A condition under which 2-homogeneity and representability are the same in continua

by

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Abstract. Our main theorem is the following: If $X$ is a 2-homogeneous continuum and $X$ admits a non-identity primitively stable homeomorphism, then $X$ is representable. The following are corollaries to this theorem: (1) If $M$ is the Menger universal curve and $X$ is a continuum, then $M \times X$ is not 2-homogeneous. (2) If $X$ and $Y$ are homeomorphically homogenous continua, then $X \rightarrow Y$ is representable.

The main result in this paper is the following theorem: If $X$ is a continuum which is 2-homogeneous and $X$ admits a primitively stable homeomorphism which is not the identity, then $X$ is representable.

C. E. Burgess [4] asked in 1955 whether, for $n \geq 2$, $n$-homogeneity implies $(n+1)$-homogeneity. The above theorem gives a partial answer to this question, since a representable continuum is $n$-homogeneous for each $n$. (This follows from results in [2], [3], [9], and [10]). Also, some corollaries follow from the theorem. One gives that if $M$ is the Menger universal curve and $X$ is a continuum, then $M \times X$ is not 2-homogeneous. This answers a question asked by K. Kuperberg, W. Kuperberg and W. R. R. Transue in [8]. Another corollary generalizes a result of G. S. Ungar which follows from results in [9] and [10].

Definitions, notation, background. In this paper a continuum is a compact, metric, connected space. If $X$ is a continuum, $H(X)$ denotes the space of all homeomorphisms from $X$ onto itself. It is well-known that $H(X)$ with the sup metric is itself a separable metric space and that it is also a complete topological space, although it may not be complete with respect to this metric.

If $n$ is a positive integer, a space $X$ is $n$-homogeneous means that if $A$ and $B$ are $n$-element subsets of $X$, then there is a homeomorphism $h \in H(X)$ such that $h(A) = B$. A space $X$ is strongly $n$-homogeneous means that if $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ are 2-element subsets of $X$ then there is a homeomorphism $h$ such that $h(a_i) = b_i$ for every $i \leq n$. $X$ is countable dense homemorphism means that if $A$ and $B$ are 2 countable dense subsets of $X$, then there is a homeomorphism $h \in H(X)$ such that $h(A) = B$. 
Let $e$ denote the sup metric on $H(X)$. For every $e > 0$ and $f \in H(X)$, let $N_e(f) = \{g \in H(X) : e(f, g) < e\}$.

A topological transformation group $(G, X)$ is a topological group $G$ together with a topological space $X$ and a continuous map $g \mapsto g(x)$ of $G \times X$ into $X$ such that $(gh)(x) = g(h(x))$, and, if $e$ is the identity of $G$, $e(x) = x$ for all $g$ and $h$ in $G$ and $x$ in $X$. $(G, X)$ is Polish if both $G$ and $X$ are separable and metrizable by a complete metric. When $X$ is a continuum, $(H(X), X)$ is a Polish transformation group.

For every $x$ in $X$ let $G_x$ denote the stabilizer subgroup of $x$ in $G$, i.e., $G_x = \{g \in G : g(x) = x\}$. Let $Gx$ denote the orbit of $x$, i.e., $Gx = \{y \in X : y = gx\}$ for some $g \in G$. If $G/G_x$ denotes the left coset space with the usual topology, Edward G. Effros [5] has proved the following theorem:

**Theorem:** Let $(G, X)$ be a Polish transformation group. Then the following are equivalent:

1. For each $x$ in $X$, the map $gG_x \rightarrow g(x)$ of $G/G_x$ onto $Gx$ is a homeomorphism.
2. Each orbit is second category in itself.
3. Each orbit is $G_x$ in $X$.
4. $X/G$ is $T_2$.

G. S. Ungar [9 and 10] has used this theorem of Effros to get some very nice results. Here, in the proofs that follow, the following theorems of his will be used:

1. If $X$ is a homogeneous continuum, then for each $x$ in $X$, the map $T_2: X/X \rightarrow X$ defined by $T_2(f)(x) = f(x)$ for $f \in H(X)$ is an open, onto map.

2. Let $F^2(X)$ denote the $2nd$ configuration space of $X$, i.e., $F^2(X) = \{(x, y) \in X^2 : x \neq y\}$. Then Ungar has shown that if $X$ is a $2$-homogeneous continuum, and $x$ and $y$ are points of $X$ such that $x \neq y$, then the map $T_2: X \rightarrow F^2(X)$ defined by $T_2(h)(x, h(y))$ is open and onto.

3. If $X$ is $2$-homogeneous, then $X$ is locally connected.

4. If $X$ is an n-homogeneous continuum, then $X$ is strongly n-homogeneous or $X$ is the circle.

A homeomorphism $h$ of $H(X)$ is said to be primitively stable if there is an open set $o$ in $X$ such that $h \upharpoonright o = id_o$. I do not know of an example of a $2$-homogeneous continuum that does not admit a primitively stable homeomorphism other than the identity, or even of an example of such a homogeneous continuum.

A space $X$ is representable means that if $x \in X$ and $u$ is open such that $x \in X$ and there is an open set $e$ such that $x \in e \subseteq u$, then there is an open set $e$ such that $x \in e \subseteq u$ and if $y \in e$, then there is $h \in H(X)$ such that $h(x) = y$ and $h(z) = z$ for every $z \in u$. (I would like to note that this notion was introduced by Peter Fletcher in [6], is equivalent to the notion of strong local homogeneity, which was introduced by L. R. Ford in [7]. John Bale, [2], proved this equivalence.)

If $a$ is a collection of subsets of $X$, then $a^* = \{x \in X : x \in A \}$ for some $A \in a$.

**Proof of the theorem.** First we need some lemmas. For all lemmas and the theorem, assume that $X$ is a $2$-homogeneous continuum and $X$ admits a primitively stable homeomorphism which is not the identity. Let $d$ denote a metric on $X$ which is compatible with the topology on $X$, and for every $e > 0$, $x \in X$, let $D_e(x) = \{y \in X : d(x, y) < e\}$.

**Lemma 1.** Suppose $x \in X$. Then there is a primitively stable homeomorphism $h \in H(X)$ such that $x \in N$ where $N = \{o \subseteq X : o$ is open and $h \upharpoonright o = id_o\}^*$. Let $f \upharpoonright o \in o$ is open and $f \upharpoonright o = id_o$. There is $z$ in $X - P$ such that $f(z) \neq z$. Suppose $p \in P$, and $y \neq x, y \neq p$. Then there is $\Phi \in H(X)$ such that $\Phi(p) = y$, and $\Phi(z) = x$, and $h = \Phi \circ \Phi^{-1}$ is the desired homeomorphism.

**Lemma 2.** Suppose $h$ is a primitively stable homeomorphism other than the identity on $X$, and $N = \{o \subseteq X : h \upharpoonright o = id_o\}^*$. Then $M$ is open such that $M \subseteq \subseteq N$. If $x \notin N$ there is $u_0$ open in $X$ such that $x \in u_0$ and if $z \in u_0$, there is $h \in H(X)$ such that $h(x) = z$, and $g \upharpoonright M = id_M$.

**Proof.** There is a positive number $\delta$ such that if $k \in H(X)$ such that $k \in N_o$, $k(M) \subseteq N$ and $k^{-1}(M) \subseteq N$. Either $x = h(x)$ or $x \neq h(x)$.

**Case 1.** Suppose $x \neq h(x)$. Now $T_{N_o}(N_o)$ is open in $F^2(X)$ and contains $(x, h(x))$. There is an open subset $U$ of $X$ such that $x \in U$, $U \times h(U) \subseteq T_{N_o}(N_o)$, and $U \cap h(U) = \emptyset$.

Then there is an open set $U$ such that $x \in U \subseteq U$ and $x \in h(U) \subseteq T_{N_o}(N_o)$. Suppose $x \notin U$, $x \notin U$. (x, h(x)) \in U \times h(U) \subseteq T_{N_o}(N_o).$ There is $\Phi \in N_o$ such that $\Phi(x) = x$ and $\Phi(h(x)) = h(x)$. Now $\Phi^{-1}(x) \neq x$, so denote $\Phi^{-1}(x) \neq x$. Since $\Phi^{-1} \in N_o$, $\Phi^{-1}(x) = r \in U$. Since $x, h(x)$ is an (h(x)), there is $\Gamma \in N_o$ such that $\Gamma(x) = z$ and $\Gamma(h(x)) = h(x)$. If $s \in M$, $\Gamma \circ h^{-1} \circ \Phi^{-1} \circ \phi \in \phi(s) = \Gamma \circ h^{-1} \circ \Phi^{-1} \circ \phi(s) = \Gamma \circ h^{-1} \circ \Phi^{-1} \circ \phi(s) = \Gamma \circ h^{-1} \circ \Phi^{-1} \circ \phi(s) = \Gamma \circ h^{-1} \circ \Phi^{-1} \circ \phi(s)$. (Recall that $\Phi(s) \in N$ and $\Gamma^{-1}(s) \in N$) Thus $g = \Gamma \circ h^{-1} \circ \Phi^{-1} \circ \phi \in \phi(h(x))$ is the desired homeomorphism and $u$ is the desired open set $u_0$.

**Case 2.** Suppose $x = h(x)$. There is an open set $M$ such that
Suppose $\omega \in A_i$ for some $i$. There is a homeomorphism $\delta$ in $H(X)$ such that $\delta(x) = \hat{w}$ and $\delta \mid N_i = id_{N_i}$. Then $\hat{w} \in A_i$, and there is an open set $O$ such that $\hat{w} \in O \subseteq T_e(N_i, \delta)$. If $\hat{w} \in O$, then there is a $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$.

Pick $\hat{w} \in O$. There is $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$. Then $\beta \circ \delta(x) = \hat{w}$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Now $\omega \in A_i$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Therefore, $\omega \in A_{i+1}$ and $A = \bigcup A_i$ is open in $X - N$.

Suppose $\omega$ is a point in $A_i$ for some $i$. There is a homeomorphism $\delta$ in $H(X)$ such that $\delta(x) = \hat{w}$ and $\delta \mid N_i = id_{N_i}$. Then $\hat{w} \in A_i$, and there is an open set $O$ such that $\hat{w} \in O \subseteq T_e(N_i, \delta)$. If $\hat{w} \in O$, then there is a $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$.

Pick $\hat{w} \in O$. There is $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$. Then $\beta \circ \delta(x) = \hat{w}$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Now $\omega \in A_i$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Therefore, $\omega \in A_{i+1}$ and $A = \bigcup A_i$ is open in $X - N$.

Let $Y = \{ x \in N \mid x \cap N_{i+1} \neq \emptyset \}$. Then $Y$ is open in $N$. Now $\omega \in A_i$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Therefore, $\omega \in A_{i+1}$ and $A = \bigcup A_i$ is open in $X - N$.

Suppose $\omega \in A_i$ for some $i$. There is a homeomorphism $\delta$ in $H(X)$ such that $\delta(x) = \hat{w}$ and $\delta \mid N_i = id_{N_i}$. Then $\hat{w} \in A_i$, and there is an open set $O$ such that $\hat{w} \in O \subseteq T_e(N_i, \delta)$. If $\hat{w} \in O$, then there is a $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$.

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Pick $\hat{w} \in O$. There is $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$. Then $\beta \circ \delta(x) = \hat{w}$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Now $\omega \in A_i$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Therefore, $\omega \in A_{i+1}$ and $A = \bigcup A_i$ is open in $X - N$.

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Pick $\hat{w} \in O$. There is $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$. Then $\beta \circ \delta(x) = \hat{w}$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Now $\omega \in A_i$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Therefore, $\omega \in A_{i+1}$ and $A = \bigcup A_i$ is open in $X - N$.

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Pick $\hat{w} \in O$. There is $\beta \in H(X)$ such that $\beta(w) = \hat{w}$ and $\beta \mid N_{i+1} = id_{N_{i+1}}$. Then $\beta \circ \delta(x) = \hat{w}$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Now $\omega \in A_i$, and $\beta \circ \delta \mid N_{i+1} = id_{N_{i+1}}$. Therefore, $\omega \in A_{i+1}$ and $A = \bigcup A_i$ is open in $X - N$.
LEMMA 5. Suppose \( x \in X \), and \( L \) is a primitive stable homeomorphism such that \( L(x) \neq x \). Let \( G = \{ \text{open in } X \mid \tau \}, \) and suppose \( M \) is open such that \( M \subseteq G \). Then there are open sets \( E \) and \( N \) in \( X \) such that (1) \( x \in E \), (2) \( N = X - E \), (3) \( E \) is connected, and (4) if \( x \in E \), then \( E \cap H(X) \) is connected such that \( h(x) = y \) and \( k \cap N = id_{\beta} \), and (5) \( M \subseteq N \). Further, if \( y \in E \), then \( y \in G \). Thus, there are open sets \( E \) and \( N \) in \( X \) such that (1) \( x \in E \), (2) \( N = X - E \), (3) \( E \) is connected, and (4) if \( x \in E \), then \( E \cap H(X) \) is connected such that \( h(x) = y \) and \( k \cap N = id_{\beta} \), and (5) \( M \subseteq N \).

Proof. This follows from the proof of Lemma 4. (Note that the proof of Lemma 4 is the proof of Lemma 5.)

LEMMA 6. If \( u \) is open in \( X \), then there is an open subset \( v \) of \( u \) such that if \( x \), \( y \) are in \( v \), there is \( h \in H(X) \) such that \( h(x) = y \) and \( h \cap u = id_{\beta} \).

Proof. If \( A \) is a subset of \( X \) let \( b \) denote the boundary of \( A \). Suppose \( u \) is open in \( X \) such that \( \partial u \neq \emptyset \). Pick \( \partial u \). For every \( v \in \partial u \), there are sets \( N_v \), \( E_v \) such that (1) \( v \in N_v \), (2) \( \partial u \cap E_v \neq \emptyset \), and (3) \( E_v \) and \( N_v \) are open in \( X \). Let \( N = \bigcup_{v \in \partial u} N_v \) and \( E = \bigcup_{v \in \partial u} E_v \). At least \( \cap E_v \neq \emptyset \).

There is a finite collection \( \{ v_1, \ldots, v_n \} \) of \( \partial u \) such that \( \bigcap_{i=1}^n N_{v_i} \) covers \( \partial u \). For \( \cap E_v \neq \emptyset \), there is \( \cap E_v \neq \emptyset \). There is a collection \( \{ M_1, \ldots, M_n \} \) of open sets \( \partial u \) such that \( \cap E_v \neq \emptyset \).

There is a finite subcollection \( \{ y_1, y_2, \ldots, y_m \} \) of \( \partial u \) such that \( \{ N_{y_1}, N_{y_2}, \ldots, N_{y_m} \} \) covers \( \partial u \). For \( \cap E_v \neq \emptyset \), there is \( \cap E_v \neq \emptyset \). There is a collection \( \{ M_1, \ldots, M_n \} \) of open sets \( \partial u \) such that \( \cap E_v \neq \emptyset \).

Either \( \cap E_v \neq \emptyset \), or \( \cap E_v \neq \emptyset \). If \( \cap E_v \neq \emptyset \), then \( \cap E_v \neq \emptyset \). If \( \cap E_v \neq \emptyset \), then \( \cap E_v \neq \emptyset \).

There is a connected and locally connected, there is an arc from \( x_1 \) to \( x_2 \), (where \( \cap E_v \neq \emptyset \)). Let \( \cap E_v \neq \emptyset \).

There is an open set \( \cap E_v \) such that \( \cap E_v \neq \emptyset \).

There is a homomorphism \( h \) such that \( h(x) = y \) and \( h \cap u = id_{\beta} \). Then there is an open set \( \cap E_v \neq \emptyset \).

Pick \( \cap E_v \neq \emptyset \). Then there is an open set \( \cap E_v \neq \emptyset \).

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There is an open subset \( \cap E_v \) such that \( \cap E_v \neq \emptyset \).

This follows from the proof of Lemma 4.
homeotopically homogeneous continua is countable dense homogeneous and n-homogeneous for each \(n\).

**Corollary 1.** Suppose that for every \(i \in A\) (where \(A\) is either a finite or a countably infinite set, but \(A\) does have at least 2 members), \(X_i\) is a homeotopically homogeneous, nondegenerate continuum. Then \(\prod_{i \in A} X_i\) is representable.

Proof. \(\prod_{i \in A} X_i\) is 2-homogeneous, connected, and it admits a primitive stable homeomorphism which is not the identity: Suppose \(a \in A\). Let \(Z = \prod_{i \in A} X_i\). Then \(Z\) is homeotopically homogeneous. Suppose \(x \neq y\) are points of \(Z\). There is an isopy \(F: Z \times [0, 1] \to Z\) such that \(F_0 = \text{id}_Z\), \(F_1(x) = y\), and \(F \in H(Z)\) for every \(t \in [0, 1]\).

Suppose \(d_a\) is a metric on \(X_a\) compatible with its topology. Pick \(e \in X_a\). There is a \(\varepsilon > 0\) such that \(D_b(e) \subseteq X_a\) where \(D_b(e) = \{x \in X_a \mid d_a(x, e) < \varepsilon\}\). There is a continuous function \(\Phi: X_a \to [0, 1]\) such that \(\Phi(e) = 1\), and \(\Phi(x) = 0\) for every \(x \in X_a - D_b(e)\). Define \(g\) as follows: \(g(d, z) = (d, F(z))\) for \((d, z) \in X_a \times Z = \prod_{i \in A} X_i\). Then \(g \in H(\prod_{i \in A} X_i, g \neq \text{id}_Z\) for \(g(e, x) = (e, y)\), and \(g \mid (X_a - D_b(e)) \times Z = \text{id}_{X_a - D_b(e)} \times Z\). Thus, the product space admits a primitive stable homeomorphism other than the identity, and so it must be representable.

**Corollary 2.** Suppose \(X\) and \(Y\) are continua, \(X \times Y\) is 2-homogeneous, and \(X\) admits a primitive stable homeomorphism other than the identity. Then \(X \times Y\) is representable.

Proof. Suppose \(h \in H(X)\) such that \(h \neq \text{id}_X\) but \(h\) is primitive stable. Then \(h \times \text{id}_Y \in H(X \times Y)\), is primitive stable, and is not the identity. Therefore \(X \times Y\) is representable.

A loop in a space \(X\) is a map from \(S_1\), the circle, into \(X\). A map is said to be essential if it is not homotopic to a constant map. Otherwise it is inessential.

In their paper [8] the Kuperbergs and Transue give the following lemma which will be needed here:

**Lemma K.K.T.** If \(X\) is a 1-dimensional continuum and if \(f_1\) and \(f_2\) are the two essential loops in \(X\) such that \(f_1(S_1) \cap f_2(S_1) = \emptyset\), then \(f_1\) and \(f_2\) are not homotopic.

**Lemma 3.** If \(M\) is the Menger universal curve and \(X\) is a continuum, then \(M \times X\) is not representable.

Proof. Suppose that \(M \times X\) is representable. Then \(M \times X\) is connected and locally connected, which implies that \(X\) is locally connected and locally arcwise connected.

Suppose that (1) \(x = (x_1, x_2) \in M \times X\), (2) \(u_i\) is an open subset of \(M\) such that \(x_1 \in u_i\) and \(u_i \neq M\), and (3) \(u_j\) is an open subset of \(X\) such that \(x_2 \in u_j\) and \(u_j \neq X\). Let \(u = u_i \cap u_j\). There is an open set \(\tau\) in \(M \times X\) such that \(x \in \tau \subseteq \emptyset \subseteq \tau\) and if \(\tau \subseteq \emptyset\), there is a homeomorphism \(\Phi \in H(M \times X)\) such that \(\Phi(x) = \emptyset\) if \(w \in (M \times X) - u\), \(\Phi(w) = w\). Pick \(y = (y_1, y_2)\) out of \(u\) such that \(x_1 \neq y_1\) and \(x_2 \neq y_2\). There is a homeomorphism \(h \in H(M \times X)\) such that \(h(x) = y\) and \(h(w) = w\) for each \(w \in (M \times X) - u\).

Pick \(z\) out of \(X - u_j\). There is an arc \(P\) from \(x_2\) to \(z\). Let \(\Pi_{\infty}\) denote the projection of \(M \times X\) onto \(M\). Now \(\Pi_{\infty} \circ h(x_1, x_2) = y_1\) and \(\Pi_{\infty} \circ h(x_1, z) = y_1\), so there is a positive number \(\varepsilon\) such that \(\Pi_{\infty} \circ h(D(x, \varepsilon)) \cap \Pi_{\infty} \circ h(D(x_1, \varepsilon)) = \emptyset\) (where \(d\) represents a metric on \(M \times X\) compatible with its topology and if \(w \in (M \times X) - u\), \(d(w) = \tau \in \emptyset\)).

There is an essential loop \(L: S \to M\) such that (1) \(x_1 \in L(S)\), (2) if \(\tau \subseteq P\), \(L(x)\) is defined by \(L(x) = (L(x), \tau)\) has the property that \(L(x) \subseteq D(\Pi_{\infty} \circ h(x_1, x_2), \varepsilon)\), and (3) \(L(x) \cap u = \emptyset\).

Now \(h \circ L = L\), and so \(\Pi_{\infty} \circ h \circ L = \Pi_{\infty} \circ L\) is essential. Since \(L_1\) and \(L_2\) are homotopic, \(\Pi_{\infty} \circ h \circ L_1\) and \(\Pi_{\infty} \circ h \circ L_2\) are homotopic. Also, each is essential, since \(\Pi_{\infty} \circ h \circ L_1\) is essential. But \(\Pi_{\infty} \circ h \circ L_1(S) \cap \Pi_{\infty} \circ h \circ L_2(S) = \emptyset\), which is a contradiction to Lemma K.K.T. Therefore the assumption that \(M \times X\) is representable must be a bad one.

**Corollary 4.** \(M \times X\) is not 2-homogeneous if \(M\) is the Menger universal curve and \(X\) is a continuum.

Proof. Assume \(M \times X\) is 2-homogeneous. Now \(M\) is itself representable [1], so it admits many primitive stable homeomorphisms. By Corollary 2, \(M \times X\) is representable. But in Lemma 3, it was proved that \(M \times X\) is not representable, so we have a contradiction.

**Last comments.** I do not know if the requirement that \(X\) admit a primitive stable homeomorphism other than the identity can be omitted. I wish I knew that. I do know that the 2-homogeneity requirement cannot be weakened to just homogeneity, or even homogeneity plus local connectedness. This is because \(M \times M\) and \(M \times S\) are both homogeneous and locally connected, but are not 2-homogeneous [3], and thus not representable.

References

A generalized version of the singular cardinals problem

by

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Abstract. We show that it is consistent, relative to the existence of an unbounded class of cardinals each of which possesses a certain degree of supercompactness, for every limit cardinal to be a strong limit cardinal and for the $i$th successor of any cardinal to violate GCH.

The behaviour of the power sets of singular cardinals has long been of interest to set theorists. Shortly after Cohen invented forcing, Easton in his thesis [2] showed that, roughly speaking, the power sets of regular cardinals could be anything desired within the technical restrictions of $2^X \leq 2^{X^2}$ if $\aleph_1 \leq \aleph_2$ and cof $(2^\kappa) > \kappa$. No such results, however, were known for singular cardinals for quite a while. Indeed, the famous singular cardinals problem asks whether it is consistent to have $2^{\aleph_1} = \aleph_{n+1}$ for all natural numbers $n$ and yet also have $2^{\aleph_0} = \aleph_{n+2}$, or more generally, whether or not it is consistent for a singular cardinal to be the least cardinal that violates GCH.

Much light has been shed on the singular cardinals problem within the last few years. It is of course now known by the work of Silver [13] that if a singular cardinal of uncountable cofinality violates GCH, then there is a stationary set of cardinals less than it which also violates GCH. This settles the generalized version of the singular cardinals problem. It is also known, by the work of Jensen [1], that if a singular cardinal (by necessity of cofinality $\omega$) is the first cardinal to violate GCH, then there is an inner model with a measurable cardinal.

Magidor was the first person who obtained positive results in the direction of the singular cardinals problem: Starting with models in which $\kappa$ possesses a certain degree of supercompactness and violated GCH, he was able to force and obtain a model in which $\aleph_1$ is a strong limit cardinal and yet violates GCH [8]. Then, starting with an enormously powerful hypothesis, namely the existence of a supercompact cardinal with a huge cardinal above it, Magidor was able to get a model in which $2^{\aleph_1} = \aleph_{n+1}$ for every natural number $n$ and yet $2^{\aleph_0} = \aleph_{n+2}$ [9], i.e., Magidor was able to solve...