On line-symmetric graphs

by

David Burns (Big Rapids, Mich.), **S. F. Kapoor** (Kalamazoo, Mich.) and **P. A. Ostrand** * (Santa Barbara, Ca.)

Abstract. The structure of line-symmetric graphs is investigated.

Introduction. Whereas, an automorphism of a graph G with vertex set $V(G)$ and edge set $E(G)$ is a bijection of $V(G)$ with itself which preserves adjacency and non-adjacency between vertices, an edge-automorphism of G is a bijection of $E(G)$ with itself which preserves adjacency and non-adjacency of edges. Following the notation of [1], $\mathscr{A}(G)$ denotes the group of all automorphisms of G and $\mathscr{A}_1(G)$ denotes the group of all edge-automorphisms of G. If G is a non-empty graph and $\alpha \in \mathscr{A}(G)$, then α induces an edge-automorphism $\hat{\alpha}$ of G, where, if $e = \{x, y\} \in E(G)$, then $\hat{\alpha}(e) = \{\alpha(x), \alpha(y)\}$. Let $\mathscr{A}^*(G)$ denote the subgroup of $\mathscr{A}_1(G)$ consisting of all the edge-automorphisms of G that are induced by the automorphisms in $\mathscr{A}(G)$. The literature of graph theory contains a variety of results that specify conditions for isomorphisms among the groups $\mathscr{A}(G)$, $\mathscr{A}_1(G)$ and $\mathscr{A}^*(G)$ [1, p. 179].

If $\mathscr{A}^*(G)$ acts transitively on $E(G)$, meaning that for all edges e and f of G there exists a λ in $\mathscr{A}^*(G)$ such that $\lambda(e) = f$, we say that G is *line-symmetric*. This concept has been studied by Bouwer [2], Dauber and Harary [6], and Folkman [7]. We observe that for all positive integers m and n, the graphs nK_m, $K(m, n)$, $K_{m(n)}$ and the line-graphs $L(K_n)$ are line-symmetric, as are all cycles, together with a number of other well-known graphs such as the Heawood and Petersen graphs and the graphs of the five regular polyhedra.

The analogous vertex related concept is point-symmetry, i.e., a graph G is *point-symmetric* if $\mathscr{A}(G)$ acts transitively on $V(G)$. Following Harary [8], a graph is *symmetric* if it is both point-symmetric and line-symmetric. Also, we call a graph *biregular* if the set of degrees of its vertices has cardinality two.

Our interest in line-symmetric graphs grew out of a recent work [3] where it was shown that a graph G is line-symmetric if and only if its complement $\bar{G}$ has the property that $\bar{G}+e \cong \bar{G}+f$ for all edges e and f of G. For this latter property we adopted a descriptive phraseology; a graph G is *uniquely edge extendible* (UEE)

* This author wishes to express his appreciation to Western Michigan University for the hospitality extended to him during the period of this research.

if $G+e \cong G+f$ for all edges e and f of $\bar{G}$. The above referenced result is then succinctly restated: a graph is line-symmetric if and only if its complement is UEE.

Unless specified otherwise we will follow the symbols and terminology as in [1].

Section I. In this section we present structural characterizations of several classes of line-symmetric graphs, where the classes are specified by various connectivity properties of either the graph or its complement.

THEOREM 1. *Let G be a disconnected graph. Then G is line-symmetric if and only if there exists a connected line-symmetric graph H such that every non-trivial component of G is isomorphic to H.*

Proof. Let G be a disconnected line-symmetric graph. It is easily seen that each component of G must be line-symmetric. If G has exactly one non-trivial component, the result is true. Otherwise, let G_1 and G_2 be distinct non-trivial components of G. Let $e_i \in E(G_i)$, $i = 1, 2$. Since G is line-symmetric, there exists $\alpha \in \mathscr{A}(G)$ such that $\hat{\alpha}(e_1) = e_2$. It follows that $\hat{\alpha}(E(G_1)) = E(G_2)$ and that α restricted to $V(G_1)$ is a $G_1 \to G_2$ isomorphism.

It is readily verified that if there exists a connected line-symmetric graph H such that every non-trivial component of G is isomorphic to H, then G is line-symmetric. ■

The analogous theorem for point symmetry also holds. We include it here because it will be needed in the proof of a subsequent characterization.

THEOREM 2. *Let G be a disconnected graph. Then G is point-symmetric if and only if there is a connected point-symmetric graph H and an integer $t \geqslant 2$ such that $G \cong tH$.*

Proof. It is readily verified that if $G \cong tH$, with $t \geqslant 2$ and H point-symmetric, then G is a disconnected point-symmetric graph. Suppose conversely that G is a disconnected point-symmetric graph. Let $G_1, \ldots, G_t$, $t \geqslant 2$, be the components of G. Let $H = G_1$ and let $u \in V(H)$ and $v \in V(G_i)$ for some $2 \leqslant i \leqslant t$. Then there exists $\pi \in \mathscr{A}(G)$ such that $\pi(v) = u$. So $\pi(V(G_i)) = V(H)$ and π restricted to $V(G_i)$ is an isomorphism of G_i with H. Thus $G_i \cong H$ for every $2 \leqslant i \leqslant t$, and $G \cong tH$. ■

THEOREM 3. *Let G be a connected graph which is not a block. Then G is line-symmetric if and only if $G \cong K(1, n)$ for some $n \geqslant 2$.*

Proof. If $G \cong K(1, n)$ for some $n \geqslant 2$, then G is a line-symmetric connected graph which is not a block.

Conversely let G be a line-symmetric connected graph which is not a block. Then G has at least one cut-vertex. If G has two or more cut-vertices, then G contains end-blocks and blocks that are not end-blocks. Let e be an edge in an end-block of G, and let f be an edge belonging to a block of G which is not an end-block. Then for all $\alpha \in \mathscr{A}^*(G)$, $\alpha(e)$ is an edge of G belonging to an end-block of G. Hence $\alpha(e) \neq f$ for any $\alpha \in \mathscr{A}^*(G)$, contradicting the fact that G is line-symmetric. Therefore G has exactly one cut-vertex v. Let g be an edge of G incident with v, and assume that G contains an edge h not incident with v. For all $\alpha \in \mathscr{A}^*(G)$, the edge $\alpha(g)$ must be incident with some cut-vertex of G; and hence $\alpha(g)$ must be incident with v, since v is the only cut-vertex of G. It follows that $\alpha(g) \neq h$ for all $\alpha \in \mathscr{A}^*(G)$, again contradicting the fact that G is line-symmetric. Thus every edge of G is incident with v and G is isomorphic with $K(1, n)$ for some $n \geqslant 1$. But since G is not a block, we must have $n \geqslant 2$. ■

We recall that the complete t-partite graph $K(n, n, \ldots, n)$ is denoted $K_{t(n)}$.

THEOREM 4. *If G is a graph whose complement is disconnected then G is line-symmetric if and only if either there exist distinct positive integers m and n such that $G \cong K(m, n)$ or there exist integers $n \geqslant 1$ and $t \geqslant 2$ such that $G \cong K_{t(n)}$.*

Proof. It is easily verified that if G has either of the specified forms then G is line-symmetric and $\bar{G}$ is disconnected. Suppose conversely that G is a line-symmetric graph whose complement is disconnected. Let $G_1, \ldots, G_t$ be the components of $\bar{G}$, $t \geqslant 2$. We claim that each G_i is complete. To see that this is so, suppose to the contrary that there are two non-adjacent vertices x and y in G_i for some $1 \leqslant i \leqslant t$. Then $e = \{x, y\}$ is not an edge of $\bar{G}$, i.e. $e \in E(G)$. Also, if we choose arbitrary vertices u in G_1 and v in G_2, then $f = \{u, v\}$ is not an edge of $\bar{G}$ but is an edge of G. So there exists $\pi \in \mathscr{A}(G)$ with $\hat{\pi}(e) = f$. But as an automorphism of G, π is also an automorphism of $\bar{G}$, so there exists an index j, $1 \leqslant j \leqslant t$, such that $\pi(G_i) = G_j$, contradicting $\hat{\pi}(e) = f$ (which requires that $\pi(G_i)$ meet both G_1 and G_2). Thus every component of $\bar{G}$ is complete. Then, if $t = 2$ and $m = |V(G_1)|$ and $n = |V(G_2)|$ we have $\bar{G} \cong K_m \cup K_n$ so that $G \cong K(m, n)$, and either $m \neq n$, or $m = n$ in which case $G \cong K_{2(n)}$. If $t \geqslant 3$ then it is easily verified that the line-symmetry of G implies $|V(G_1)| = |V(G_2)| = \ldots = |V(G_t)|$. If n denotes the common values of $|V(G_i)|$, then $\bar{G} \cong tK_n$, and $G \cong K_{t(n)}$. ■

For graphs G with sufficiently many isolated vertices a suitable modification of Theorem 4 also characterizes line symmetry.

COROLLARY 4a. *For any positive integer s let G be a graph with at least s isolated vertices such that $\varkappa(\bar{G}) = s$. Then G is line-symmetric if and only if either $G \cong \overline{K_{s+1}}$ or there exist distinct positive integers m and n such that $G \cong \overline{K_s} \cup K(m, n)$ or there exist integers $n \geqslant 1$ and $t \geqslant 2$ such that $G \cong \overline{K_s} \cup K_{t(n)}$.*

Proof. It is easily seen that if G has any of the three forms specified in the conclusion of the corollary then G is line-symmetric. Suppose conversely that G is a line-symmetric graph with at least s isolated vertices and that $\varkappa(\bar{G}) = s$. Then either $\bar{G} = K_{s+1}$ or $\bar{G}$ contains a cut-set S of cardinality s. In the former case we are done. So we suppose that $\bar{G}$ has a cut-set S of cardinality s. Being a cut-set of $\bar{G}$, S must contain all isolated vertices of G. Thus G has exactly s isolated vertices which constitute the cut-set S. Then $G-S$ is a line-symmetric graph and its complement $\overline{G-S} = \bar{G}-S$ is disconnected. Now by Theorem 4 either $G-S \cong K(m, n)$ or $G-S \cong K_{t(n)}$; and $G \cong \overline{K_s} \cup K(m, n)$ or $G \cong \overline{K_s} \cup K_{t(n)}$ for appropriate values of m and n or t and n. ■

A similar result holds for graphs whose complements have isolated vertices.

Corollary 4b. *Let G be a graph such that $\bar{G}$ has an isolated vertex. Then G is line-symmetric if and only if either $G \cong K_t$ for some positive integer t or $G \cong K(1, n)$ for some integer $n \geq 2$.*

Proof. It is easily seen that complete graphs and stars are line-symmetric and that their complements each contain an isolated vertex. Conversely, let G be a line-symmetric graph whose complement contains an isolated vertex. If G is of order 1 then G is K_1 and we are done. So we suppose that the order of G is at least 2. Then $\bar{G}$ is disconnected, so by Theorem 4 we need consider only two cases.

Case 1. $G \cong K(m, n)$ for some positive integers $m<n$.

Since G has a vertex which is isolated in $\bar{G}$, we must have $m = 1$, and $G \cong K(1, n)$ for some $n \geq 2$.

Case 2. $G \cong K_{t(n)}$ for some integers $n \geq 1$ and $t \geq 2$.

Then $\bar{G} \cong tK_n$ and because $\bar{G}$ has an isolated vertex we must have $n = 1$, and $G \cong K_{t(1)} = K_t$. ■

If α is an automorphism of a graph G and e and f are edges of G such that $\hat{\alpha}(e) = f$, then α is also an isomorphism of $G-e$ with $G-f$, which suggests another characterization of line symmetry.

Theorem 5. *A graph G is line-symmetric if and only if for any two edges e and f of G, $G-e \cong G-f$.*

Proof. In [3] it is shown that a graph G is line-symmetric if and only if $\bar{G}$ is UEE, that is, if and only if for any two edges e and f of G, $\bar{G}+e \cong \bar{G}+f$. But $\bar{G}+e = \overline{G-e}$ and $\bar{G}+f = \overline{G-f}$, so $\bar{G}+e \cong \bar{G}+f$ if and only if $\overline{G-e} \cong \overline{G-f}$; and this is so if and only if $G-e \cong G-f$. ■

Information about the edges of a graph can often be equivalently interpreted as information about the vertices of its line graph. Our next two theorems do so for the problem at hand.

Theorem 6. *If G is a non-empty graph which does not contain both K_3 and $K(1, 3)$ as components then G is line-symmetric if and only if $L(G)$ is point-symmetric.*

Proof. First, suppose that G is a non-empty line-symmetric graph. Let v, w be arbitrary vertices of $L(G)$ and let e, f be the corresponding edges of G. There exists λ in $\mathscr{A}^*(G)$ such that $\lambda(e) = f$. Now $\mathscr{A}^*(G) \subset \mathscr{A}_1(G) \cong \mathscr{A}(L(G))$ so there is a corresponding α in $\mathscr{A}(L(G))$ and $\alpha(v) = w$ because v and w are the vertices of $L(G)$ corresponding to edges e and f of G. Thus the line symmetry of G implies that $L(G)$ is point-symmetric.

Now, suppose that G is a non-empty graph which does not contain both K_3 and $K(1, 3)$ as components such that $L(G)$ is point-symmetric. We consider two cases according as G is connected or not.

Case 1. G is connected.

We note that $L(C_4+e) \cong K_1+C_4$ which is not point-symmetric and neither is $L(K(1, 3)+e) \cong C_4+f$, so G cannot be either of the graphs C_4+e, $K(1, 3)+e$. G could be K_4, but in that case we are done since K_4 is line-symmetric, so we may assume that G is not K_4 either. Thus $\mathscr{A}_1(G) = \mathscr{A}^*(G)$ [1, p. 179]. Now let e, f be arbitrary edges of G and v, w the corresponding vertices of $L(G)$. There is an α in $\mathscr{A}(L(G))$ such that $\alpha(v) = w$. Let λ be the corresponding mapping in $\mathscr{A}_1(G) = \mathscr{A}^*(G)$. Then $\lambda(e) = f$. Thus G is line-symmetric.

Case 2. G is disconnected.

Let $G_1, G_2, \ldots, G_k$ be the components of G ($k \geq 2$). If $L(G)$ is connected, then G has exactly one non-trivial component to which Case 1 may be applied to show that G is line-symmetric. If $L(G)$ is disconnected then its components, without loss of generality, are $L(G_1), L(G_2), \ldots, L(G_m)$ for some $2 \leq m \leq k$. By Theorem 2, $L(G_1) \cong L(G_2) \cong \ldots \cong L(G_m)$. Since not both of K_3 and $K(1, 3)$ are components of G it follows by Whitney's Theorem [1, p. 188] that $G_1 \cong G_2 \cong \ldots \cong G_m$. Also, $G_{m+1} \cong G_{m+2} \cong \ldots \cong G_k \cong K_1$. Hence, if $G_1 \cong H$, then $G \cong mH \cup (k-m)K_1$, and $L(G) \cong mL(H)$. By Theorem 2, since $L(G)$ is point-symmetric we know that $L(H)$ is point-symmetric. Then, by applying the argument of Case 1 to H, we conclude that H is line-symmetric. Hence G is line-symmetric by Theorem 1. ■

Theorem 7. *Let G be a connected bridgeless graph. Then G is line-symmetric if and only if for any two vertices v and w of $L(G)$, the graphs $L(G)-v$ and $L(G)-w$ are isomorphic.*

Proof. First let G be a connected line-symmetric graph. By Theorem 5 we know that for any two edges g_1 and g_2 of G, the graphs $G-g_1$ and $G-g_2$ are isomorphic. Let v and w be any two vertices of $L(G)$ and let e and f be the corresponding edges of G. Then $G-e \cong G-f$ which yields $L(G-e) \cong L(G-f)$. But $L(G-e) \cong L(G)-v$ and $L(G-f) \cong L(G)-w$ [1, p. 198] so $L(G)-v \cong L(G)-w$.

Conversely assume that G is a connected bridgeless graph where $L(G)-v_1 \cong L(G)-v_2$ for any pair v_1, v_2 of vertices of $L(G)$. Let e and f be any two edges of G and let v and w be the corresponding vertices of $L(G)$. Then $L(G-e) \cong L(G)-v \cong L(G)-w \cong L(G-f)$. Since G is connected and has no bridges, both $G-e$ and $G-f$ are non-trivial connected graphs different from K_3. By Whitney's Theorem [1, p. 188] we conclude $G-e \cong G-f$, and by Theorem 5 we conclude that G is line-symmetric. ■

We note that the restriction in Theorem 6 that G should not contain both K_3 and $K(1, 3)$ as components is essential since if $G \cong K_3 \cup K(1, 3)$ then G is not line-symmetric but $L(G) \cong 2K_3$ is point-symmetric. By comparison, the restriction in Theorem 7 that G be connected and have no bridges is added only because it is useful in our proof. We know of no example which shows that this condition is necessary for the conclusion.

Section II. As the results of Section I indicate, there is great diversity among the line-symmetric graphs. To bring some order to this state of chaos we now present some results which categorize line-symmetric graphs. To facilitate the description of the three essentially different types it is convenient to adopt some additional terminology. A *transitive bipartition* of a bipartite graph G is a bipartition of G such that $\mathscr{A}(G)$ acts transitively on each of the partite sets.

The next two results presented are attributed by Folkman [7] to an unpublished manuscript of Dauber and Harary [6], and they are attributed by Harary [8] to Dauber alone. We provide their proofs for the sake of completeness.

THEOREM 8. *Let G be a line-symmetric graph which has no isolated vertices.*
(i) *If G is bipartite, then G has a transitive bipartition.*
(ii) *If G is not bipartite, then G is point-symmetric.*

Proof. Let u_1 and u_2 be adjacent vertices of G and let $V_i = \{\pi(u_i)|\ \pi \in \mathscr{A}(G)\}$ for $i = 1, 2$. Then of course, $\mathscr{A}(G)$ acts transitively on each of the sets V_1 and V_2. We consider two cases.

Case 1. V_1 and V_2 are disjoint.

Let $e_0 = (u_1, u_2)$, and let $e = (u, v)$ be an arbitrary edge of G. Then there exists an automorphism α of G such that $\hat{\alpha}(e_0) = e$. Hence $\{\alpha(u_1), \alpha(u_2)\} = \{u, v\}$, so e is incident with one vertex in each of the sets V_1 and V_2. Moreover for an arbitrary vertex z of G, z is not isolated so there is an edge f of G which is incident with z. Then there exists an automorphism π of G such that $\hat{\pi}(e_0) = f$, so that $z \in \{\pi(u_1), \pi(u_2)\} \subset V_1 \cup V_2$. Thus $V(G) = V_1 \cup V_2$ is a bipartition of G and as noted above $\mathscr{A}(G)$ acts transitively on each of the sets V_1 and V_2. In Case 1 G is bipartite and has a transitive bipartition.

Case 2. V_1 meets V_2.

It is readily verified that $V_1 = V_2 = V(G)$ so $\mathscr{A}(G)$ acts transitively on $V(G)$. Thus G is point-symmetric; and in the event that G is also bipartite, every bipartition of G is transitive because G is point-symmetric. ■

We note from Case 2 of the preceding proof that if G is a line-symmetric graph it is possible for G to be both bipartite and point-symmetric. Indeed all cycles of even order are of this type. Regularity or lack thereof provides an alternative scheme for categorizing line-symmetric graphs.

THEOREM 9. *If G is a line-symmetric graph which has no isolated vertices then either G is regular or G is biregular.*

Proof. If G is bipartite, then by Theorem 8 there is a bipartition of G such that $\mathscr{A}(G)$ acts transitively on each of the two partite sets. Then it is apparent that vertices which lie in the same partite set must have the same degree, and thus at most two different degrees occur in G. On the other hand, if G is not bipartite, then again by Theorem 8, G is point-symmetric, and hence regular. ■

The common hypothesis in Theorems 8 and 9 that G have no isolated vertices is not a significant hindrance since if G is a graph which has isolated vertices and H is the subgraph of G resulting from the deletion of the isolated vertices then G is line-symmetric if and only if H is line-symmetric. In short, the isolated vertices do not affect the line-symmetry of a graph; they only complicate the description of its structure.

The two classification criteria of the preceding theorems may be combined to produce a three-way classification of line-symmetric graphs.

THEOREM 10. *If G is a line-symmetric graph which has no isolated vertices then G satisfies exactly one of the following conditions.*

(i) *G is biregular and bipartite, in which case G has a unique transitive bipartition namely, the unique partition of $V(G)$ into two subsets so that vertices of equal degree are in the same subset.*

(ii) *G is regular and bipartite, in which case G might be or might not be point-symmetric.*

(iii) *G is regular but not bipartite, in which case G must be point-symmetric.*

Proof. By Theorem 9 we know that G is either regular or biregular. If G is biregular then, by Theorem 8, G must be bipartite since G cannot be point-symmetric. For any transitive bipartition of G, vertices in the same partite set must have the same degree. Since two different degrees occur in G there can only be one such partition of $V(G)$. Thus if G is biregular it satisfies condition (i) but neither (ii) nor (iii).

If G is regular then either G is bipartite or it is not. If G is bipartite it may be point-symmetric, as in the case of even cycles, or it may fail to be point-symmetric as in the case of the graphs constructed by Folkman [7] and the graphs described in Corollary 16a below. If G is regular and not bipartite then, by Theorem 8, G must be point-symmetric. Thus regular line-symmetric graphs without isolated vertices satisfy exactly one of conditions (ii) and (iii) and, of course, do not satisfy condition (i). ■

Among the regular line-symmetric graphs, the condition of also being point-symmetric is certainly the rule rather than the exception. So one-sided is this bias that the existing literature deals mainly with techniques for constructing regular line-symmetric graphs which are not point-symmetric. Folkman [7] provides a set of necessary conditions together with a set of sufficient conditions for the existence of regular line-symmetric graphs of a specified order which are not point-symmetric. The vertex splitting construction below enables us to extend Folkman's result to include a new infinite family of possible orders.

Let G be a graph and let $\mathscr{S} = \{S_v|\ v \in V(G)\}$ be a family of pairwise disjoint non-empty finite sets. In this case we will say that $\mathscr{S}$ is a *vertex splitting family for* G. For each edge $e = \{u, v\}$ of G let $S_e = \{\{x, y\}|\ x \in S_u \text{ and } y \in S_v\}$. For each $v \in V(G)$ and each $x \in S_v$ we say that x is a *fragment* of v and that v is the *source* of x. Likewise for each $e \in E(G)$ and each $f \in S_e$ we say that f is a *fragment* of e and that e is the *source* of f. Let $\mathscr{S}^*G$ denote the graph with vertex set $\bigcup_{v \in V(G)} S_v$ and edge set $\bigcup_{e \in E(G)} S_e$. We say that $\mathscr{S}^*G$ is a *vertex-splitting* of G. Note that if every set in $\mathscr{S}$ is of cardinality 1 then $\mathscr{S}^*G \cong G$. In this case we refer to $\mathscr{S}^*G$ as the *trivial* vertex-splitting of G.

EXAMPLE 1. Let G be the graph K_3 with $V(G) = \{u, v, w\}$. Let $\mathscr{S}$ be the vertex splitting family with $S_u = \{1, 2\}$, $S_v = \{3\}$, and $S_w = \{4\}$. The graphs G and $\mathscr{S}^*G$ are illustrated in Figure 1.

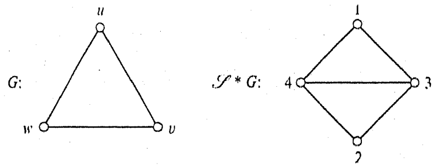

Fig. 1

EXAMPLE 2. For the very same graph G as in Example 1 now let $\mathscr{S}$ be the family, $S_u = \{1\}$, $S_v = \{2, 3\}$, $S_w = \{4\}$. Then $\mathscr{S}^*G \cong C_4$ as shown in Figure 2.

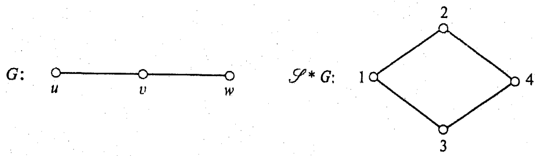

Fig. 2

EXAMPLE 3. Let G be the graph $C_4 \cup K(1, 2)$ and let $\mathscr{S}$ be the family which splits the C_4 component of G trivially and splits the $K(1, 2)$ as in Example 2. Then $\mathscr{S}^*G \cong 2C_4$.

We note that in Example 1 the graph G is line-symmetric but $\mathscr{S}^*G$ is not line-symmetric, while in Example 3 G is not line-symmetric but $\mathscr{S}^*G$ is line-symmetric. In Example 2 both G and $\mathscr{S}^*G$ are line-symmetric, while if G is any graph which is not line-symmetric then neither G nor its trivial vertex splitting are line-symmetric. Thus a graph and its vertex splittings are independent with respect to the property of line-symmetry. The same is true with respect to point-symmetry. However in the presence of other conditions the line-symmetry and point-symmetry of G and $\mathscr{S}^*G$ are related. Our first such condition is the following. A graph is called *elementary* if no two of its vertices have the same neighborhoods.

LEMMA 11. *Let G be an elementary graph and let $\mathscr{S}$ be a vertex splitting family for G. Then for vertices x and y of $\mathscr{S}^*G$, $N(x) = N(y)$ if and only if x and y are fragments of the same vertex in G.*

Proof. Let G be an elementary graph and let $\mathscr{S} = \{S_v | \ v \in V(G)\}$ be a vertex splitting family for G. Let x, y be vertices of $\mathscr{S}^*G$. Let u and v be the sources of x and y respectively. Then $N(x) = \bigcup_{w \in N(u)} S_w$ and $N(y) = \bigcup_{w \in N(v)} S_w$. Thus $N(x) = N(y)$ if and only if $N(u) = N(y)$, which is so if and only if $u = v$, in which case x and y are both fragments of $u = v$. ■

THEOREM 12. *Let G be an elementary graph and let $\mathscr{S}$ be a vertex splitting family for G such that $\mathscr{S}^*G$ is line-symmetric. Then G is line-symmetric.*

Proof. Let G and $\mathscr{S}$ be as in the hypothesis of the theorem. Let $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$ be arbitrary edges of G. Let f_1 and f_2 be fragments of e_1 and e_2 respectively in $\mathscr{S}^*G$. Then $f_i = \{x_i, y_i\}$ where $x_i \in S_{u_i}$ and $y_i \in S_{v_i}$ for $i = 1, 2$. There exists $\alpha \in \mathscr{A}(\mathscr{S}^*G)$ such that $\hat{\alpha}(f_1) = f_2$. Without loss of generality we may assume that $\alpha(x_1) = x_2$ and $\alpha(y_1) = y_2$. Define π: $V(G) \to V(G)$ as follows. For arbitrary $v \in V(G)$ let x be an arbitrary fragment of v and let w be the source of $\alpha(x)$. Define $\pi(v)$ to be this w. Note that if y is some other fragment of v then by Lemma 11 $N(x) = N(y)$ and $N(\alpha(x)) = \alpha(N(x)) = \alpha(N(y)) = N(\alpha(y))$, so by Lemma 11 again $\alpha(x)$ and $\alpha(y)$ have the same source w. Thus π is well-defined. If v and v' are adjacent vertices of G then their fragments x and x' are adjacent vertices of $\mathscr{S}^*G$ so $\alpha(x)$ and $\alpha(x')$ are adjacent vertices of $\mathscr{S}^*G$ and their sources $\pi(v)$ and $\pi(v')$ are adjacent vertices of G. Thus π is an automorphism of G and it is apparent that $(\hat{\pi}e_1) = e_2$. Thus G is line-symmetric. ■

The analogous result holds also for point-symmetry with the very same construction as proof.

THEOREM 13. *Let G be an elementary graph and let $\mathscr{S}$ be a vertex splitting family for G such that $\mathscr{S}^*G$ is point-symmetric. Then G is point-symmetric.*

Proof. Let $\mathscr{S}$ and G be as in the hypothesis of the theorem. Let u_1 and u_2 be arbitrary vertices of G and let x_1 and x_2 be fragments of u_1 and u_2 respectively. There is an $\alpha \in \mathscr{A}(\mathscr{S}^*G)$ such that $\alpha(x_1) = x_2$. Construct π: $V(G) \to V(G)$ from α as in the proof of Theorem 12. Then π is an automorphism of G and $\pi(u_1) = u_2$. Thus G is point-symmetric. ■

If $\mathscr{S}$ satisfies the right conditions line- or point-symmetry of G can also be carried over to $\mathscr{S}^*G$.

THEOREM 14. *Let G be a line-symmetric graph and let $\mathscr{S} = \{S_v | \ v \in V(G)\}$ be a vertex splitting family for G and let Γ be a subgroup of $\mathscr{A}(G)$ such that Γ^* acts transitively on the edges of G, and such that $|S_v| = |S_{\alpha(v)}|$ for every $v \in V(G)$ and every $\alpha \in \Gamma$. Then $\mathscr{S}^*G$ is line-symmetric.*

Proof. Let G, $\mathscr{S}$, and Γ be as in the hypothesis of the theorem. Let $f_1 = \{x_1, y_1\}$ and $f_2 = \{x_2, y_2\}$ be arbitrary edges of $\mathscr{S}^*G$. Let $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$ respectively be the sources of f_1 and f_2. Then without loss of generality we may assume that u_i is the source of x_i and v_i the source of y_i for $i = 1, 2$. There exists α in Γ such that $\hat{\alpha}(e_1) = e_2$. Again, without loss of generality we may assume that $\alpha(u_1) = u_2$ and $\alpha(v_1) = v_2$. For each vertex v of G $|S_v| = |S_{\alpha(v)}|$ so there is a bijection π_v: $S_v \to S_{\alpha(v)}$, where, without loss of generality, we may assume that $\pi_{u_1}(x_1) = x_2$ and $\pi_{v_1}(y_1) = y_2$. Define π: $V(\mathscr{S}^*G) \to V(\mathscr{S}^*G)$ by letting π restricted to S_v be π_v for each $v \in V(G)$. It is apparent that π is an automorphism of $\mathscr{S}^*G$ and that $\hat{\pi}(f_1) = f_2$. ■

Theorem 14 is principally of value for the following corollaries which illustrate how the conditions involving the subgroup Γ can be applied.

COROLLARY 14a. *If G is a line-symmetric graph and $\mathscr{S}$ is a vertex splitting family for G such that all members of $\mathscr{S}$ are of the same cardinality, then $\mathscr{S}^*G$ is line-symmetric.*

Proof. Let G and $\mathscr{S}$ satisfy the hypothesis of the corollary. Let $\Gamma = \mathscr{A}(G)$. Then G, $\mathscr{S}$, and Γ satisfy the hypothesis of Theorem 14, so $\mathscr{S}^* G$ is line-symmetric. ∎

COROLLARY 14b. *If G is a biregular line-symmetric graph and $\mathscr{S} = \{S_v \mid v \in V(G)\}$ is a vertex splitting family for G such that for vertices u and v of G of equal degree $|S_u| = |S_v|$, then $\mathscr{S}^*G$ is line-symmetric.* ∎

We omit the proof of Corollary 14b since it is identical to that of Corollary 14a. Corollary 14c is a bit different, however.

COROLLARY 14c. *Let n be an even positive integer and let G be the cycle of length n so that G has a unique bipartition. Let $\mathscr{S} = \{S_v \mid v \in V(G)\}$ be a vertex splitting family for G such that for any two vertices u and v of G in the same partite set, $|S_u| = |S_v|$. Then $\mathscr{S}^*G$ is line-symmetric.*

Proof. Let G and $\mathscr{S}$ be as in the hypothesis of the corollary. Let $V(G) = V_1 \cup V_2$ be the unique bipartition of G. Let $\Gamma = \{\alpha \in \mathscr{A}(G) \mid \alpha(V_1) \subset V_1\}$. Then G, $\mathscr{S}$, and Γ satisfy the hypothesis of Theorem 14, so $\mathscr{S}^*G$ is line-symmetric. ∎

The even cycles are not only graphs to which the preceding argument applies. It may be used for any connected bipartite line-symmetric graph such that for Γ as defined in the proof, Γ^* acts transitively on $E(G)$. Other examples of such graphs include the n-cubes Q_n for $n \geqslant 1$.

As in the case of Theorem 12, Theorem 14 also has its point-symmetric analog which we state without proof because it is analogous to that of Theorem 14.

THEOREM 15. *Let G be a point-symmetric graph and let $\mathscr{S}$ be a vertex splitting family for G and let Γ be a subgroup of $\mathscr{A}(G)$ which acts transitively on $V(G)$ such that $|S_v| = |S_{\alpha(v)}|$ for every $v \in V(G)$ and every $\alpha \in \Gamma$. Then $\mathscr{S}^*G$ is point-symmetric.* ∎

By selecting $\Gamma = \mathscr{A}(G)$, Corollary 15a is established. The analogs of 14b and 14c in terms of point-symmetry are not meaningful.

COROLLARY 15a. *If G is a point-symmetric graph and $\mathscr{S}$ is a vertex splitting family for G such that all members of $\mathscr{S}$ are of equal cardinality, then $\mathscr{S}^*G$ is point-symmetric.* ∎

EXAMPLE 4. Let $G = K_2$ with $V(G) = \{u, v\}$. Let $S_u = \{1\}$, $S_v = \{2, 3\}$. Then $\mathscr{S}^*G$ is $K(1, 2)$. (See Figure 3.)

As Example 4 indicates, the converse of Corollary 14a is false. The graph G in this example is elementary and $\mathscr{S}$ contains members of different cardinalities and yet $\mathscr{S}^*G$ is line-symmetric. However the converse of Corollary 15a is valid.

THEOREM 16. *If G is an elementary graph and if $\mathscr{S} = \{S_v \mid v \in V(G)\}$ is a family of pairwise disjoint non-empty finite sets such that $\mathscr{S}^*G$ is point-symmetric then all members of $\mathscr{S}$ are of the same cardinality.*

Proof. Let G and $\mathscr{S}$ satisfy the hypothesis of the theorem. Let u and v be arbitrary vertices of G. Then there exists $x \in S_u$ and $\alpha \in \mathscr{A}(\mathscr{S}^*G)$ such that $\alpha(x) \in S_v$. Thus for all $y \in S_u$ we have $N(y) = N(x)$ by Lemma 11 and therefore we have $N(\alpha(y)) = \alpha(N(y)) = \alpha(N(x)) = N(\alpha(x))$ so that $\alpha(y) \in S_v$ by Lemma 11. Then $\alpha(S_u) \subset S_v$. Since α is injective we conclude that $|S_u| \leqslant |S_v|$. The same argument applied to the automorphism α^{-1} and the vertex $\alpha(x) \in S_v$ yields the reverse inequality $|S_v| \leqslant |S_u|$. Thus $|S_u| = |S_v|$ for arbitrary vertices u and v of G. ∎

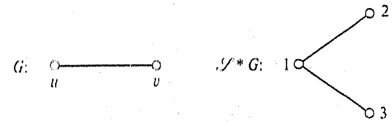

Fig. 3

COROLLARY 16a. *For every integer $t \geqslant 2$ there exists a regular line-symmetric graph of order $2t(2t+1)$ which is not point-symmetric.*

Proof. It will be shown in Section III that for every integer $t \geqslant 2$ the subdivision graph of K_{2t+1} is an elementary biregular line-symmetric graph which has $t(2t+1)$ vertices of degree 2 and also $2t+1$ vertices of degree $2t$. Take this as the graph G, and let $\mathscr{S} = \{S_v \mid v \in V(G)\}$ be a family of pairwise disjoint sets with $|S_v| = 1$ whenever $\deg v = 2$ and $|S_v| = t$ when $\deg v = 2t$. Then $\mathscr{S}^*G$ is regular of degree $2t$ and order $2t(2t+1)$ and by Corollary 14b $\mathscr{S}^*G$ is line-symmetric but, by Theorem 16, it is not point-symmetric. ∎

Corollary 16a extends the results of Folkman since it includes some orders which are not covered by Folkman's results. Specifically for $t = 9$ we have a graph of order 342 which is not covered by Folkman's constructions. It is interesting to note that for $t = 2$ we obtain the very same graph of order 20 which Folkman [7] used (see Figure 4) to illustrate his construction. In [7], Folkman proves the following result, for which we offer an alternative proof based on the preceding theorems.

THEOREM 17. *Let G be an elementary graph of order p which is line-symmetric and regular of degree d but is not point-symmetric. Let r be a positive integer. Then there is a line-symmetric graph $\tilde{G}$ of order rp which is regular of degree rd but is not point-symmetric.*

Proof. Let G be a graph which satisfies the hypothesis of the theorem and let $\mathscr{S} = \{S_v \mid v \in V(G)\}$ be a vertex splitting family for G consisting of sets of cardinality r. Let $\tilde{G} = \mathscr{S}^*G$. Apparently $\tilde{G}$ is of order rp and regular of degree rd. By Corollary 14a $\tilde{G}$ is line-symmetric and by Theorem 13 $\tilde{G}$ is not point-symmetric. ∎

In fact the graph $\tilde{G}$ constructed by Folkman in his proof is the very same one we construct, but described differently. What is interesting is that our proof that $\tilde{G}$ has the desired properties comes essentially for free when $\tilde{G}$ is expressed as a vertex-splitting.

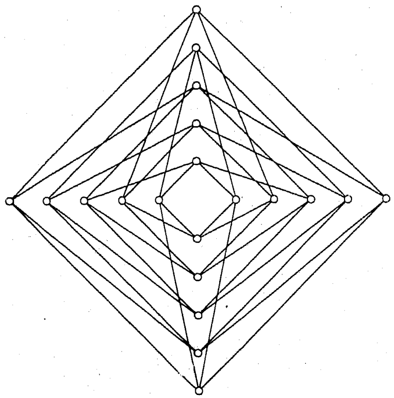

Fig. 4

We conclude this section by noting that regular symmetric graphs which are either bipartite or non-bipartite are easy to find, as the following infinite families of examples indicate. Regular symmetric bipartite graphs include the even cycles and more generally the graphs tC_n for $t\geqslant 1$ and n even, the graphs tK_2 for any $t\geqslant 1$, and the regular complete bipartite graphs $K(n,n)$, $n\geqslant 1$. The regular symmetric non-bipartite graphs include tC_n for n odd and $t\geqslant 1$, tK_n for $t\geqslant 1$ and $n\geqslant 3$, $K_{t(n)}$ for $n\geqslant 1$ and $t\geqslant 3$, $L(K_n)$ and $\overline{L(K_n)}$ for $n\geqslant 5$, and $K_n\times K_n$ and $\overline{K_n\times K_n}$ for $n\geqslant 3$. In addition to these families there are a number of sporadic examples such as the graphs of the five regular polyhedra.

Section III. In this section we collect results which deal primarily with the structure of line-symmetric graphs in the biregular bipartite category. We begin by observing that one way to construct a biregular bipartite graph is to subdivide a regular graph.

If $G = (V, E)$ is a graph, the *subdivision* of G, denoted $S(G)$, is the graph with vertex set $E\cup V$ and edge set $\{\{e, v\}|\ e\in E,\ v\in V$, and e and v are incident$\}$. It is apparent that $S(G)$ must always be bipartite with one of the partite sets consisting entirely of vertices of degree 2 introduced by the subdivision process, and that distinct vertices in said partite set cannot have the same pair of neighbors. In fact these properties characterize subdivisions when we exclude the uninteresting case of empty graphs. (If G is empty then $S(G) = G$.)

EXAMPLE 5. The graph G and its subdivision shown in Figure 5 explain the use of the term subdivision in this context. $S(G)$ is essentially a copy of G in which an extra vertex has been added in the middle of each edge.

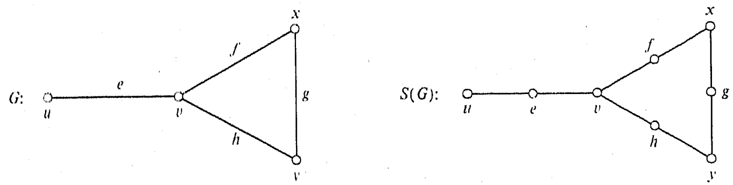

Fig. 5

THEOREM 18. *A non-empty graph G is a subdivision, meaning that there exists a graph H with $S(H)\cong G$, if and only if G is bipartite and G has a bipartition $V(G) = V_1\cup V_2$ such that each vertex in V_1 has degree 2 and no two vertices in V_1 have the same pair of neighbors.*

Proof. It is apparent that for any non-empty graph H, $S(H)$ is bipartite and the partition $V(S(H)) = E(H)\cup V(H)$ is a bipartition of $S(H)$ with the properties specified in the theorem. If $G\cong S(H)$ then G must have a bipartition.

Suppose, conversely, that G is a bipartite graph with a bipartition $V(G) = V_1\cup V_2$ such that every vertex in V_1 has degree 2 and no two vertices in V_1 have the same pair of neighbors. Construct a graph H as follows: $V(H) = V_2$, and $E(H) = \{N(u)|\ u\in V_1\}$. Define $\varphi\colon V_1\cup V_2\to V_2\cup E(H)$ as follows. For $x\in V_2$, let $\varphi(x) = x$ and for $u\in V_1$, let $\varphi(u) = N(u)$. The hypothesis that no two vertices in V_1 have the same pair of neighbors is just what is needed to guarantee that H is a graph. It is easy to verify that φ is an isomorphism of G with $S(H)$. Thu- G is a subdivision. ■

Before proceeding to the applications of Theorem 18 in the context of lines symmetry we must establish some results concerning the automorphisms of subdivisions.

LEMMA 19. *Let H be a graph and let $\alpha\in\mathscr{A}(H)$ and let $\beta = \hat{\alpha}\in\mathscr{A}^*(H)$. Define $\pi\colon V(H)\cup E(H)\to V(H)\cup E(H)$ by $\pi(x) = \alpha(x)$ for $x\in V(H)$ and $\pi(e) = \beta(e)$ for $e\in E(H)$. Then π is an automorphism of $S(H)$.*

Proof. Let $v\in V(H)$ and let $e\in E(H)$ be such that in $S(H)$ v and e are adjacent. Then in H, v and e are incident, so in H $\alpha(v)$ and $\beta(e)$ are incident. Then in $S(H)$, $\pi(v)$ and $\pi(e)$ are adjacent. Thus π is an automorphism of $S(H)$. ■

We note that the subdivision automorphism π constructed in the Lemma has a special property, namely that $\pi(V(H)) = V(H)$ and $\pi(E(H)) = E(H)$. For some graphs there are subdivision automorphisms which do not have this property.

EXAMPLE 6. Let $G = K_3$. Then G and $S(G)$ are shown in Figure 6. The automorphism γ of $S(G)$ given by $\gamma(u) = e$, $\gamma(v) = f$, $\gamma(w) = g$ and $\gamma(e) = v$, $\gamma(f) = w$, $\gamma(g) = u$ interchanges the partite sets $V(G)$ and $E(G)$ of $S(G)$.

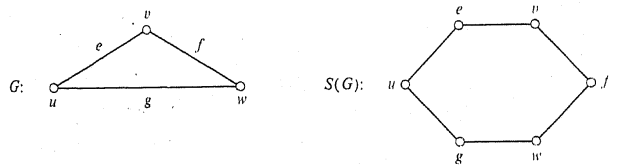

Fig. 6

For automorphisms of $S(G)$ which do not interchange the partite sets of $S(G)$ the converse of Lemma 19 is valid.

LEMMA 20. *Let H be a graph and let π be an automorphism of $S(H)$ such that $\pi(V(H))\subset V(H)$ and $\pi(E(H))\subset E(H)$, and let α and β be the restrictions of π to $V(H)$ and to $E(H)$ respectively. Then $\alpha\in \mathscr{A}(H)$ and $\beta = \hat{\alpha} \in \mathscr{A}^*(H)$.*

Proof. Let $\pi\in\mathscr{A}(S(H))$ be such that $\pi(V(H))\subset V(H)$ and $\pi(E(H))\subset E(H)$. Let α and β be the restrictions of π to $V(H)$ and to $E(H)$ respectively. Then α and β are permutations of $V(H)$ and $E(H)$ respectively.

For adjacent vertices x and y of H, let $e = \{x, y\}\in E(H)$. Then, in $S(H)$: e is adjacent to both x and y, $\pi(e)$ is adjacent to both $\pi(x)$ and $\pi(y)$. Thus $\pi(e) = \{\pi(x), \pi(y)\}\in E(H)$. So $\alpha\in\mathscr{A}(H)$ and $\beta = \hat{\alpha}$. ■

From Theorems 10 and 18 we can infer that many, if not all, of the biregular line-symmetric graphs which contain vertices of degree 2 are subdivisions of regular graphs. Our next two results concern line-symmetry of such graphs.

THEOREM 21. *If G is a regular graph and $S(G)$ is line-symmetric then G is symmetric.*

Proof. Let G be a regular graph for which $S(G)$ is line-symmetric. We consider 3 cases depending on the degree of regularity of G.

Case 1. G is regular of degree 0.

In this case G is certainly symmetric.

Case 2. G is regular of degree 2.

In this case G is either a cycle or the union of cycles, i.e., $G\cong \bigcup_{i=1}^{k} C_{n_i}$ for some $k\geqslant 1$ and some sequence of integers $n_1\geqslant n_2\geqslant \ldots \geqslant n_k\geqslant 3$. Then $S(G) = \bigcup_{i=1}^{k} S(C_{n_i}) \cong \bigcup_{i=1}^{k} C_{2n_i}$ is also a cycle or the union of cycles. Thus, by Theorem 1, the line-symmetry of $S(G)$ implies that $2n_1 = 2n_2 = \ldots = 2n_k$, in which case it is also true (by Theorems 1 and 2) that G is symmetric.

Case 3. G is regular of degree $d\neq 0, 2$.

If $e, f\in E(G)$ and $x, y\in V(G)$ such that x is incident with e and y is incident with f, then $\{e, x\}, \{f, y\}\in E(S(G))$. So there is a $\pi\in\mathscr{A}(S(G))$ such that $\hat{\pi} = (\{e, x\}) = \{f, y\}$. Then $\{\pi(e), \pi(x)\} = \{f, y\}$. Each vertex in $V(G)$ is of degree $d\neq 2$ as a vertex of $S(G)$ and each edge in $E(G)$ is of degree 2 as a vertex of $S(G)$. Thus $\pi(V(G))\subset V(G)$ and $\pi(E(G)) \subset E(G)$. Let α and β be the restrictions of π to $V(G)$ and to $E(G)$ respectively. By Lemma 20, $\alpha\in\mathscr{A}(G)$ and $\beta\in\mathscr{A}^*(G)$ with $\alpha(x) = y$ and $\beta(e) = f$. Since every edge of G is incident with two vertices of G and every vertex of G is incident with $d\neq 0$ edges of G we deduce that G is both point-symmetric and line-symmetric, that is, G is symmetric. ■

THEOREM 22. *If G is a regular graph, then $S(G)$ is line-symmetric if and only if G is line-symmetric and for every edge $e = \{x, y\}$ of G there exists $\pi_e\in\mathscr{A}(G)$ such that $\pi_e(x) = y$ and $\pi_e(y) = x$.*

Proof. Let G be a regular graph. We consider three cases according to the degree of regularity of G.

Case 1. G is regular of degree 0.

The result is apparent.

Case 2. G is regular of degree 2.

In this case $G = \bigcup_{i=1}^{k} C_{n_i}$ for some integer $k\geqslant 1$ and some sequence $n_1, n_2, \ldots, n_k$ of positive integers. Then $S(G)\cong \bigcup_{i=1}^{k} C_{2n_i}$. By Theorem 1, $S(G)$ is line-symmetric if and only if $2n_1 = 2n_2 = \ldots = 2n_k$, in which case G is also line-symmetric by Theorem 1, and in any case G, as the union of cycles, satisfies the interchange conditions of the theorem.

Case 3. G is regular of degree $d\neq 0, 2$.

Suppose that $S(G)$ is line-symmetric. By Theorem 21, G is also line-symmetric. Let $e = \{x, y\}\in E(G)$. Then $\{e, x\}$ and $\{e, y\}$ are edges of $S(G)$ so there exists π in $\mathscr{A}(S(G))$ such that $\hat{\pi}(\{e, x\}) = \{e, y\}$. Thus $\{\pi(e), \pi(x)\} = \{e, y\}$. Every vertex of G is of degree $d\neq 2$ in $S(G)$ and every edge of G is a vertex of degree 2 in $S(G)$ so $\pi(V(G))\subset V(G)$ and $\pi(E(G))\subset E(G)$. Let π_e be the restriction of π to $V(G)$. Then by Lemma 20, $\pi_e\in\mathscr{A}(G)$ and $\pi_e(x) = y$. Since $\pi(e) = e$ and $\pi(x) = y$, $\pi(y)$ must be x, the only other vertex of G incident with e. Thus $\pi_e(y) = x$.

Suppose conversely that G is line-symmetric and has the interchange property specified in the theorem. Let $e, f\in E(G)$ and $x, y\in V(G)$ with $\{e, x\}, \{f, y\} \in E(S(G))$. Then there exist $u, v\in V(G)$ such that $e = \{x, u\}$ and $f = \{y, v\}$. There exists $\alpha\in\mathscr{A}(G)$ such that $\hat{\alpha}(e) = f$. We consider two subcases.

Subcase 3A. $\alpha(x) = y$ and $\alpha(u) = v$.

Define π: $V(S(G))\to V(S(G))$ by $\pi(z) = \alpha(z)$ for $z\in V(G)$ and $\pi(z) = \hat{\alpha}(z)$ for $z\in E(G)$. By Lemma 19, π is an automorphism of $S(G)$ and $\hat{\pi}(\{e, x\}) = \{f, y\}$.

Subcase 3B. $\alpha(x) = v$ and $\alpha(u) = y$.

There exists $\pi_f \in \mathscr{A}(G)$ such that $\pi_f(y) = v$ and $\pi_f(v) = y$. Let $\beta = \pi_f\alpha$. Then $\hat{\beta}(e) = f$, $\beta(x) = y$ and $\beta(u) = v$, so β is covered by Subcase 3A and there exists π in $\mathscr{A}(S(G))$ such that $\hat{\pi}(\{e, x\}) = \{f, y\}$.

Thus $S(G)$ is line-symmetric. ■

In fact we can improve upon Theorem 22 a bit by reducing the interchange requirement to a single edge.

THEOREM 23. *If G is a regular non-empty graph then $S(G)$ is line-symmetric if and only if G is line-symmetric and there exists π in $\mathscr{A}(G)$ and $e = (x, y)$ in $E(G)$ such that $\pi(x) = y$ and $\pi(y) = x$.*

Proof. By Theorem 22 if G is regular and $S(G)$ is line-symmetric then G has the specified properties. Thus it suffices to prove the reverse implication. To that end, let G be a regular line-symmetric graph such that there exists $\pi \in \mathscr{A}(G)$ and $e = \{x, y\} \in E(G)$ for which $\pi(x) = y$ and $\pi(y) = x$. For an arbitrary edge $f = \{u, v\}$ of G there exists $\alpha \in \mathscr{A}(G)$ such that $\hat{\alpha}(e) = f$. Let $\pi_f = \alpha\pi\alpha^{-1}$. It is readily verified that $\pi_f(u) = v$ and $\pi_f(v) = u$. Thus G has the full interchange property specified in Theorem 22 so $S(G)$ is line-symmetric. ■

As a consequence of either Theorem 22 or Theorem 23 it is easily seen that $S(K_p)$ is line-symmetric for every $p \geqslant 2$, which fact was used for odd p in the proof of Corollary 16a in Section II.

It is interesting to note that if G is a regular bipartite line-symmetric graph then the limited interchange property of Theorem 23 is exactly what is needed to make G point-symmetric since the interchange automorphism must interchange the two partite sets. Combined with the transitivity of the bipartition, this implies that $\mathscr{A}(G)$ acts transitively on all of $V(G)$. All of this is quite compatible with Theorem 21 which requires G to be point-symmetric as a necessary condition for $S(G)$ to be line-symmetric.

We complete our consideration of subdivisions by noting that there are biregular line-symmetric graphs which are not subdivisions, namely the complete bipartite graphs $K(2, n)$ for $n \geqslant 3$. This does not contradict Theorems 10 and 18 because the hypothesis of Theorem 18 requires that no two vertices in the degree 2 partite set have the same pair of neighbors. In $K(2, n)$, $n \geqslant 3$, all n vertices of degree 2 have the same pair of neighbors. In fact the n vertices of degree 2 are graphically indistinguishable from each other. With regard to many graphical properties the presence of all n of these vertices should have no more consequence than the presence of just one such vertex. This is certainly the case with regard to line-symmetry with $K(2, 1)$ which has just one such vertex being just as line-symmetric as $K(2, n)$ for any other n. This leads to our next result which is essentially the reverse of the vertex-splitting construction of Section II.

We say that two vertices x and y of a graph G are *indistinguishable* and write $x \equiv y$ if $N(x) = N(y)$. It is apparent that $\equiv$ is an equivalence relation on $V(G)$. Let $\bar{x}$ denote the equivalence class containing the vertex x.

LEMMA 24. *Let x and y be vertices of a graph G. If x and y are adjacent, then every vertex in $\bar{x}$ is adjacent to every vertex in $\bar{y}$. If x and y are non-adjacent then no vertex in $\bar{x}$ is adjacent to any vertex in $\bar{y}$.*

Proof. Assume that x and y are adjacent and that $u \in \bar{x}$, $v \in \bar{y}$. Then $x \in N(y) = N(v)$, so $v \in N(x) = N(u)$, so u and v are adjacent. The second half of the lemma is a logical consequence of the first half. ■

LEMMA 25. *Let x and y be vertices of a graph G and let $\alpha \in \mathscr{A}(G)$ be such that $\alpha(x) = y$. Then $|\bar{x}| = |\bar{y}|$.*

Proof. We will show that $|\bar{x}| \leqslant |\bar{y}|$. Since $\alpha^{-1} \in \mathscr{A}(G)$ and $\alpha^{-1}(y) = x$, the same argument establishes the reverse inequality. Let $z \in \bar{x}$. Then $N(\alpha(z)) = \alpha(N(z)) = \alpha(N(x)) = N(\alpha(x)) = N(y)$, so $\alpha(z) \in \bar{y}$. Thus $\alpha(x) \subset \bar{y}$ for the injective mapping α, so $|\bar{x}| \leqslant |\bar{y}|$. ■

THEOREM 26. *If G is a biregular line-symmetric graph with no isolated vertices and if x and y are vertices of G of equal degree then $|\bar{x}| = |\bar{y}|$.*

Proof. By Theorem 10, G has a transitive bipartition with vertices of equal degree in the same partite set. Then the result follows from Lemma 25. ■

EXAMPLE 7. Consider the graphs G and H shown in Figure 7.

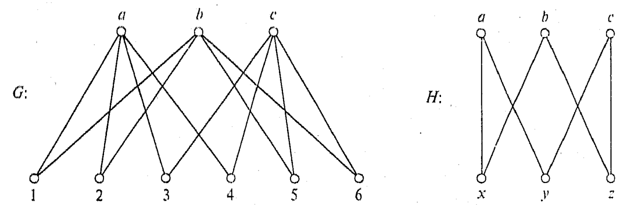

Fig. 7

Both G and H are bipartite line-symmetric graphs with essentially the same structure. They differ only in that the vertex x of H has been replaced in G by the two indistinguishable vertices 1 and 2 with the same neighbors a and b. Likewise y is replaced by 3 and 4 and z by 5 and 6. In short $G = \mathscr{S}^*H$ where $S_x = \{1, 2\}$, $S_y = \{3, 4\}$, $S_z = \{5, 6\}$ and $S_a = \{a\}$, $S_b = \{b\}$ and $S_c = \{c\}$. With regard to the bipartite and line-symmetry properties the extra vertices in G with the same neighborhoods are of no consequence. With respect to these properties G and H do indeed represent essentially the same structure, which suggests that reducing a graph such as G which has indistinguishable vertices to a relatively more simple graph such as H which does not have indistinguishable vertices but otherwise represents the same graphical structure might be useful in the study of line-symmetry.

Let G be a graph. The *nucleus* of G, denoted $\gamma(G)$ is the graph with vertex set $\{\bar{x} |\ x \in V(G)\}$ and edge set $\{\{\bar{x}, \bar{y}\} |\ \{x, y\} \in E(G)\}$. By Lemma 24, $E(\gamma(G))$ is well defined. Recall that in Section II we called a graph elementary if no two of its vertices have the same neighborhood. In our present terminology G is elementary if no two vertices of G are indistinguishable. The nucleus of a graph is just the elementary graph with essentially the same structure. The process of reducing a graph to its nucleus is just the reverse of vertex-splitting.

THEOREM 27. *The nucleus of a graph G is the unique elementary graph H such that G is a vertex splitting of H.*

Proof. Let $H = \gamma(G)$. For each $\bar{x} \in V(H)$ let $S_{\bar{x}} = \bar{x}$, and $\mathscr{S} = \{S_{\bar{x}} |\ \bar{x} \in V(H)\}$. Then $\mathscr{S}$ is a vertex splitting family for H and $G = \mathscr{S}^*H$. Suppose conversely that H is an elementary graph and that $G = \mathscr{S}^*H$ for some vertex splitting family $\mathscr{S} = \{S_v |\ v \in V(H)\}$. By Lemma 11 two vertices of G are indistinguishable if and only if they are fragments of the same vertex in H. Thus for $x \in S_v$, $\bar{x} = S_v$. Thus $V(\gamma(G)) = \{S_v |\ v \in V(H)\}$ and $E(\gamma(G)) = \{\{S_v, S_w\} |\ \{v, w\} \in E(H)\}$. So the function $\pi\colon V(H) \to V(\gamma(G))$ defined by $\pi(v) = S_v$ is an isomorphism of $\gamma(G)$ with H and, up to isomorphism, H is $\gamma(G)$. ■

Many of the results in Section II can be reinterpreted in terms of the nucleus. Theorem 28 follows from Theorems 12 and 13, Theorem 29 from Theorem 14, Corollary 29a from Corollary 14a, and Corollary 29b from Corollary 14b.

THEOREM 28. *If G is line-symmetric (point-symmetric) then $\gamma(G)$ is also line-symmetric (point-symmetric).*

THEOREM 29. *Let G be a graph such that $\gamma(G)$ is line-symmetric and there exists a subgroup Γ of $\mathscr{A}(\gamma(G))$ such that Γ^* acts transitively on the edges of $\gamma(G)$ and $|\bar{x}| = |\alpha(\bar{x})|$ for all $x \in V(G)$ and $\alpha \in \Gamma$. Then G is line-symmetric.*

COROLLARY 29a. *If G is a graph such that $\gamma(G)$ is line-symmetric and $|\bar{x}| = |\bar{y}|$ for all $x, y \in V(G)$ then G is line-symmetric.*

COROLLARY 29b. *If G is such that $\gamma(G)$ is biregular and line-symmetric and if $|\bar{x}| = |\bar{y}|$ for all $x, y \in V(G)$ such that $\bar{x}$ and $\bar{y}$ have equal degree as vertices of $\gamma(G)$, then G is line-symmetric.*

COROLLARY 29c. *If G is a connected bipartite graph such that $\gamma(G)$ is an even cycle and if $|\bar{x}| = |\bar{y}|$ whenever x and y are vertices in the same partite set of G, then G is line-symmetric.*

Proof. This follows by Corollary 14c and the fact that $\gamma(G)$ is connected and bipartite and if $V(G) = V_1 \cup V_2$ is the unique bipartition of G, then $V(\gamma(G)) = \{\bar{x} |\ x \in V_1\} \cup \{\bar{x} |\ x \in V_2\}$ is the unique bipartition of $\gamma(G)$. ■

Theorem 30 follows from Theorem 15 by applying this to Theorem 30. Theorem 31 is established using Theorem 16 and Corollary 15a.

THEOREM 30. *If $\gamma(G)$ is point-symmetric and if Γ is a subgroup of $\mathscr{A}(\gamma(G))$ such that Γ acts transitively on $V(\gamma(G))$ and if $|\bar{x}| = |\alpha(\bar{x})|$ for every $x \in V(G)$ and every $\alpha \in \Gamma$, then G is point-symmetric.*

THEOREM 31. *If $\gamma(G)$ is point-symmetric then G is point-symmetric if and only if $|\bar{x}| = |\bar{y}|$ for every $x, y \in V(G)$.*

Our next result is the natural one which we might expect about isomorphisms of graphs and their nuclei.

LEMMA 32. *Let G and H be isomorphic graphs and $\pi\colon V(G) \to V(H)$ be an isomorphism of G with H. Define $\bar{\pi}\colon V(\gamma(G)) \to V(\gamma(H))$ by $\bar{\pi}(\bar{x}) = \overline{\pi(x)}$. Then $\bar{\pi}$ is well defined and $\bar{\pi}$ is an isomorphism of $\gamma(G)$ with $\gamma(H)$.*

Proof. If $\bar{x} = \bar{y}$ then $N(x) = N(y)$, so $N(\pi(x)) = \pi(N(x)) = \pi(N(y)) = N(\pi(y))$ so $\overline{\pi(x)} = \overline{\pi(y)}$. Thus π is well-defined. Similarly, if $\bar{\pi}(\bar{x}) = \bar{\pi}(\bar{y})$ then $\overline{\pi(x)} = \overline{\pi(y)}$, so $\pi(N(x)) = N(\pi(x)) = N(\pi(y)) = \pi(N(y))$. Thus, since π is injective $N(x) = N(y)$ and $\bar{x} = \bar{y}$. So $\bar{\pi}$ is injective. It is apparent that $\bar{\pi}$ is surjective because π is. For $x, y \in V(G)$, $\bar{x}$ and $\bar{y}$ are adjacent vertices in $\gamma(G)$ if and only if x and y are adjacent in G, in which case $\pi(x)$ and $\pi(y)$ are adjacent in H, so that $\bar{\pi}(\bar{x})$ and $\bar{\pi}(\bar{y})$ are adjacent in $\gamma(H)$. Thus $\bar{\pi}$ is an isomorphism. ■

The nucleus has a bearing on the position of the anomalous biregular line-symmetric graphs $K(2, n)$, $n \geqslant 3$, which, although they have vertices of degree 2, are not subdivisions of regular line-symmetric graphs. All of these graphs have the same nucleus K_2, as does the graph $K(2, 1)$ which is a subdivision, namely $S(K_2)$. So in some sense the graphs $K(2, n)$ which are not themselves subdivision have the same structure as $K(2, 1)$ which is a subdivision.

We can produce other examples of non-elementary biregular line-symmetric graphs with vertices of degree 2 which are not subdivisions by splitting only the "edge" vertices of some line-symmetric subdivision. One example is shown in Figure 8.

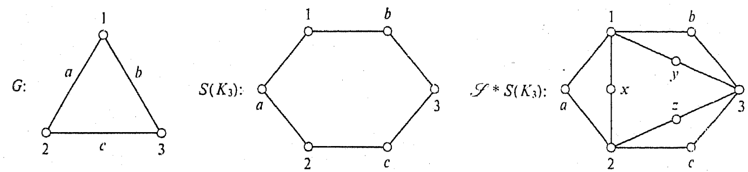

Fig. 8

Let $t \geqslant 2$ and $p \geqslant 2$ be integers such that $\max\{p, t\} \geqslant 3$. Let $\mathscr{S}$ be a vertex splitting family for $S(K_p)$ with $|S_x| = 1$ for all $x \in V(K_p)$ and $|S_x| = t$ for all $x \in E(K_p)$ and let $G = \mathscr{S}^*S(K_p)$. Then G is a biregular line-symmetric graph which contains vertices of degree 2. For $p = 3$ and $t = 2$ the construction is shown in Figure 8. Note that the restriction $\max\{p, t\} \geqslant 3$ is necessary for biregularity of G since for $p = t = 2$, G is C_4. We note that all of the examples obtained in this manner have the subdivision $S(K_p)$ as their nucleus.

We now move in the direction of another description of biregular line-symmetric graphs, for which another construction is needed, a construction which was inspired by the idea of an intersection graph. Let T be a finite set and let $\mathscr{S}$ be a family of subsets of T. The *inclusion graph* associated with $\mathscr{S}$, denoted $G(\mathscr{S})$, is the graph with vertex set $\mathscr{S}$ and edge set $\{\{S_1, S_2\} \subset \mathscr{S} |\ S_1 \subsetneq S_2\}$.

Certain isomorphisms of inclusion graphs are easy to describe.

LEMMA 33. *Let T_1, T_2 be finite sets with $|T_1| = |T_2|$ and let $\mathscr{S}_1, \mathscr{S}_2$ be families of subsets of T_1 and of T_2 respectively, with $|\mathscr{S}_1| = |\mathscr{S}_2|$. Let $\pi\colon T_1 \to T_2$ be a bijective mapping such that for each $S \in \mathscr{S}_1$, $\{\pi(x)|\ x \in S\} \in \mathscr{S}_2$. Define $\pi'\colon \mathscr{S}_1 \to \mathscr{S}_2$ by $\pi'(S) = \{\pi(x)|\ x \in S\}$. Then π' is an isomorphism of $G(\mathscr{S}_1)$ with $G(\mathscr{S}_2)$.*

Proof. The mapping π' is injective because π is injective. Then π' is surjective because $|\mathscr{S}_2| = |\mathscr{S}_1|$ and π' is injective. It is apparent that for $S_1, S_2 \in \mathscr{S}_1$, $S_1 \subsetneqq S_2$ if and only if $\pi'(S_1) \subsetneqq \pi'(S_2)$. Thus π' is an isomorphism of $G(\mathscr{S}_1)$ with $G(\mathscr{S}_2)$. ■

Let m, n and t be positive integers with $m \leqslant n \leqslant t$. Let T be a set with $|T| = t$. Let $\mathscr{S}(T, m, n) = \{S \subset T|\ |S| = m$ or $|S| = n\}$. The *complete* (t, m, n) *inclusion graph*, denoted $G(t, m, n)$ is the graph $G(\mathscr{S}(T, m, n))$. A graph G is a (t, m, n) *inclusion graph* if there exists a family $\mathscr{S} \subset \mathscr{S}(T, m, n)$ such that $G \cong G(\mathscr{S})$.

We are now prepared for the results which explain our interest in inclusion graphs.

THEOREM 34. *Let t, m, n be positive integers with $m < n \leqslant t$. Then the complete inclusion graph $G(t, m, n)$ is line-symmetric.*

Proof. Let $m < n \leqslant t$. Let T be a set with $|T| = t$ and $\mathscr{S} = \mathscr{S}(T, m, n)$. Then $G(t, m, n) = G(\mathscr{S})$. Let $e = \{R_1, S_1\}$ and $f = \{R_2, S_2\}$ be edges of $G(\mathscr{S})$. Without loss of generality we may assume that $|R_1| = |R_2| = m$ and $|S_1| = |S_2| = n$, and $R_1 \subset S_1$, $R_2 \subset S_2$ so that $|S_i - R_i| = n - m$ and $|T - S_i| = t - n$, for $i = 1, 2$. Then there exist bijections $\alpha\colon R_1 \to R_2$ and $\beta\colon (S_1 - R_1) \to (S_2 - R_2)$ and $\gamma\colon (T - S_1) \to (T - S_2)$. Let $\pi\colon T \to T$ be the mapping $\alpha \cup \beta \cup \gamma$; i.e., $\pi(x) = \alpha(x)$ for $x \in R_1$ and $\pi(x) = \beta(x)$ for $x \in (S_1 - R_1)$ and $\pi(x) = \gamma(x)$ for $x \in (T - S_1)$. Then π is a bijection of T. By Lemma 33 the induced map π' is an automorphism of $G(\mathscr{S})$; with $\pi'(R_1) = R_2$ and $\pi'(S_1) = S_2$ so $\hat{\pi}'(e) = f$. Thus $G(\mathscr{S}) = G(t, m, n)$ is line-symmetric. ■

We note that the complete (t, m, n) inclusion graphs are always biregular and elementary.

Our next result is not quite the converse of the preceding one, but it comes close.

THEOREM 35. *Let G be an elementary biregular line-symmetric graph. Then there exist positive integers n and t with $t \geqslant n \geqslant 2$ such that G is a $(t, 1, n)$ inclusion graph.*

Proof. Let G be an elementary biregular line-symmetric graph. Let m and n be the degrees which occur in G where $m < n$. Let $S = \{x \in V(G)|\ \deg x = n\}$ and $T = \{x \in V(G)|\ \deg x = m\}$. By Theorem 10, $V(G) = S \cup T$ is a bipartition of G. Let $\mathscr{S} = \{\{x\}|\ x \in T\} \cup \{N(x)|\ x \in S\}$. Then $\mathscr{S}$ is a family of subsets of T and $\mathscr{S} \subset \mathscr{S}(T, 1, n)$. Define $\varphi\colon V(G) \to \mathscr{S}$ by $\varphi(x) = \{x\}$ for $x \in T$ and $\varphi(x) = N(x)$ for $x \in S$. Since G is elementary and because $n > m$ which implies that $n \geqslant 2$, φ is injective. By the construction of $\mathscr{S}$, φ is surjective. Let $x \in S$, $y \in T$. Then, in G, x is adjacent to y if and only if $y \in N(x)$, in which case $\{y\} \subset N(x)$ and, in $G(\mathscr{S})$, $\varphi(y)$ is adjacent to $\varphi(x)$. Thus φ is an isomorphism of G with $G(\mathscr{S})$ so that G is a $(t, 1, n)$ inclusion graph with $t = |T|$. ■

We note again that Theorems 34 and 35 are not quite converses. In fact the full converse of Theorem 35 is false as the next example illustrates.

EXAMPLE 8. Let $T = \{a, b, c\}$ and let $\mathscr{S}$ consist of $\{a\}$, $\{b\}$, $\{a, b\}$ and $\{a, c\}$. Then $G(\mathscr{S}) \cong P_4$ is a $(3, 1, 2)$ inclusion graph which is not line-symmetric (see Fig. 9).

Fig. 9

We close by noting that in the proof of Theorem 35 we could just as well have used S in place of T and m in place of n provided that $m \neq 1$. Thus if G is an elementary biregular line-symmetric graph which contains vertices of degree 2 then G is a $(T, 1, 2)$ inclusion graph for some finite set T. The special case of Theorem 34 when vertices of degree 2 are present is that for every integer $t \geqslant 2$, the complete $(T, 1, 2)$ inclusion graph $G(t, 1, 2)$ is an elementary biregular line-symmetric graph. Actually it is readily verified that $G(t, 1, 2) \cong S(K_t)$. Thus the complete inclusion graphs do not provide any new examples of biregular line-symmetric graphs with degree 2 vertices with which we were previously not familiar.

References

[1] M. Behzad, G. Chartrand and L. Lesniak-Foster, *Graphs and Digraphs*, Boston 1979.

[2] I. Z. Bouwer, *On edge but not vertex transitive regular graphs*, J. Combinatorial Theory Ser. B 12 (1972), pp. 32–40.

[3] D. Burns, S. F. Kapoor and P. A. Ostrand, *Uniquely edge extendible graphs: I* (to appear).

[4] — — — *Edge Cosymmetric Graphs*, Discrete Math. 32 (1980), pp. 99–103.

[5] — — — *Uniquely edge extendible graphs: II* (to appear).

[6] E. Dauber and F. Harary, *On point-symmetric and line-symmetric graphs* (unpublished manuscript).

[7] J. Folkman, *Regular line-symmetric graphs*, J. Combinatorial Theory 3 (1967), pp. 215–232.

[8] F. Harary, *Graph Theory*, Reading 1969.

FERRIS STATE COLLEGE
Big Rapids, Michigan 49307
WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan 49008
UNIVERSITY OF CALIFORNIA
Santa Barbara, California 93106

Received 25 September 1979;
in revised form 16 June 1980