability of $(X,\mathscr{C})$ can be used to obtain a sequence $(C_{i,k})_{k=1}^\infty$ of convex closed neighbourhoods of z_i such that for each $k\in N$,

$$C_{i,k} \subset \operatorname{int} C_{i,k+1} \subset C_{i,k+1} \subset U_i.$$

Then

$$V_i = \bigcup_{k=1}^{\infty} C_{i,k} \subset U_i$$

and V_i is a convex open neighbourhood of z_i. By [V_3, 4.6],

$$\operatorname{ind} V_i = \sup\{\mathit{ind}\, C_{i,k} |\ k \in N\}$$

(recall that *ind* stands for convex dimension). Now note that (5) is valid if $\dim \mathscr{C}^* = \infty$. We may therefore assume that $\dim \mathscr{C}^* < \infty$. Then

$$\mathit{ind}\, X = \dim X \leqslant \dim \mathscr{C}^* < \infty ,$$

and consequently there is a $k \in N$ with

$$\mathit{ind}\, V_i = \mathit{ind}\, C_{i,k} = m_i, \text{ say}.$$

Having fixed such a compact convex neighbourhood $C_i = C_{i,k}$ for each i, we put

$$\mathscr{Y} = f(C_1 \times \ldots \times C_n).$$

Note that $C_1 \times \ldots \times C_n$ and $\mathscr{Y}$ are compact metric spaces, and f restricts to a closed map g between these spaces. By a theorem of Hurewicz (cf. [En, 1.12.4]),

(6) $$\dim(C_1 \times \ldots \times C_n) \leqslant \dim \mathscr{Y} + \sup\{\dim g^{-1}(y) |\ y \in \mathscr{Y}\}.$$

We now determine the various quantities appearing in (6).

First, $C_1 \times \ldots \times C_n$ can be equipped with the product convexity (see [J_1, p. 21]), which is easily seen to be metrizable S_4, and to have connected convex sets. As convex dimension behaves additively under the formation of products (cf. [V_3, 2.6]) we obtain from the equality of topological and convex dimension that

(7) $$\dim(C_1 \times \ldots \times C_n) = \dim C_1 + \ldots + \dim C_n ,$$

where $\dim C_i = \mathit{ind}\, C_i = m_i$.

In order to determine $\dim g^{-1}(y)$ for each $y \in \mathscr{Y}$ we first derive the following fact. Let $C \in \mathscr{Y}$, say:

$$C = h\{y_1, \ldots, y_n\} \quad \text{with} \quad y_1 \in C_1, \ldots, y_n \in C_n .$$

By semi-regularity, and as $n > 1$, we can extend the above introduced P_i to an open half-space Q_i of X with $y_i \in \dot{Q}_i = \bar{Q}_i \setminus Q_i$. Then

(8) $$\prod_{i=1}^{n} C_i \cap f^{-1}(C) \subset \prod_{i=1}^{n} (\dot{Q}_i \cap C_i).$$

Indeed, let $(y'_1, \ldots, y'_n)$ be in the left hand set of (8), and let $i \neq j$. Then

$$y'_j \in C_j \subset U_j \subset P_i \subset Q_i .$$

Hence if y'_i were a member of Q_i too, then

$$y_i \in C = h\{y'_1, \ldots, y'_n\} \subset Q_i ,$$

contradicting that $y_i \in \dot{Q}_i$. Hence $y'_i \notin Q_i$. Also,

$$y'_i \in C = h\{y_1, \ldots, y_n\} \subset \bar{Q}_i ,$$

whence $y'_i \in \dot{Q}_i$, $i = 1, \ldots, n$, establishing (8). Now note that since V_i is open,

$$\dot{Q}_i \cap C_i \subset \dot{Q}_i \cap V_i = \text{relative boundary of } Q_i \cap V_i \text{ in } V_i .$$

By [V_3, 2.8] the convex dimension of the *hyperplane* $\dot{Q}_i \cap V_i$ is at most $\mathit{ind}\, V_i - 1$, i.e.:

$$\mathit{ind}(\dot{Q}_i \cap C_i) \leqslant \mathit{ind}(\dot{Q}_i \cap V_i) \leqslant m_i - 1 .$$

We again use the product convexity and the equality of convex and topological dimension to conclude that

$$\dim \prod_{i=1}^{n} (\dot{Q}_i \cap C_i) \leqslant \sum_{i=1}^{n} (m_i - 1) ,$$

whence by (8),

(9) $$\dim \prod_{i=1}^{n} C_i \cap f^{-1}(C) \leqslant \sum_{i=1}^{n} (m_i - 1) .$$

Evaluating (7) and (9) in the Hurewicz inequality (6) we find that

$$\sum_{i=1}^{n} m_i \leqslant \dim \mathscr{Y} + \sum_{i=1}^{n} (m_i - 1) ,$$

and hence that

$$n \leqslant \dim \mathscr{Y} \leqslant \dim \mathscr{C}^*,$$

establishing the theorem. ■

If one interpretes rank as the degree of variation that convex sets are allowed to have, and if one regards dimension as a degree of freedom, then Theorem 2.3 is made plausible. However, the "philosophy" that dimension is some sort of freedom degree is accurate for "nice" spaces only. The following example illustrates that rank and convex hyperspace dimension are quite different quantities if one drops the connectedness condition on convex sets.

2.4. Example. Let X be the *Cantor discontinuum*, represented as the countable product of a discrete 3-point space $\{0, 1, 2\}$. If the latter is equipped with the order-convexity, then X can be equipped with the product convexity $\mathscr{C}$, which is metrizable (use a "product" metric) and S_4 (see [J_1, I. 10]).

It follows from an argument in [J_1, III.4] that the convex closed sets in X are exactly the products of convex sets in $\{0, 1, 2\}$. This implies that $\mathscr{C}^*$ is also a Cantor discontinuum, and $\dim \mathscr{C}^* = 0$.

On the other hand, the collection F, consisting of all points of X whose co-ordinates are all equal to 1 except for some nth coordinate which is 0 or 2, is free, whence $d(X, \mathscr{C}) = \infty$.

3. Some applications and problems. For convexities on *noncompact* spaces, no criteria for uniformizability or metrizability are known at present, except for particular classes of convexity spaces such as convex sets of linear spaces and tree-like spaces (see [V_5, 3.9]). Our next result is an extension of the main theorem to a better controllable class of convexities.

3.1. COROLLARY. *Let X be a separable metric space, and let $\mathscr{C}$ be a regular convexity on X with connected convex sets and with compact polytopes, such that $\mathscr{C}_c^*$ is closed in $H_c(X)$. Then*

$$\dim \mathscr{C}_c^* = d(X, \mathscr{C}) \geqslant 2\cdot \operatorname{ind}(X, \mathscr{C}).$$

If X is also locally convex, then $\operatorname{ind}(X, \mathscr{C}) = \dim X$.

We note that if $(X, \mathscr{C})$ is uniformizable, then $\mathscr{C}_c^*$ is closed in $H_c(X)$, and in fact, $\mathscr{C}^*$ is closed in $H(X)$: this follows from [V_2, 2.6] and from the fact that the convex hull operator of $(X, \mathscr{C})$ is continuous on finite sets.

Proof of 3.1. It is clear by definition that

$$(1) \qquad d(X, \mathscr{C}) = \sup\{d(P, \mathscr{C} \restriction P) |\ P \subset X \text{ a polytope}\}.$$

Note that a polytope P is compact, and that the collection

$$(\mathscr{C} \restriction P)^* = \mathscr{C}_c^* \cap \langle P\rangle$$

is closed in $H_c(X)$ and hence in $\langle P\rangle$. It follows from [V_5, 3.3] that $(P, \mathscr{C} \restriction P)$ is metrizable, whereas it is clear that $\mathscr{C} \restriction P$ is an S_4-convexity with connected convex sets and with compact polytopes. By the main theorem,

$$(2) \qquad \dim(\mathscr{C} \restriction P)^* = d(P, \mathscr{C} \restriction P).$$

It is obvious that

$$(3) \qquad \dim \mathscr{C}_c^* \geqslant \sup\{\dim(\mathscr{C} \restriction P)^* |\ P \subset X \text{ a polytope}\}.$$

By (1), (2), (3) and [V_6, 4.2] we already obtain that

$$\dim \mathscr{C}_c^* \geqslant d(X, \mathscr{C}) \geqslant 2\cdot \operatorname{ind}(X, \mathscr{C}).$$

If $d(X, \mathscr{C}) = \infty$, then the first inequality becomes an equality. Assume $d(X, \mathscr{C}) < \infty$. Then by [$V_6$, 4.8], X is the convex hull of a countable set

$$A = \{a_n |\ n \in N\}.$$

Put

$$D_n = h\{a_i |\ i \leqslant n\}.$$

If $C \subset X$ is compact convex, then by [V_6, 4.7], C is a polytope. As

$$X = \bigcup_{n=1}^{\infty} D_n; \quad (D_n)_{n=1}^{\infty} \text{ increasing},$$

we find a sufficiently large n such that D_n includes all vertices of the polytope C, and hence it includes C. This shows that

$$\mathscr{C}_c^* = \bigcup_{n=1}^{\infty} (\mathscr{C} \restriction D_n)^*$$

and in (3), one can restrict to a *countable* number of polytopes P. Hence by [En, 1.5.3], (3) becomes an equality, and (1), (2) and (3) together give the desired result again.

The second part of the theorem is quoted from [V_5, 5.3]. ■

3.2. COROLLARY. *Let X be a separable metric and locally connected tree-like space and let $C(X)$ be the hyperspace of subcontinua of X. Then $\dim C(X)$ equals the sum of* (i) *the number of endpoints of X and* (ii) *the number of free ultrafilters of closed connected subsets of X.*

See [D, 7.4] or [K, 5.5] for a corresponding result concerning finite graphs.

Proof of 3.2. By [V_5, 2.2] the collection $\mathscr{C}$ of all connected subsets of X is a metrizable S_4-convexity with compact polytopes. Then apply [V_6, 3.1]. ■

Our final application may require a word of explanation. Let $(X, \mathscr{C})$ be a (set-theoretic) convexity. Then the *Radon number* of $(X, \mathscr{C})$ is the number

$$r(X, \mathscr{C}) \in \{0, 1, 2, \ldots, \infty\}$$

such that for each $n < \infty$,

$r(X, \mathscr{C}) \leqslant n$ iff no finite subset of X with more than n elements is independent.

A set $F \subset X$ is called *independent* if for each pair of disjoint subsets F', F'' it is true that

$$h(F') \cap h(F'') = \emptyset.$$

Note that an independent set is also free.

For reasons explained in [V_4] we prefer the above definition, giving a value of $r(X, \mathscr{C})$ which is one *lower* than the value it has under the "classical" definition. Our definition agrees with the one in [L].

It was shown in [V_4, 2.12] that if $(X, \mathscr{C})$ is an n-dimensional binary normal convexity with connected convex sets and with compact polytopes, then $r(X, \mathscr{C})$ equals r_n or r_n+1, where r_n denotes the Radon number of the n-cube with subcube convexity. Also, the latter possibility can occur only if n is a member of

$$E = \{n |\ r_n \text{ is even and } C(r_n, -1+r_n/2) \leqslant n\}.$$

Here, $C(p, q)$ denotes the number of combinations of q elements out of p elements. With the use of Eckhoff's formula for r_n (see below; we adapted the formula to our definition) one can see that E is a rather thin, irregular sequence of "exceptional" dimensions. However, we only have an example with $r(X, \mathscr{C}) = r_n+1$ in case $n = 1 \in E$ [1].

In our next theorem we present a condition to ensure that $r(X, \mathscr{C}) = r_n$ even if $n \in E$:

---

[1] We have recently obtained examples in all other "exceptional" dimensions.

3.3. THEOREM. *Let $(X, \mathscr{C})$ be a normal binary convex structure with connected convex sets and with compact polytopes. If the rank of $(X, \mathscr{C})$ is minimal, that is, if*

$$d(X, \mathscr{C}) = 2\cdot ind(X, \mathscr{C}),$$

then $r(X, \mathscr{C})$ equals the Radon number of the $ind(X, \mathscr{C})$-cube.

A binary normal convexity with compact polytopes is closure-stable by [V_2, 2.9], and its convex hyperspace is normal and binary again by [vMV_1, 5.4] or [V_7, 3.9].

Proof of 3.3. We first present a simple reduction of the problem to metrizable convexities. We note that the theorem is trivial if $d(X, \mathscr{C}) = 2\cdot ind(X, \mathscr{C})$ is infinite. So assume $d(X, \mathscr{C}) < \infty$. Then also $ind(X, \mathscr{C}) < \infty$, and by [$V_4$, 2.13], $r(X, \mathscr{C}) < \infty$. Let $F \subset X$ be a free set with $d(X, \mathscr{C})$ many points, let $G \subset X$ be an independent set with $r(X, \mathscr{C})$ many points, and let

$$f\colon X \to [0, 1]^n \quad (n = ind(X, \mathscr{C}))$$

be a C.P. map onto. Then there is a finite set $H \subset X$ such that $f(H)$ equals the corner point set of $[0, 1]^n$, and by [V_3, 4.3],

$$fh(H) = [0, 1]^n.$$

It follows that $h(F \cup G \cup H)$ is a compact convex set with the same dimension, rank and Radon number as X. For each $x \in F$ let $f_x\colon X \to [0, 1]$ be a C.P. map separating $F \setminus \{x\}$ from x, and for each subset $G' \subset G$ let $f_{G'}\colon X \to [0, 1]$ be a C.P. map separating $h(G')$ from $h(G \setminus G')$. Then let

$$g\colon h(F \cup G \cup H) \to [0, 1]^p \quad (p = n + d(X, \mathscr{C}) + 2^{r(X, \mathscr{C})})$$

be the C.P. map determined by f, f_x $(x \in F)$ and $f_{G'}$ $(G' \subset G)$ on $h(F \cup G \cup H)$. The image set

$$Y = g\big(h(F \cup G \cup H)\big)$$

carries a normal binary (trace) convexity by [vMW, 3.4]. Clearly, $g(F)$ is free, $g(G)$ is independent, and $g\big(h(H)\big)$ is already n-dimensional (use the coordinate maps of g determined by f). As a C.P. map does not raise convex dimension, [V_3, 4.8], nor rank, [V_6, 2.5], nor the Radon number, [V_4, 1.4], we find that this convex structure is n-dimensional, and that its rank and its Radon number equal the one of $(X, \mathscr{C})$. Finally, a normal binary convexity on a compact metric space is metrizable, as follows from [vMV_1, 3.8] and [V_5, 3.4].

So we may assume henceforth that $(X, \mathscr{C})$ is metrizable and, if the reader wishes, that X is even compact. Next we derive the following two auxilliary results (the first one is stated in full generality).

STATEMENT 1. *Let X be connected and nontrivial, and let $\mathscr{C}$ be a semiregular closure-stable convexity on X with compact polytopes. Then*

$$r(X, \mathscr{C}) \leqslant r\big(\mathscr{C}_c^*, \mathscr{H}(C)\big) - 1.$$

Indeed, let $F \subset X$ be a finite independent set with at least two points, and fix a point $c \in F$. As F is also free, there exists a relative open half-space O of $h(F)$ with

$$F \setminus \{c\} \subset O, \quad c \in \dot{O} = \bar{O} \setminus O.$$

Also, there exists a neighbourhood U of c such that for each $x \in U$, $F \cup \{x\} \setminus \{c\}$ is independent. Then choose $c' \in U \cap O$, and put

$$F' = F \cup \{c'\} \setminus \{c\}.$$

Note that $h(F') \subset O$, whereas $c \in h(F) \setminus O$. Hence

(1) $$h(F) \notin \langle h(F') \rangle \cap \mathscr{C}_c^*.$$

We show that the collection

(2) $$\{\{y\} \mid y \in F'\} \cup \{h(F)\}$$

is independent in $\mathscr{C}_c^*$: let G, G' be disjoint subsets of the set (2). We may assume that $h(F) \in G$ (it actually suffices to consider 2-partitions of the above set).

If $h(F)$ is the only member of G, then

$$h(G') \subset \langle h(F') \rangle \cap \mathscr{C}_c^*$$

and $h(G) \cap h(G')$ is empty by (1). Suppose next that

$$G = \{h(F), \{y_1\}, \ldots, \{y_k\}\} \quad (k \geqslant 1)$$
$$G' = \{\{y_{k+1}\}, \ldots, \{y_p\}\}.$$

If $C \in h(G')$, then

(3) $$C \subset h\{y_{k+1}, \ldots, y_p\}.$$

If C were also in $h(G)$, then (cf. (2) of 1.1) for each $z \in h(F)$, C meets $h\{y_1, \ldots, y_k, z\}$. Now

$$h\{y_1, \ldots, y_k\} \subset h(F') \subset h(F),$$

and we choose z in the former set. Then C meets

$$h\{y_1, \ldots, y_k, z\} = h\{y_1, \ldots, y_k\},$$

contradicting with (3) and the independency of F'.

STATEMENT 2. *If $r = r(I^n)$ is even, then $r(I^{2n}) = r+1$ $(n \geqslant 1)$.*

As was proved in [E, Satz 3], the Radon number of I^n $(n \geqslant 1)$ is the largest possible $r \in N$ with $C(r, [r/2]) \leqslant 2n$ ($[r/2]$ is the integer closest to $r/2$). If r is even, then

$$C(r+1, r/2) < 2C(r, r/2).$$

If $r = r(I^n)$, then the latter number is at most $2\cdot 2n$, whence $r+1 \leqslant r(I^{2n})$. If $r+2 \leqslant r(I^{2n})$, then

(4) $$C(r+2, 1+r/2) \leqslant 4n,$$

and as r is even we have

(5) $$C(r+2, 1+r/2) = 2C(r+1, r/2).$$

As $r+1 > r(I^n)$, we also have

(6) $$2n < C(r+1, r/2).$$

These three (in)equalities together contradict.

The theorem is clear if X has one point. Otherwise, we find by Theorem 2.3 that

$$ind(\mathscr{C}^*_c, \mathscr{H}(\mathscr{C})) = dim\mathscr{C}^*_c = d(X, \mathscr{C}) = 2n,$$

where $n = ind(X, \mathscr{C}) \geqslant 1$. Suppose that

$$r(I^n)+1 = r(X).$$

Then $n \in E$, and in particular, $r = r(I^n)$ is even. By Statement 2, $r(I^{2n}) = r+1$, which is odd, whence $2n \notin E$. Then by Statement 1, and by [V_4, 2.12],

$$r+1 = r(X) < r(\mathscr{C}^*_c) = r(I^{2n}) = r+1,$$

contradiction. ■

3.4. Problem. It appears from [V_6] that rank and generating degree of a convexity are often equal, and that at least in the case of binary convexities, the generating degree is determined by the subbase of all closed half-spaces. It follows directly from Theorem 1.2 that then the generating degree of the hyperspace convexity is *at most twice* the generating degree of the original space: every closed half-space C gives rise to two convex sets $\langle C\rangle \cap \mathscr{C}^*_c$ and $\langle C, X\rangle \cap \mathscr{C}^*_c$, and the closed half-spaces of $\mathscr{C}^*_c$ are among the latter.

Is it true that, for a (sufficiently nice) convex structure $(X, \mathscr{C})$,

$$d(\mathscr{C}^*_c, \mathscr{H}(\mathscr{C})) = 2d(X, \mathscr{C})?$$

In case of an affirmative answer, one finds that the rank of $\mathscr{C}^*_c$ is twice its dimension, and this would also enlarge the applicability of Theorem 3.3 above.

3.5. Problem. The *rank of a convexity* $\mathscr{C}$ *at a point* $p \in X$ can be defined as follows:

$$d_p(X, \mathscr{C}) = \inf\{d(U, \mathscr{C} \mid U) |\ U \text{ a convex neighbourhood of } p\}.$$

It follows from [V_6, 3.2] that if a convex structure $(X, \mathscr{C})$ (with the "usual" conditions) satisfies Fuchssteiner's property, and if $C \subset X$ is convex and nowhere 1-dimensional, then the rank of C at any $p \in C$ is infinite. Does this imply (for compact metric C) that the convex hyperspace of C is homeomorphic to the Hilbert cube?

An affirmative answer would give a very natural extension of the Nadler–Quinn–Stavrakis theorem on compact linearly convex sets of dimension >1 (cf. [NQS]).

It is also natural to ask for conditions implying that a convex hyperspace is homeomorphic to an n-cube, or, more generally, to a polyhedron. Such a question was investigated at length for finite graphs in [D].

3.6. Problem. Is Corollary 3.1 a *proper* extension of the main theorem? More concretely, let $\mathscr{C}$ be a regular convexity on X with compact polytopes, such that $\mathscr{C}^*_c$ is closed in $H_c(X)$. If X is uniformizable (metrizable), is it then true that $(X, \mathscr{C})$ is uniformizable (metrizable)? In case of an affirmative answer, Corollary 3.1 reduces to the main theorem.

Added in proof. The main result of this paper has been extended to compact non-metric spaces in a recent paper of the author (Comp. Math. 50 (1983), pp. 95–108).

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