A condition under which 2-homogeneity and representability are the same in continua

by

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Abstract. Our main theorem is the following: If X is a 2-homogeneous continuum and X admits a non-identity primitively stable homeomorphism, then X is representable. The following are corollaries to this theorem: (1) If M is the Menger universal curve and X is a continuum, then $M \times X$ is not 2-homogeneous. (2) If X and Y are homeotopically homogeneous continua, then $X \to Y$ is representable.

The main result in this paper is the following theorem: If X is a continuum which is 2-homogeneous and X admits a primitively stable homeomorphism which is not the identity, then X is representable.

C. E. Burgess [4] asked in 1955 whether, for $n \geqslant 2$, n-homogeneity implies $(n+1)$-homogeneity. The above theorem gives a partial answer to this question, since a representable continuum is n-homogeneous for each n. (This follows from results in [2], [3], [9], and [10].) Also, some corollaries follow from the theorem. One gives that if M is the Menger universal curve and X is a continuum, then $M \times X$ is not 2-homogeneous. This answers a question asked by K. Kuperberg, W. Kuperberg and W. R. R. Transue in [8]. Another corollary generalizes a result of G. S. Ungar which follows from results in [9] and [10].

Definitions, notation, background. In this paper a continuum is a compact, metric, connected space. If X is a continuum, $H(X)$ denotes the space of all homeomorphisms from X onto itself. It is well-known that $H(X)$ with the sup metric is itself a separable metric space and that it is also a complete topological space, although it may not be complete with respect to this metric.

If n is a positive integer, a space X is *n-homogeneous* means that if A and B are 2 n-element subsets of X, then there is a homeomorphism $h \in H(X)$ such that $h(A) = B$. A space X is *strongly n-homogeneous* means that if $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ are 2 n-element subsets of X then there is a homeomorphism h such that $h(a_i) = b_i$ for every $i \leqslant n$. X is *countable dense homogeneous* means that if A and B are 2 countable dense subsets of X, then there is a homeomorphism $h \in H(X)$ such that $h(A) = B$.

Let e denote the sup metric on $H(X)$. For every $\varepsilon > 0$ and $f \in H(X)$, let $N_\varepsilon(f) = \{g \in H(X) |\ e(f, g) < \varepsilon\}$.

A *topological transformation group* (G, X) is a topological group G together with a topological space X and a continuous map $(g, x) \to g(x)$ of $G \times X$ into X such that $(gh)(x) = g(h(x))$, and, if $\bar{e}$ is the identity of G, $\bar{e}(x) = x$ for all g and h in G and x in X. (G, X) is polish if both G and X are separable and metrizable by a complete metric. When X is a continuum, $(H(X), X)$ is a polish transformation group.

For every x in X let G_x denote the *stabilizer subgroup* of x in G, i.e., $G_x = \{g \in G |\ g(x) = x\}$. Let Gx denote the *orbit of* x, i.e., $Gx = \{y \in X |$ there is $g \in G$ such that $g(x) = y\}$. If G/G_x denotes the left coset space with the usual topology, Edward G. Effros [5] has proved the following theorem:

Let (G, X) be a polish transformation group. Then the following are equivalent:

(1) For each x in X, the map $gG_x \to g(x)$ of G/G_x onto Gx is a homeomorphism.

(2) Each orbit is second category in itself.

(3) Each orbit is G_δ in X.

(4) X/G is T_0.

G. S. Ungar [9 and 10] has used this theorem of Effros to get some very nice results. Here, in the proofs that follow, the following theorems of his will be used:

(1) If X is a homogeneous continuum, then for each x in X, the map T_x: $H(X) \to X$ defined by $T_x(f) = f(x)$ for f in $H(X)$ is an open, onto map.

(2) Let $F^2(X)$ denote the *2nd configuration space of* X, i.e., $F^2(X) = \{(x, y) \in X^2 |\ x \neq y\}$. Then Ungar has shown that if X is a 2-homogeneous continuum, and x and y are points of X such that $x \neq y$, then the map $T_{x,y}$: $H(X) \to F^2(X)$ defined by $T_{x,y}(h) = (h(x), h(y))$ is open and onto.

(3) If X is 2-homogeneous, then X is locally connected.

(4) If X is an n-homogeneous continuum, then X is strongly n-homogeneous or X is the circle.

A homeomorphism h of $H(X)$ is said to be *primitively stable* if there is an open set o in X such that $h \restriction o = \mathrm{id}_o$. I do not know of an example of a 2-homogeneous continuum that does not admit a primitively stable homeomorphism other than the identity, or even of an example of such a homogeneous continuum.

A space X is *representable* means that if $x \in X$ and u is open such that $x \in u$, then there is an open set v such that $x \in v \subseteq u$ and if $y \in v$, then there is $h \in H(X)$ such that $h(x) = y$ and $h(z) = z$ for every $z \notin u$. (I would like to note that this notion, which was introduced by Peter Fletcher in [6], is equivalent to the notion of strong local homogeneity, which was introduced by L. R. Ford in [7]. John Bales, [2], proved this equivalence.)

If a is a collection of subsets of X, then $a^* = \{x \in X |\ x \in A$ for some $A \in a\}$.

Proof of the theorem. First we need some lemmas. For all lemmas and the theorem, assume that X is a 2-homogeneous continuum and X admits a primitively stable homeomorphism which is *not* the identity. Let d denote a metric on X which is compatible with the topology on X, and for every $\varepsilon > 0$, $x \in X$, let $D_\varepsilon(x) = \{y \in X |\ d(x, y) < \varepsilon\}$.

LEMMA 1. *Suppose $x \in X$. Then there is a primitively stable homeomorphism $h \in H(X)$ such that $x \notin N$ where $N = \{o \subseteq X |\ o$ is open and $h \restriction o = \mathrm{id}_o\}^*$.*

Proof. Let f denote a primitively stable homeomorphism of $H(X)$ such that $f \neq \mathrm{id}_X$. Let $P = \{o \subseteq X |\ o$ is open and $f \restriction o = \mathrm{id}_o\}^*$. There is z in $X - P$ such that $f(z) \neq z$. Suppose $p \in P$, and $y \neq x$, $y \neq p$. Then there is $\Phi \in H(X)$ such that $\Phi(p) = y$, and $\Phi(z) = x$, and $h = \Phi \circ f \circ \Phi^{-1}$ is the desired homeomorphism.

LEMMA 2. *Suppose h is a primitively stable homeomorphism other than the identity on X, and $N = \{o$ open in $X |\ h \restriction o = \mathrm{id}_o\}^*$. Then suppose M is open such that $M \subseteq \bar{M} \subseteq N$. If $x \notin \bar{N}$ there is u_x open in X such that $x \in u_x$ and if $z \in u_x$ there is $g \in H(X)$ such that $g(x) = z$, and $g \restriction \bar{M} = \mathrm{id}_{\bar{M}}$.*

Proof. There is a positive number δ such that if $k \in H(X)$ such that $k \in N_\delta(\mathrm{id})$, $k(\bar{M}) \subseteq N$ and $k^{-1}(\bar{M}) \subseteq N$. Either $x = h(x)$ or $x \neq h(x)$.

Case i. Suppose $x \neq h(x)$. Now $T_{x,h(x)}(N_\delta(\mathrm{id}))$ is open in $F^2(X)$ and contains $(x, h(x))$. There is an open subset u' of X such that $x \in u'$, $u' \times h(u') \subseteq T_{x,h(x)}(N_\delta(\mathrm{id}))$, and $u' \cap h(u') = \emptyset$. Then there is an open set $\hat{u}$ such that $x \in \hat{u} \subseteq \bar{\hat{u}} \subseteq u'$. There is a positive number σ less than δ such that if $k \in N_\delta(\mathrm{id})$, $k(\bar{\hat{u}}) \subseteq u'$, and $h \circ k(\bar{\hat{u}}) \subseteq h(u')$ and, finally, there is an open set u such that $x \in \bar{u} \subseteq \hat{u}$ and $u \times h(u) \subseteq T_{x,h(x)}(N_\sigma(\mathrm{id}))$.

Suppose $z \in u$, $z \neq x$. $(x, h(z)) \in u \times h(u) \subseteq T_{x,h(x)}(N_\sigma(\mathrm{id}))$. There is Φ in $N_\sigma(\mathrm{id})$ such that $\Phi(x) = x$ and $\Phi(h(x)) = h(z)$. Now $\Phi^{-1}(z) \neq x$, so denote $\Phi^{-1}(z)$ by r. Since $\Phi^{-1} \in N_\sigma(\mathrm{id})$, $\Phi^{-1}(z) = r \in u'$. Since $(z, h(r)) \in u' \times h(u')$, there is $\Gamma \in N_\delta(\mathrm{id})$ such that $\Gamma(x) = z$ and $\Gamma(h(x)) = h(r)$. Then

$$\begin{aligned} \Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h \circ \Phi^{-1} \circ h^{-1} \circ \Phi \circ h(x) &= \Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h \circ \Phi^{-1} \circ h^{-1}(h(z)) \\ &= \Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h \circ \Phi^{-1}(z) = \Gamma \circ h^{-1} \circ \Gamma^{-1}(h(r)) = \Gamma \circ h^{-1}(h(x)) = z. \end{aligned}$$

If $s \in \bar{M}$,

$$\begin{aligned} &\Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h \circ \Phi^{-1} \circ h^{-1} \circ \Phi \circ h(s) \\ &\quad = \Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h \circ \Phi^{-1} \circ h^{-1} \circ \Phi(s) = \Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h(s) \\ &\quad = \Gamma \circ h^{-1} \circ \Gamma^{-1}(s) = s. \end{aligned}$$

(Recall that $\Phi(s) \in N$ and $\Gamma^{-1}(s) \in N$.)

Thus $g = \Gamma \circ h^{-1} \circ \Gamma^{-1} \circ h \circ \Phi^{-1} \circ h^{-1} \circ \Phi \circ h$ is the desired homeomorphism and u is the desired open set u_x.

Case ii. Suppose $x = h(x)$. There is an open set M' such that

$\bar{M} \subseteq M' \subseteq \bar{M} \subseteq N$. There is a positive number δ' such that if $k \in N_{\delta'}(\mathrm{id})$, $k(\bar{M}') \subseteq N$ and $k^{-1}(\bar{M}') \subseteq N$. Since $T_x(N_{\delta'}(\mathrm{id}))$ is open in X, there is an open set u' such that $x \in u' \subseteq \bar{u}' \subseteq T_x(N_{\delta'}(\mathrm{id}))$. There is u open in X such that $x \in u \subseteq \bar{u} \subseteq u'$ and if $w \in u$, $h(w) \in u'$. Pick z out of u such that $z \neq h(z)$. Then there is $\Phi' \in H(X)$ such that $\Phi'(x) = h(z)$ and $\Phi' \in N_{\delta'}(\mathrm{id})$. Now $h(z) \neq x$ and $\Phi'^{-1}(z) \neq x$. Further

$$\Phi'^{-1} \circ h^{-1} \circ \Phi' \circ h(x) = \Phi'^{-1} \circ h^{-1} \circ \Phi'(x) = \Phi'^{-1} \circ h^{-1}\big(h(z)\big) = \Phi'^{-1}(z) \neq x$$

and if $s \in \bar{M}'$,

$$\Phi'^{-1} \circ h^{-1} \circ \Phi' \circ h(s) = \Phi'^{-1} \circ h^{-1} \circ \Phi'(s) = \Phi'^{-1} \circ \Phi'(s) = s.$$

Let $\alpha = \Phi'^{-1} \circ h^{-1} \circ \Phi' \circ h$. Then $\alpha \upharpoonright \bar{M}' = \mathrm{id}_{\bar{M}'}$, $\alpha(x) \neq x$ and $M \subseteq \bar{M} \subseteq M'$. Now apply case (i) to α, M, and M' to finish the proof of the lemma.

LEMMA 3. *Suppose that h is a primitively stable homeomorphism other than the identity, and $N = \{o \text{ open in } X \mid h \upharpoonright o = \mathrm{id}_o\}^*$. Suppose further that M is open such that $\bar{M} \subseteq N$. Then if D is a component of $X - \bar{N}$ and x and z are points of D, there is g in $H(X)$ such that $g(x) = z$ and $g \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$.*

Proof. Since X is locally connected, D is locally connected and open. Suppose $x \in D$. Let $A_x = \{y \in D \mid$ there is $f \in H(X)$ such that $f(x) = y$ and $f \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}\}$. $A_x \neq \Phi$ for A_x contains the open set $u_x \cap D$ where u_x is as in Lemma 2. In fact, this gives that A_x is open in D, for if $y \in A_x$, $u_y \cap D \subseteq A_x$: If $w \in u_y \cap D$, there is $\Phi \in H(X)$ such that $\Phi(y) = w$ and $\Phi \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$. There is $\beta \in H(X)$ such that $\beta(x) = y$ and $\beta \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$. Then $\Phi \circ \beta(x) = w$ and $\Phi \circ \beta \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$, so $w \in A_x$.

Also, A_x is closed in D: $z \in \bar{A}_x \cap D$ implies that there is a point $y \in u_z \cap A_x$. There is $\alpha \in H(X)$ such that $\alpha(x) = y$ and $\alpha \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$ and there is σ such that $\sigma(z) = y$ and $\sigma \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$. Then $\sigma^{-1} \circ \alpha(x) = z$ and $\sigma^{-1} \circ \alpha \upharpoonright \bar{M} = \mathrm{id}_{\bar{M}}$. Thus $z \in A_x$.

A_x is both open and closed in a connected set D, so $A_x = D$.

LEMMA 4. *Suppose $x \neq y$. Then there is an open set E such that (1) $\bar{E} \neq X$, (2) $y \in N^0$ (N^0 denotes the interior of N) where $N = X - E$, (3) $X \in E$, (4) E is connected, and (5) if $z \in E$ there is $h \in H(X)$ such that $h(x) = z$ and $h \upharpoonright N = \mathrm{id}_N$. Further, if $w \in E$, $\varepsilon > 0$, there is an open set o such that $w \in o$ and if $t \in o$, there is $f \in H(X)$ such that $f(w) = t$, $f \in N_\varepsilon(\mathrm{id})$, and $f \upharpoonright N = \mathrm{id}_N$.*

Proof. From Lemma 1 it follows that there is a primitively stable homeomorphism Γ in $H(X)$ which is not the identity such that $\Gamma(x) \neq x$ and $y \in G = \{o \text{ open in } X \mid \Gamma \upharpoonright o = \mathrm{id}_o\}^*$.

Suppose that N' is an open set such that $\bar{N}' \subseteq G$ and $y \in N'$. There is a positive number ε such that $D_\varepsilon(\bar{N}') \subseteq G$. For every positive integer i, let $N_i = D_{\varepsilon/i}(\bar{N}')$, and let $A_i = \{w \in X - \bar{N}' \mid$ there is $f \in H(X)$ such that $f(x) = w$ and $f \upharpoonright \bar{N}_i = \mathrm{id}_{\bar{N}_i}\}$. Let D be the component of $X - \bar{G}$ that contains x. Then $x \in D \subseteq A_i$ for every i and D is an open set.

Suppose $\hat{w} \in A_i$ for some i. There is a homeomorphism δ in $H(X)$ such that $\delta(x) = \hat{w}$ and $\delta \upharpoonright \bar{N}_i = \mathrm{id}_{\bar{N}_i}$. Then $\hat{w} \notin \bar{N}_i$, and there is an open set $o_{\hat{w}}$ such that $\hat{w} \in o_{\hat{w}}$, and $\bar{o}_{\hat{w}} \cap \bar{N}_i = \emptyset$, and if $w' \in o_{\hat{w}}$, there is $\beta \in H(X)$ such that $\beta(\hat{w}) = w'$ and $\beta \upharpoonright \bar{N}_{i+1} = \mathrm{id}_{\bar{N}_{i+1}}$.

Pick $\hat{w}' \in o_{\hat{w}}$. There is $\beta' \in H(X)$ such that $\beta'(\hat{w}) = \hat{w}'$ and $\beta' \upharpoonright \overline{N_{i+1}} = \mathrm{id}_{\overline{N_{i+1}}}$. Then $\beta \circ \delta(x) = \hat{w}'$, and $\beta \circ \delta \upharpoonright \overline{N_{i+1}} = \mathrm{id}_{\overline{N_{i+1}}}$. Then $o_{\hat{w}} \subseteq A_{i+1}$ and $A = \bigcup_{i=1}^{\infty} A_i$ is open in $X - \bar{N}'$.

Now suppose $z \in \bar{A} - A$. Suppose further that $g \in H(X)$ such that $g \upharpoonright \bar{N}_i = \mathrm{id}_{\bar{N}_i}$ for some i. Then $g(z) = z$:

Suppose not. Then there is $g' \in H(X)$ such that $g' \upharpoonright \bar{N}_i = \mathrm{id}_{\bar{N}_i}$ for some i and $g'(z) = l' \neq z$. Applying Lemma 2 we get that there is an open set v_z of X such that $z \in v_z$ and if $z' \in v_z$ there is $\hat{\alpha} \in H(X)$ such that $\hat{\alpha}(z) = z'$ and $\hat{\alpha} \upharpoonright \overline{N_{i+1}} = \mathrm{id}_{\overline{N_{i+1}}}$. The fact that there is $\hat{w}$ in $A \cap v_z$ means that $\hat{w} \in A_j$ for some j and there is $\hat{\beta} \in H(X)$ such that $\hat{\beta}(x) = \hat{w}$ and $\beta \upharpoonright \bar{N}_j = \mathrm{id}_{\bar{N}_j}$. There is $\hat{\hat{\alpha}} \in H(X)$ such that $\hat{\hat{\alpha}}(z) = \hat{w}$ and $\hat{\hat{\alpha}} \upharpoonright \overline{N_{i+1}} = \mathrm{id}_{\overline{N_{i+1}}}$. Then $\hat{\hat{\alpha}}^{-1} \circ \hat{\beta}(x) = z$ and $\hat{\hat{\alpha}}^{-1} \circ \hat{\beta} \upharpoonright \bar{N}_{j'} = \mathrm{id}_{N_{j'}}$ for some positive integer j'. Thus $z \in A_{j'} \subseteq A$, which is a contradiction.

Let E denote the component of A that contains x. E is both open and closed in A. Then $\bar{E} - E \subseteq \bar{A} - A$ and if $t \in \bar{E} - E$ and $f \in H(X)$ such that $f \upharpoonright \bar{N}_i = \mathrm{id}_{\bar{N}_i}$ for some i, then $f(t) = t$. Let $N'' = X - \bar{E}$. Now $y \in N''$, but possibly, $X - \bar{N}'' \neq E$, so let N^* denote $X - E$.

Suppose $z \in E$. There is $\hat{h} \in H(X)$ such that $\hat{h}(x) = z$ and $\hat{h} \upharpoonright N_i = \mathrm{id}_{\bar{N}_i}$ for some i. Define h as follows: If $s \in X$,

$$h(s) = \begin{cases} \hat{h}(s) & \text{if } s \in E, \\ s & \text{if } s \notin E. \end{cases}$$

Therefore, if $z \in E$, there is $h \in H(X)$ such that $h(x) = z$ and $h \upharpoonright (X - E) = \mathrm{id}_{(X-E)}$. The first part of the lemma is proved.

Suppose $w \in E$ and $\varepsilon > 0$. Let $H' = \{h \in H(X) \mid h \upharpoonright N^* = \mathrm{id}_{N^*}\}$. H' is a closed subgroup of $H(X)$; so (H', X) is a transformation group with both H' and X complete separable metric spaces. For every $x \in X$, let $G'_x = \{g \in H' \mid g(x) = x\}$ and $Gx' = \{a(x) \mid a \in H'\}$. Now $x \in X$ means that either $x \in N^*$ or $x \in E$; and $x \in N^*$ means that $Gx' = \{x\}$, $x \in E$ means that $Gx' = E$. Then each orbit Gx' is a G_δ-set in X, so the map $gG'_x \to g(x)$ of H'/G'_x onto Gx' is a homeomorphism for every $x \in X$ (Effros' Theorem).

Then the map $T'_x\colon H' \to E$ defined for $x \in E$ by $T'_x(g) = g(x)$ (for $g \in H'$) is open. (Recall that the map $\Phi\colon H' \to H'/G'_x$ is open where $\Phi(g) = gG'_x$ and $T'_x(g) = \Gamma \circ \Phi(g)$ where $\Gamma(gG'_x) = g(x)$. Thus Γ is a homeomorphism, Φ is open given that T'_x is open.)

Pick $w \in E = X - N^*$. $T'_w(N_\varepsilon(\mathrm{id}) \cap H')$ is open in E and contains w. Hence if $z \in T'_w(N_\varepsilon(\mathrm{id}) \cap H')$, there is $f \in H' \cap N_\varepsilon(\mathrm{id})$ such that $f(w) = z$. $T_{w'}(N_\varepsilon(\mathrm{id}) \cap H')$ is the desired o.

LEMMA 5. *Suppose $x \in X$, and Γ is a primitively stable homeomorphism such that $\Gamma(x) \neq x$. Let $G = \{o \text{ open in } X \mid \Gamma \restriction o = \mathrm{id}\, o\}^*$, and suppose M is open such that $\bar{M} \subseteq G$. Then there are open sets E and N in X such that* (1) *$x \in E$,* (2) *$N = X - \bar{E}$,* (3) *E is connected, and* (4) *if $z \in E$ there is $h \in H(X)$ such that $h(x) = z$ and $h \restriction N = \mathrm{id}_N$, and* (5) *$\bar{M} \subseteq N$. Further, if $w \in E$, $\varepsilon > 0$, there is an open set o such that $w \in o$ and if $t \in o$, there is $f \in H(X)$ such that $f(w) = t$, $f \in N_\varepsilon(\mathrm{id})$, and $f \restriction N = \mathrm{id}_N$.*

Proof. This follows from the proof of Lemma 4. (Note that the N^0 of Lemma 4 is the N of this lemma).

LEMMA 6. *If u is open in X, there is an open subset v of u such that if x, y are in v, there is $h \in H(X)$ such that $h(x) = y$ and $h \restriction (X-u) = \mathrm{id}_{(X-u)}$.*

Proof. If A is a subset of X let bd A denote the boundary of A. Suppose u is open in X such that $\bar{u} \neq X$. Pick $x \in u$. For every y in bd u there are sets Ny, Ey such that (1) $y \in Ny^0$, (2) $x \in Ey$, and (3) Ey and Ny are as the E and N in Lemma 4. Let $Ny^* = X - Ey$ for every y.

There is a finite subcollection $\{y_1, y_2, \ldots, y_m\}$ of the bd u such that $\{Ny_1^0, Ny_2^0, \ldots, Ny_m^0\}$ covers bd u. For convenience rename $\{Ny_1^0, \ldots, Ny_m^0\}$, $\{N_1, \ldots, N_m\}$; $\{Ny_1^*, \ldots, Ny_m^*\}$, $\{N_1^\#, \ldots, N_m^\#\}$; $\{Ey_1, \ldots, Ey_m\}$, $\{E_1, \ldots, E_m\}$. There is a collection $\{M_1', \ldots, M_m'\}$ of open sets that covers bd u such that for every i, $\bar{M}_i' \subseteq N_i$.

Either bd $u \subseteq N_1^\#$, or bd $u \nsubseteq N_1^\#$. If bd $u \subseteq N_1^\#$, $E_1 \subseteq u$ (remember that it is connected) and E_1 is the desired v. Otherwise bd $u \nsubseteq N_1^\#$. There is $\hat{j}_2$ in $\{2, \ldots, m\}$ such that $M_{\hat{j}_2} \cap \mathrm{bd}\, u \cap (X - N_1^\#) \neq \emptyset$. Let $u_1 = [X - (\bigcup_{i \neq 1} \bar{N}_i \cup N_i^\#)] \cap u$. Pick $z_1 \in M_{\hat{j}_2} \cap \mathrm{bd}\, u \cap E_1$. Since E_1 is connected and locally connected, there is an arc from x to z_1 in E_1, and thus, there is a point $w_1 \in E_1 \cap \mathrm{bd}\, u_1 \cap \bar{N}_{j_2}$ for some $j_2 \neq 1$.

There is an open set o_{w_1} such that $w_1 \in o_{w_1} \subseteq \bar{o}_{w_1} \subseteq E_1$ and if t, w are in o_{w_1}, then there is $h \in H(X)$ such that $h(w) = t$, $h \restriction N_1^\# = \mathrm{id}_{N_1^\#}$, $h(\bar{M}_{j_2}') \subseteq N_{j_2}$, and $h^{-1}(\bar{M}_{j_2}') \subseteq N_{j_2}$. Pick $t_1 \in o_{w_1} \cap (u_1 - N_{j_2}^\#)$. Then there is a homeomorphism h_1 such that $h_1(t_1) = w_1$, $h_1 \restriction N_1^\# = \mathrm{id}_{N_1^\#}$, $h_1(\bar{M}_{j_2}') \subseteq N_{j_2}$ and $h_1^{-1}(\bar{M}_{j_2}') \subseteq N_{j_2}$. There is an open set o_{t_1} such that $t_1 \in o_{t_1} \subseteq \bar{o}_{t_1} \subseteq (u_1 - N_{j_2}^\#) \cap o_{w_1}$ and such that if $s \in o_{t_1}$, there is $k \in H(X)$ such that $k(t_1) = s$, $k(\bar{M}_1') \subseteq N_1$, $k^{-1}(\bar{M}_i) \subseteq N_1$, and $k \restriction N_{j_2}^\# = \mathrm{id}_{N_{j_2}^\#}$.

Pick s_1 from o_{t_1} such that $s_1 \neq t_1$. There is $k_1 \in H(X)$ such that $k_1(t_1) = s_1$, $k_1(\bar{M}_1') \subseteq N_1$ and $k_1^{-1}(\bar{M}_1') \subseteq N_1$, $k_1 \restriction N_{j_2}^\# = \mathrm{id}_{N_{j_2}^\#}$. Then

$$h_1^{-1} \circ k_1 \circ h_1 \circ k_1^{-1}(s_1) = h_1^{-1} \circ k_1 \circ h_1(t_1) = h_1^{-1} \circ k_1(w_1) = h_1^{-1}(w_1) = t_1.$$

If $s \in M_1'$, $h_1^{-1} \circ k_1 \circ h_1 \circ k_1^{-1}(s) = h_1^{-1} \circ k_1 \circ k_1^{-1}(s) = s$; and if $s \in M_{j_2}'$, $h_1^{-1} \circ k_1 \circ h_1 \circ k_1^{-1}(s) = h_1^{-1} \circ k_1 \circ h_1(s) = s$.

Let $\{M_1^2, \ldots, M_m^2\}$ be an open cover of bd u such that for every

$i \leqslant m$, $\bar{M}_i^2 \subseteq M_i^1$. From Lemma 5 it follows that there are open sets P_1 and F_1 such that (1) F_1 is connected, (2) $t_1 \in F_1$, (3) $P_1 \cup \bar{F}_1 = X$, (4) $P_1 \cap F_1 = \emptyset$, (5) $(\bar{M}_1^2 \cup \bar{M}_{j_2}^2) \subseteq P_1$; and (6) if $\hat{x}$, $\hat{y}$ are in F_1 there is a homeomorphism $l \in H(X)$ such that $l(\hat{x}) = \hat{y}$, $l \restriction (X - F_1) = \mathrm{id}_{(X-F_1)}$.

Let $P_1^\# = X - F_1$, $P_1' = M_1^2 \cup M_{j_2}^2$. Either bd $u \subseteq P_1^\#$ or bd $u \nsubseteq P_1^\#$. If bd $u \subseteq P_1^\#$, $F_1 \subseteq u$ and is the desired v. Otherwise bd $u \nsubseteq P_1^\#$. Let $\{\tilde{M}_1^2, \ldots, \tilde{M}_m^2\}$ be an open cover of bd u such that for every $i \leqslant m$ $\bar{\tilde{M}}_i^2 \subseteq M_i^2$.

There is an integer $\hat{j}_3$ such that $\tilde{M}_{\hat{j}_3}^2 \cap \mathrm{bd}\, u \cap (X - P_1^\#) = \emptyset$ and $\hat{j}_3 \neq 1$ or j_2. Let $u_2 = u_1 \cap (X - \bigcup_{i \notin \{1, j_2\}} N_i - P_1^\#)$. Then $u_2 \subseteq u_1 \subseteq u$, $u_2 \subseteq F_1$ and $t_1 \in u_2$.

Now since F_1 is connected and locally connected, there is an arc from t_1 to z_2 (where $z_2 \in \tilde{M}_{\hat{j}_3}^2 \cap \mathrm{bd}\, u \cap F_1$) and there is a point w_2 in $F_1 \cap \mathrm{bd}\, u_2 \cap \bar{N}_{j_3}$ for some $j_3 \notin \{1, j_2\}$. There is an open set o_{w_2} such that $w_2 \in o_{w_2} \subseteq \bar{o}_{w_2} \subseteq F_1$ such that if t, w are in o_{w_2}, there is $h \in H(X)$ such that $h(w) = t$, $h \restriction P_1^\# = \mathrm{id}_{P_1^\#}$, $h(\bar{\tilde{M}}_{j_3}^2) \subseteq M_{j_3}^2$ and $h^{-1}(\bar{\tilde{M}}_{j_3}^2) \subseteq M_{j_3}^2$.

There is $t_2 \in o_{w_2} \cap (u_2 - N_{j_3}^\#)$, and thus there is a homeomorphism h_2 such that $h_2(t_2) = w_2$, $h_2 \restriction P_1^\# = \mathrm{id}_{P_1^\#}$, $h_2(\bar{\tilde{M}}_{j_3}^2) \subseteq M_{j_3}^2$, and $h_2^{-1}(\bar{\tilde{M}}_{j_3}^2) \subseteq M_{j_3}^2$.

Then there is an open set o_{t_2} such that $t_2 \in o_{t_2} \subseteq \bar{o}_{t_2} \subseteq (u_2 - N_{j_3}^\#) \cap o_{w_2}$ and such that if $s \in o_{t_2}$, there is $k \in H(X)$ such that $k(t_2) = s$, $k(\bar{P}_1') \subseteq P_1$, $k^{-1}(\bar{P}_1^1) \subseteq P_1$, and $k \restriction N_{j_3}^\# = \mathrm{id}_{N_{j_3}^\#}$. Pick $s_2 \in o_{t_2}$ such that $s_2 \neq t_2$ and let k_2 denote the guaranteed homeomorphism above. Then $h_2^{-1} \circ k_2 \circ h_2 \circ k_2^{-1}(s_2) = t_2$, and if $s \in P_1' \cup \tilde{M}_{j_3}^2$, $h_2^{-1} \circ k_2 \circ h_2 \circ k_2^{-1}(s) = s$.

Let $\{M_1^3, \ldots, M_m^3\}$ be an open cover of bd u such that for every $i \leqslant m$, $\bar{M}_i^3 \subseteq \tilde{M}_i^2$. From Lemma 5 it follows that there are open sets P_2 and F_2 such that (1) F_2 is connected, (2) $t_2 \in F_2$, (3) $P_2 \cup \bar{F}_2 = X$, (4) $P_2 \cap F_2 = \emptyset$; and (5) if $\hat{x}$, $\hat{y}$ are in F_2 there is a homeomorphism $l \in H(X)$ such that $l(\hat{x}) = \hat{y}$, $l \restriction (X - F_2) = \mathrm{id}_{(X-F_2)}$, and $\bar{M}_1^3 \cup \bar{M}_{j_2}^3 \cup \bar{M}_{j_3}^3 \subseteq P_2 \subseteq X - F_2$.

Continue this process. It is a finite one with at most m steps: for some $i \leqslant m-1$, bd $u \subseteq P_i^\#$ and F_i is the desired v.

Proof of the theorem. Since X is complete and homogeneous, it now follows easily that X is representable.

The corollaries. Gerald Ungar has defined a space X to be *homeotopically homogeneous* if for every x and y in X there is an isotopy $F\colon X \times I \to X$ such that (1) $F_0 = \mathrm{id}_X$, (2) $F_1(x) = y$, and (3) for every $t \in I$, $F_t \in H(X)$. Ungar [11] proved that a countable product of compact homeotopically homogeneous spaces, or a countable product of locally compact, locally connected, homeotopically homogeneous spaces is n-homogeneous for all n. With his results in [10] he gets the theorem that follows: A countable product of homeotopically homogeneous continua is countable dense homogeneous and n-homogeneous for each n.

COROLLARY 1. *Suppose that for every $i\in A$ (where A is either a finite or a countably infinite set, but A does have at least 2 members), X_i is a homeotopically homogeneous, nondegenerate continuum. Then $\prod_{i\in A} X_i$ is representable.*

Proof. $\prod_{i\in A} X_i$ is 2-homogeneous, connected, and it admits a primitively stable homeomorphism which is not the identity: Suppose $a\in A$. Let $Z = \prod_{i\in A-\{a\}} X_i$. Then Z is homeotopically homogeneous. Suppose $x\neq y$ are points of Z. There is an isotopy $F\colon Z\times[0,1]\to Z$ such that $F_0 = \mathrm{id}_z$, $F_1(x) = y$, and $F_t\in H(Z)$ for every $t\in[0,1]$.

Suppose d_a is a metric on X_a compatible with its topology. Pick $e\in X_a$. There is $\varepsilon>0$ such that $\overline{D_\varepsilon(e)}\subsetneq X_a$ where $D_\varepsilon(e) = \{x\in X_a \mid d_a(x, e)<\varepsilon\}$. There is a continuous function $\Phi\colon X_a\to[0,1]$ such that $\Phi(e) = 1$, and $\Phi(t) = 0$ for every $t\in X_a-\overline{D_\varepsilon(e)}$. Define g as follows: $g(d, z) = \big(d, F_{\Phi(d)}(z)\big)$ for $(d, z)\in X_a\times Z = \prod_{i\in A} X_i$. Then $g\in H\big(\prod_{i\in A} X_i\big)$, $g\neq \mathrm{id}_{\prod_{i\in A} X_i}$ (for $g(e, x) = (e, y)$), and $g\restriction\big[\big(X_a-\overline{D_\varepsilon(e)}\big)\times Z\big] = \mathrm{id}_{(X_a-\overline{D_\varepsilon(e)})\times Z}$. Thus, the product space admits a primitively stable homeomorphism other than the identity, and so it must be representable.

COROLLARY 2. *Suppose X and Y are continua, $X\times Y$ is 2-homogeneous, and X admits a primitively stable homeomorphism other than the identity. Then $X\times Y$ is representable.*

Proof. Suppose $h\in H(X)$ such that $h\neq\mathrm{id}_X$ but h is primitively stable. Then $h\times\mathrm{id}_Y\in H(X\times Y)$, is primitively stable, and is not the identity. Thus $X\times Y$ is representable.

A *loop* in a space X is a map from S, the circle, into X. A map is said to be *essential* if it is not homotopic to a constant map. Otherwise it is inessential.

In their paper [8] the Kuperbergs and Transue give the following lemma, which will be needed here:

LEMMA K,K,T. *If X is a 1-dimensional continuum and if f_1 and f_2 are the two essential loops in X such that $f_1(S)\cap f_2(S) = \emptyset$, then f_1 and f_2 are not homotopic.*

LEMMA 3. *If M is the Menger universal curve and X is a continuum, then $M\times X$ is not representable.*

Proof. Suppose that $M\times X$ is representable. Then $M\times X$ is connected and locally connected, which implies that X is locally connected and locally arcwise connected.

Suppose that (1) $x = (x_1, x_2)\in M\times X$, (2) u_1 is an open subset of M such that $x_1\in u_1$ and $\bar u_1\neq M$, and (3) u_2 is an open subset of X such that $x_2\in u_2$ and $\bar u_2\neq X$. Let $u = u_1\times u_2$. There is an open set v in $M\times X$ such that $x\in v\subseteq\bar v\subseteq u$ and if $t\in v$, there is a homeomorphism $\Phi\in H(M\times X)$ such that $\Phi(x) = t$ and if $w\in(M\times X)-u$, $\Phi(w) = w$. Pick $y = (y_1, y_2)$ out of v such that $x_1\neq y_1$ and $x_2\neq y_2$. There is a homeomorphism $h\in H(X\times M)$ such that $h(x) = y$ and $h(w) = w$ for each $w\in(M\times X)-u$.

Pick z out of $X-u_2$. There is an arc P from x_2 to z. Let Π_M denote the projection of $M\times X$ onto M. Now $\Pi_M\circ h(x_1, x_2) = y_1$ and $\Pi_M\circ h(x_1, z) = x_1$, so there is a positive number ε such that $\Pi_M\circ h\big(D_\varepsilon(x)\big)\cap\Pi_M\circ h\big(D_\varepsilon(x_1, z)\big) = \emptyset$ (where d represents a metric on $M\times X$ compatible with its topology and if $w\in M\times X$, $\varepsilon>0$, $D_\varepsilon(w) = \{t\in M\times X\mid d(t, w)<\varepsilon\}$.

There is an essential loop $L\colon S\to M$ such that (1) $x_1\in L(S)$, (2) if $p\in P$, $L_p\colon S\to M\times X$ defined by $L_p(s) = \big(L(s), p\big)$ has the property that $L_p(S)\subseteq D_\varepsilon(x_1, p)$, and (3) $L_z(s)\cap u = \emptyset$.

Now $h\circ L_z = L_z$, and so $\Pi_M\circ h\circ L_z = \Pi_M\circ L_z$ is essential. Since L_z and L_{x_2} are homotopic, $\Pi_M\circ h\circ L_{x_2}$ and $\Pi_M\circ h\circ L_z$ are homotopic. Also, each is essential, since $\Pi_M\circ h\circ L_z$ is essential. But $\Pi_M\circ h\circ L_z(S)\cap\Pi_M\circ h\circ L_{x_2}(S) = \emptyset$, which is a contradiction to Lemma K,K,T. Then the assumption that $M\times X$ is representable must be a bad one.

COROLLARY 4. *$M\times X$ is not 2-homogeneous if M is the Menger universal curve and X is a continuum.*

Proof. Assume $M\times X$ is 2-homogeneous. Now M is itself representable [1], so it admits many primitively stable homeomorphisms. By Corollary 2, $M\times X$ is representable. But in Lemma 3, it was proved that $M\times X$ is not representable, so we have a contradiction.

Last comments. I do not know if the requirement that X admit a primitively stable homeomorphism other than the identity can be omitted. I wish I knew that. I do know that the 2-homogeneity requirement cannot be weakened to just homogeneity, or even homogeneity plus local connectedness. This is because $M\times M$ and $M\times S$ are both homogeneous and locally connected, but are not 2-homogeneous [8], and thus not representable.

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A generalized version of the singular cardinals problem

by

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Abstract. We show that it is consistent, relative to the existence of an unbounded class of cardinals each of which possesses a certain degree of supercompactness, for every limit cardinal to be a strong limit cardinal and for the ωth successor of any cardinal to violate GCH.

The behaviour of the power sets of singular cardinals has long been of interest to set theorists. Shortly after Cohen invented forcing, Easton in his thesis [2] showed that, roughly speaking, the power sets of regular cardinals could be anything desired within the technical restrictions of $2^{\varkappa_1} \leqslant 2^{\varkappa_2}$ if $\varkappa_1 \leqslant \varkappa_2$ and cof $(2^{\varkappa}) > \varkappa$. No such results, however, were known for singular cardinals for quite a while. Indeed, the famous singular cardinals problem asks whether it is consistent to have $2^{\aleph_n} = \aleph_{n+1}$ for all natural numbers n and yet also have $2^{\aleph_\omega} = \aleph_{\omega+2}$, or more generally, whether or not it is consistent for a singular cardinal to be the least cardinal that violates GCH.

Much light has been shed on the singular cardinals problem within the last few years. It is of course now known by the work of Silver [13] that if a singular cardinal of uncountable cofinality violates GCH, then there is a stationary set of cardinals less than it which also violates GCH. This settles the generalized version of the singular cardinals problem. It is also known, by the work of Jensen [1], that if a singular cardinal (by necessity of cofinality ω) is the first cardinal to violate GCH, then there is an inner model with a measurable cardinal.

Magidor was the first person who obtained positive results in the direction of the singular cardinals problem: Starting with models in which $\varkappa$ possessed a certain degree of supercompactness and violated GCH, he was able to force and obtain a model in which $\aleph_\omega$ is a strong limit cardinal and yet violates GCH [8]. Then, starting with an enormously powerful hypothesis, namely the existence of a supercompact cardinal with a huge cardinal above it, Magidor was able to get a model in which $2^{\aleph_n} = \aleph_{n+1}$ for every natural number n and yet $2^{\aleph_\omega} = \aleph_{\omega+2}$ [9], i.e., Magidor was able to solve