Fix-finite approximation of n-valued multifunctions

by

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Abstract. We call a multifunction $\varphi\colon X \to Y$ *n-valued* if $\varphi(x)$ consists, for all $x \in X$, of n points, and show that an n-valued continuous multifunction from a compact polyhedron to itself has an arbitrarily close approximation by an n-valued continuous multifunction which has only finitely many fixed points. The proof includes multivalued analogues of the Simplicial Approximation Theorem and of the Hopf construction. A basic tool is the Splitting Lemma, which shows that n-valued continuous multifunctions are locally equivalent to n single-valued continuous functions, and implies that every n-valued continuous multifunction from a compact, Hausdorff, path connected and simply connected space with the fixed point property to itself has at least n fixed points.

1. Introduction. B. O'Neill [5] studied continuous multifunctions $\varphi\colon X \to Y$ which have the property that, for all $x \in X$, $\varphi(x)$ is either acyclic or consists of n acyclic components. He showed that if X and Y are compact polyhedra, then an induced homomorphism $\varphi_*\colon H_*(X) \to H_*(Y)$ can be defined. Hence if $X = Y$, there exists a Lefschetz number $L(\varphi)$ with the usual property that $L(\varphi) \neq 0$ implies that the fixed point set $\operatorname{Fix}\varphi = \{x \in X \mid x \in \varphi(x)\}$ is not empty.

Here we consider a special case of such multifunctions, namely those where, for all $x \in X$, $\varphi(x)$ consists of n points. Such multifunctions, which we call *n-valued*, inherit some of the properties of single-valued functions. Our main purpose is to show that $\operatorname{Fix}\varphi$ is generically finite (Theorem 6). To accomplish it, we first prove a multivalued analogue of the simplicial approximation theorem (Theorem 4), and then carry out a "Hopf construction" for n-valued simplicial multifunctions (Lemma 5). A crucial tool in all proofs is the Splitting Lemma (Lemma 1), which shows that n-valued continuous multifunctions are locally (but not necessarily globally) equivalent continuous multifunctions are locally equivalent to n single-valued continuous functions. The proof of the Splitting Lemma uses a lemma by O'Neill [5], but otherwise this paper is independent of his work. Theorem 3 could, in fact, be used to define for an n-valued continuous multifunction φ an induced homomorphism $\varphi_*\colon H_*(X) \to H_*(Y)$ in a way different from the one given by O'Neill, but we do not persue this topic.

It seems likely that the Simplicial Approximation Theorem (Theorem 4) and the Fix-Finite Approximation Theorem (Theorem 6) still hold if the n-valued multifunction is only upper semi-continuous rather than continuous, but our method of proof fails in this case. For consider the 2-valued upper semi-continuous multifunction ψ: $[-1, 1] \to [-1, 1]$ given by

$$\psi(x) = \begin{cases} \{x+1, 1\} & \text{if } -1 \leqslant x \leqslant 0, \\ \{-1, x-1\} & \text{if } \quad 0 \leqslant x \leqslant 1. \end{cases}$$

The Splitting Lemma is false for ψ, and so is Theorem 2, which states that if X is a compact, Hausdorff, path connected and simply connected space with the fixed point property, then every n-valued continuous multifunction φ: $X \to X$ has at least n fixed points.

It may also be possible to extend Theorems 4 and 6 to a continuous multifunction φ for which the number of points in $\varphi(x)$ is finite but not independent of x. But again there is no analogue of the Splitting Lemma and of Theorem 2, as an unpublished example by R. Dunn (see O'Neill [5], p. 1183) shows that there exists such a multifunction from the 2-cell to itself without a fixed point.

2. Splitting n-valued multifunctions. The proofs in this paper depend heavily on the fact that the multifunctions considered here can locally be "split" into single-valued functions. We give the necessary background in this paragraph. A basic reference for multifunctions is e.g. the book by C. Berge [1], especially Chapter VI.

A multifunction φ: $X \to Y$ from a topological space X to a topological space Y is a correspondence which assigns to each point of X at least one point of Y. The multifunction φ: $X \to Y$ is called *upper semi-continuous* if $\varphi(x)$ is closed for all $x \in X$ and if for each open set $V \subset Y$ with $\varphi(x) \subset V$ there exists an open set $U \subset X$ with $x \in U$ and $\varphi(U) \subset V$. It is called *lower semi-continuous* if for every $x \in X$ and open set $V \subset Y$ with $\varphi(x) \cap V \neq \emptyset$ there exists an open set $U \subset X$ with $x \in U$ and $\varphi(x') \cap V \neq \emptyset$ for all $x' \in U$. If φ is both upper and lower semi-continuous, then it is called *continuous*. If X and Y are compact Hausdorff spaces, then φ: $X \to Y$ is upper semi-continuous if and only if its graph $\Gamma_\varphi = \{(x, y) \in X \times Y |\ y \in \varphi(x)\}$ is closed in $X \times Y$ (see [1], p. 112). We will reserve the term map for a single-valued continuous function.

We define that a multifunction φ: $X \to Y$ *splits into maps* if there exist finitely many maps f_i: $X \to Y$, where $i = 1, 2, \ldots, n$, so that $\varphi(x) = \{f_1(x), f_2(x), \ldots, f_n(x)\}$ for all $x \in X$. If $f_i(x) \neq f_j(x)$ for all $x \in X$ and $i, j = 1, 2, \ldots, n$ with $i \neq j$, then we say that φ *splits into distinct maps*, and write $\varphi = \{f_1, f_2, \ldots, f_n\}$. Clearly a multifunction which splits into maps is continuous.

LEMMA 1 (Splitting Lemma). *Let X and Y be compact Hausdorff. If X is path connected and simply connected and φ: $X \to Y$ is n-valued and continuous, then φ splits into distinct maps.*

Proof. It follows from [5], Lemma 4 that the graph $\Gamma_\varphi \subset X \times Y$ consists of n components, say C_i, with $i = 1, 2, \ldots, n$. As φ is continuous, each projection p_i: $C_i \to X$ induced by the projection of $X \times Y$ onto X is a surjection. Hence we can define functions f_i: $X \to Y$ by $f_i(x) = p_i^{-1}(x)$, and as φ is n-valued, the f_i are single-valued. The graphs C_i of the f_i are closed, so the f_i are continuous, and therefore $\varphi = \{f_1, f_2, \ldots, f_n\}$ splits into distinct maps.

O'Neill [5], Theorem 9 proved that if X is an acyclic compact polyhedron and φ: $X \to X$ a continuous multifunction such that, for all $x \in X$, $\varphi(x)$ is acyclic or consists of n acyclic components, then $\text{Fix}\,\varphi \neq \emptyset$. We can strengthen this result if φ is n-valued.

THEOREM 2. *Let X be a compact Hausdorff, path connected and simply connected space with the fixed point property. Then every n-valued continuous multifunction φ: $X \to X$ has at least n fixed points.*

Proof. Lemma 1 implies that $\varphi = \{f_1, f_2, \ldots, f_n\}$. Each f_i has at least one fixed point, and these fixed points are distinct.

Remark. Using [5], Lemma 4 and generalizations of Kakutani's Theorem it is possible to obtain analogues for Lemma 1 and Theorem 2 for continuous multifunctions φ: $X \to X$, where for all $x \in X$, $\varphi(x)$ consists of n acyclic components. It is, however, not possible to drop the assumption in Lemma 1 that X is simply connected. To see this, let $S^1 = \{e^{it} |\ 0 \leqslant t < 2\pi\}$ be the unit circle in the complex plane and ψ: $S^1 \to S^1$ be the multifunction given by $\psi(e^{it}) = \{e^{it/2}, e^{it/2+i\pi}\}$ for all $0 \leqslant t < 2\pi$. Then ψ is 2-value and continuous, but does not split into distinct maps. It has one fixed point.

3. Simplicial approximation of n-valued multifunctions. We now consider n-valued continuous multifunctions φ: $X \to Y$ in the case where X and Y are compact polyhedra, and show that they have a simplicial approximation. Denote by $|K|$ a polyhedron which is the realization of a finite simplicial complex K, by σ an open simplex of $|K|$, by $\bar{\sigma}$ the corresponding closed simplex, and by $\dot{\sigma}$ the boundary of σ. The (open) star $\mathrm{st}_K\sigma$ is the union of all simplexes of $|K|$ which have σ as a face. We shall write $\overline{\mathrm{st}_K\sigma}$ for the closure of $\mathrm{st}_K\sigma$.

We call a multifunction φ: $|K| \to |L|$ from a polyhedron $|K|$ to a polyhedron $|L|$ a *simplical multifunction* if, for every $\bar{\sigma} \in |K|$, the restriction $\varphi|\bar{\sigma}$ splits into maps $f_1, f_2, \ldots, f_n$ so that each f_i maps $\bar{\sigma}$ affinely onto a simplex $\bar{\tau}_i \in |L|$. It is easy to check that a simplicial multifunction is continuous. We use d_K and d_L to denote the barycentric metric of $|K|$ and $|L|$, ϱ_L to denote the Hausdorff metric on $|L|$ induced by d_L, and

$$\bar{d}(\varphi, \varphi') = \sup\{\varrho_L(\varphi(x), \varphi'(x)) |\ x \in |K|\}$$

for the distance between two multifunctions φ, φ': $|K| \to |L|$.

If $\varphi\colon |K| \to |L|$ is n-valued and continuous, then we define a function from $|K|$ into the reals by

$$\gamma(x) = \inf\{d_L(y_i, y_j) \mid y_i, y_j \in \varphi(x),\ y_i \neq y_j\}$$

for all $x \in |K|$, and the *gap* of φ by

$$\gamma(\varphi) = \inf\{\gamma(x) \mid x \in |K|\}.$$

The continuity of φ implies that $\gamma(x)$ depends continuously on x. Hence if $|K|$ is compact, then it follows from $\gamma(x) > 0$ for all $x \in |K|$ that $\gamma(\varphi) > 0$.

The symbol $\mu(K)$ denotes the mesh of $|K|$. We first describe the simplicial approximation of φ in the form of a detailed lemma needed later on.

LEMMA 3. *Let $|K|$ and $|L|$ be compact polyhedra and $\varphi\colon |K| \to |L|$ an n-valued continuous multifunction. If L^s is a subdivision of L with $\mu(L^s) < \frac{1}{4}\gamma(\varphi)$, then there exist a subdivision K^r of K and an n-valued simplicial multifunction $\varphi'\colon |K^r| \to |L^s|$ so that*

$$\text{i) } \gamma(\varphi') \geqslant \gamma(\varphi) - 2\mu(L^s), \qquad \text{ii) } \bar{d}(\varphi, \varphi') \leqslant \mu(L^s).$$

Proof. According to Lemma 1 the restriction $\varphi|\mathrm{st}_K v$ splits into n distinct maps for every vertex $v \in |K|$. Write $\varphi|\overline{\mathrm{st}_K v} = \{f_{v,1}, f_{v,2}, \ldots, f_{v,n}\}$, let $\lambda_i(v)$, for $i = 1, 2, \ldots, n$, be the Lebesgue number of the open cover

$$\{f_{v,i}^{-1}(\mathrm{st}_{L^s} b) \mid b \text{ is a vertex of } |L^s|\}$$

of $\overline{\mathrm{st}_K v}$, and let

$$\lambda = \inf\{\lambda_i(v) \mid v \text{ is a vertex of } |K| \text{ and } i = 1, 2, \ldots, n\}.$$

As $|K|$ is compact, we have $\lambda > 0$. Now each $f_{v,i}\colon \overline{\mathrm{st}_K v} \to |L|$ is uniformly continuous, therefore we can for every $v \in |K|$ choose $\delta(v) > 0$ so that $d_L(f_{v,i}(x), f_{v,i}(x')) < \frac{1}{4}\gamma(\varphi)$ for all $x, x' \in \overline{\mathrm{st}_K v}$ with $d_K(x, x') < \delta(v)$ and all $i = 1, 2, \ldots, n$. Let

$$\delta = \inf\{\delta(v) \mid v \text{ is a vertex of } |K|\},$$

and let K^r be a subdivision of K so that $\mu(K^r) < \min(\delta, \lambda/2)$.

We now construct φ' on the vertices of $|K^r|$. Select for every vertex $a \in |K^r|$ a vertex $v = v(a) \in |K|$ so that $\mathrm{st}_{K^r} a \subset \mathrm{st}_K v$. As the diameter diam $(\mathrm{st}_{K^r} a) < \lambda \leqslant \lambda_i(v)$ for all $i = 1, 2, \ldots, n$, we see that

$= \{f_1'(a_l), f_2'(a_l), \ldots, f_n'(a_l)\}$ so that

$$f_{v,i}|\overline{\mathrm{st}_K v} \cap \overline{\mathrm{st}_K v(a_l)} = f_{v(a_l),i}|\overline{\mathrm{st}_K v} \cap \overline{\mathrm{st}_K v(a_l)}$$

for $i = 1, 2, \ldots, n$. Then

$$\bigcap_{l=0}^{p} \mathrm{st}_{K^r} a_l = \mathrm{st}_{K^r} \sigma \neq \emptyset$$

implies

$$\emptyset \neq f_{v,i}\Big(\bigcap_{l=0}^{p} \mathrm{st}_{K^r} a_l\Big) \subseteq \bigcap_{l=0}^{p} f_{v(a_l),i}(\mathrm{st}_{K^r} a_l).$$

But by construction of $f_i'(a_l)$ we have

$$\mathrm{st}_{K^r} a_l \subset f_{v(a_l),i}^{-1}(\mathrm{st}_{L^s} f_i'(a_l)),$$

so

$$\bigcap_{l=0}^{p} \mathrm{st}_{L^s} f_i'(a_l) \neq \emptyset.$$

Thus the set $\{f_i'(a_0), f_i'(a_1), \ldots, f_i'(a_p)\}$ spans, for each $i = 1, 2, \ldots, n$, a simplex $\tau_i \in |L^s|$, and hence we can extend each f_i' from the vertices of σ to an affine map $f_{\sigma,i}'\colon \bar{\sigma} \to \bar{\tau}_i$. Define $\varphi'\colon |K^r| \to |L^s|$ by $\varphi'(x) = \{f_{\sigma,1}'(x), f_{\sigma,2}'(x), \ldots, f_{\sigma,n}'(x)\}$ if $x \in \bar{\sigma}$.

To show that φ' is well defined, it is necessary to show that if $\bar{\sigma} \subset \bar{\sigma}^*$, where σ, σ^* are simplexes of $|K^r|$, then $f_{\sigma,i}$ and $f_{\sigma^*,j}'$ can be indexed so that $f_{\sigma,i}'(x) = f_{\sigma^*,i}'(x)$ for all $x \in \bar{\sigma}$. The construction of the $f_{\sigma,i}'$ from the $\varphi'(a)$ depends only on the choice of the vertex $v \in |K|$ with $\mathrm{st}_{K^r} \sigma \subset \mathrm{st}_K v$. Hence it suffices to show that if $\varphi'(\bar{\sigma}) = \{\bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_n\}$, then the simplexes $\tau_i \in |L^s|$ are independent of v up to order. To see this, take a vertex $f_i'(a_l) \in \bar{\tau}_i$. As $f_{v(a_l),i}(\mathrm{st}_{K^r} a_l) \subset \mathrm{st}_{L^s} f_i'(a_l)$, we have

$$d_L(f_{v(a_l),i}(a_l), f_i'(a_l)) \leqslant \mu(L^s).$$

So if $f_j'(a_k) \in \bar{\tau}_j$ and $i \neq j$, then $f_{v(a_l),i}(a_l) = f_{v,i}(a_l)$ and $f_{v(a_k),j}(a_k) = f_{v,k}(a_k)$ imply

$$\begin{aligned} d_L(f_i'(a_l), f_j'(a_k)) &\geqslant d_L(f_{v,i}(a_l), f_{v,j}(a_l)) - d_L(f_{v,j}(a_l), f_{v,j}(a_k)) - \\ &\quad - d_L(f_{v,i}(a_l), f_i'(a_l)) - d_L(f_{v,j}(a_k), f_j'(a_k)) \\ &\geqslant \gamma(\varphi) - \tfrac{1}{4}\gamma(\varphi) - 2\cdot\tfrac{1}{4}\gamma(\varphi) = \tfrac{1}{4}\gamma(\varphi) > \mu(L^s). \end{aligned}$$

$$f_{v,i}(x)\in \bigcap_{l=0}^{p} f_{v,i}(\mathrm{st}_{K^r}(a_l)) \subseteq \bigcap_{l=0}^{p} \mathrm{st}_{L^s} f_i'(a_l) = \mathrm{st}_{L^s}\tau_i$$

for $i = 1, 2, \ldots, n$, and as $f'_{\sigma,i}(x)\in\tau_i$, we have

$$d_L(f'_{\sigma,i}(x), f_{v,i}(x)) \leqslant \mu(L^s)$$

and therefore if $i\neq j$

$$d_L(f'_{\sigma,i}(x), f'_{\sigma,j}(x)) \geqslant d_L(f_{v,i}(x), f_{v,j}(x)) - 2\mu(L^s).$$

As $\varphi(x) = \{f_{v,1}(x), f_{v,2}(x), \ldots, f_{v,n}(x)\}$, $\varphi'(x) = \{f'_{\sigma,1}(x), f'_{\sigma,2}(x), \ldots, f'_{\sigma,n}(x)\}$, it follows that $\gamma(\varphi')\geqslant\gamma(\varphi)-2\mu(L^s)$ and $\bar{d}(\varphi, \varphi')\leqslant\mu(L^s)$.

An immediate consequence of Lemma 3 is

THEOREM 4 (Simplicial approximation for n-valued multifunctions). *Let $|K|$ and $|L|$ be compact polyhedra and $\varphi\colon |K|\to|L|$ an n-valued continuous multifunction. Given $\varepsilon>0$, there exist subdivisions K^r of K and L^s of L and an n-valued simplicial multifunction $\varphi'\colon |K^r|\to|L^s|$ such that $\bar{d}(\varphi, \varphi')<\varepsilon$.*

Proof. In Lemma 3 choose L^s so that $\mu(L^s) < \min(\frac{1}{4}\gamma(\varphi), \varepsilon)$.

Remark. A simplicial approximation theorem for certain finite-valued multifunctions – the so-called m-functions – was proved by R. Jerrard [3], Theorem 5.2, but he only showed that piecewise linear m-functions (defined as m-functions with a polyhedral graph) have a simplicial approximation.

4. Fix-finite approximation of n-valued multifunctions. It is the purpose of this paragraph to show that every n-valued continuous multifunction $\varphi\colon |K|\to|K|$ from a compact polyhedron to itself has an arbitrarily close approximation by an n-valued continuous multifunction $\varphi\colon |K|\to|K|$ which has only finitely many fixed points. Our method consists of a repeated use of a Hopf construction performed on an n-valued simplicial multifunction approximating φ. The Hopf construction for n-valued continuous multifunctions, described in the rather technical Lemma 5, is modelled on the Hopf construction for maps which can be found in [2], pp. 117–118.

We use K'_L to denote the barycentric subdivision of the simplicial complex K modulo the subcomplex L (see e.g. [2], p. 116, or [4], p. 49). A *refinement* of K is a simplicial complex obtained by means of a finite number of subdivisions modulo subcomplexes. A simplex $\sigma\in|K|$ is *maximal* if $\mathrm{st}_K\sigma=\sigma$. K^p denotes the p-skeleton of K, and hence K'^p_L the p-skeleton of K'_L.

LEMMA 5 (Hopf construction for n-valued multifunctions). *Let K' be a refinement of the finite simplicial complex K, let $\varphi\colon |K'|\to|K|$ be an n-valued simplicial multifunction with $\gamma(\varphi')>4\mu(K)$, and let $\sigma\in|K'|$ be a p-dimensional simplex which is not maximal. If $\dot{\sigma}\cap\mathrm{Fix}\,\varphi'=\emptyset$ but $\sigma\cap\mathrm{Fix}\,\varphi'\neq\emptyset$, then there exists an n-valued simplicial multifunction $\varphi''\colon |K'_L|\to|K|$, where $|L| = |K'|-\mathrm{st}_{K'}\sigma$, so that*

i) *all fixed points of $\varphi''||K'^p_L|$ lie in $|L|$,*

ii) *$\varphi'(x)=\varphi''(x)$ for all $x\in|L|$,*

iii) *$\gamma(\varphi'')\geqslant\gamma(\varphi')-4\mu(K)$,*

iv) *$\bar{d}(\varphi', \varphi'')\leqslant 2\mu(K)$.*

Proof. Select a vertex $v\in|K'|$ so that $\mathrm{st}_{K'}\sigma\subset\mathrm{st}_{K'}v$. It follows from Lemma 1 that $\varphi'|\overline{\mathrm{st}_{K'}v} = \{f_1', f_2', \ldots, f_n'\}$ splits into n distinct maps, and as φ' is simplicial, each f_i' maps $\bar{\sigma}$ affinely onto a simplex $\bar{\tau}_i\in|K|$. Note that the $\bar{\tau}_i$ are distinct, for if $x, x'\in\bar{\sigma}$ and $i\neq j$, then

$$\begin{aligned} d_L(f_i'(x), f_j'(x')) &\geqslant d_L(f_i'(x), f_j'(x)) - d_L(f_j'(x), f_j'(x')) \\ &\geqslant \gamma(\varphi')-\mu(K) > 3\mu(K). \end{aligned}$$

But $\bar{\tau}_i\cap\bar{\tau}_j\neq\emptyset$ would imply

$$d_L(f_i'(x), f_j'(x'))\leqslant 2\mu(K),$$

so this is impossible. Hence it follows from $\sigma\cap\mathrm{Fix}\,\varphi'\neq\emptyset$ that there exists exactly one $k\in\{1, 2, \ldots, n\}$ with $\sigma\cap\mathrm{Fix} f_K'\neq\emptyset$. Let S be the subcomplex of K'_L underlying $\overline{\mathrm{st}_{K'}v}$. If we apply the Hopf construction [2], pp. 117–118 to the map $f_k'\colon \overline{\mathrm{st}_{K'}v}\to|K|$, then we obtain a simplicial map $f_k''\colon |S|\to|K|$ so that $f_k'(x)=f_k''(x)$ for all $x\in|L|\cap|S|$, all fixed points of $f_k''||S^p|$ lie in $|L|$, and $d_L(f_k'(x), f_k''(x))\leqslant 2\mu(K)$ for all $x\in|S|$.

Now take any map $f_i'\colon \overline{\mathrm{st}_{K'}v}\to|K|$ with $i=1, 2, \ldots, n$, and $i\neq k$, and define a function f_i'' on the vertices of $|S|$ as follows: if $v\in|L|$, let $f_i''(v)=f_i'(v)$, and if $v\notin|L|$, let $f_i''(v)$ be any vertex of $\bar{\tau}_i$. It is easy to check that f_i'' can be extended to a simplicial map $f_i''\colon |S|\to|K|$.

f_i'' has no fixed points on $\bar{\sigma}$, for if $x\in\bar{\sigma}$, then $f_i''(x)\in\bar{\tau}_i$. But one can see as in [2], p. 117 that $\mathrm{Fix} f_k'\cap\sigma\neq\emptyset$ implies $\bar{\sigma}\subset\bar{\tau}_k$, and so it follows from $\bar{\tau}_i\cap\bar{\tau}_k=\emptyset$ that $f_i''(x)\neq x$. If we now consider a point $x\in\mathrm{st}_{K'}\sigma-\sigma$ which lies in $|S^p|$, then $x\notin|K^p|$ but $f_i''(x)\in|K^p|$, so $f_i''(x)\neq x$ also. Hence all fixed points of f_i'' on $|S^p|$ lie in $|L|$. Note that $f_i''(x)=f_i'(x)$ for all $x\in|L|\cap|S|$, and that $d_L(f_i'(x), f_i''(x))\leqslant 2\mu(K)$ for all $x\in|S|$.

As $d_L(f_i''(x), f_j''(x))\geqslant\gamma(\varphi')-4\mu(K)>0$ for all $i, j=1, 2, \ldots, n$, with $i\neq j$, the multifunction $\varphi''\colon |K'_L|\to|K|$ given by

$$\varphi''(x) = \begin{cases} \varphi'(x) & \text{if } x\in|L|, \\ \{f_1''(x), f_2''(x), \ldots, f_n''(x)\} & \text{if } x\notin|L| \end{cases}$$

shows that Lemma 5 is true.

THEOREM 6. (Fix-finite approximation of n-valued multifunctions). *Let X be a compact polyhedron and $\varphi\colon X\to X$ be an n-valued continuous multifunction. Given $\varepsilon>0$, there exists an n-valued continuous multifunction $\psi\colon X\to X$ so that*

i) *ψ has only finitely many fixed points,*

ii) *there exists a triangulation of X so that each fixed point of ψ lies in a maximal simplex,*

iii) *$\bar{d}(\varphi, \psi)<\varepsilon$.*

Proof. Let m be the dimension of X; we can assume $m>0$. Choose a simplicial complex K with $|K|=X$ and

$$\mu(K)<\min\left(\frac{\gamma(\varphi)}{2(1+2m)}, \frac{\varepsilon}{4m}\right),$$

and use Lemma 3 to obtain an n-valued simplicial multifunction

$$\varphi'\colon |K^r|\to|K| \quad \text{with} \quad \gamma(\varphi')\geqslant\gamma(\varphi)-2\mu(K) \quad \text{and} \quad \bar{d}(\varphi, \varphi')<\mu(K).$$

Now proceed as in the proof of Theorem 2, pp. 118–119, of [2], i.e. apply the Hopf construction of Lemma 5 repeatedly on simplexes of increasing dimension until a simplicial multifunction $\psi\colon |K'|\to|K|$ is obtained, where K' is a refinement of K so that ψ is fixed point free on all non-maximal simplexes. An argument parallel to the one in [2], p. 119 implies that the image of each point is changed at most m times. Hence Lemma 5 iii) shows that

$$\gamma(\psi)\geqslant\gamma(\varphi')-4m\mu(K)\geqslant\gamma(\varphi)-2\mu(K)-4m\mu(K)>0,$$

so ψ is n-valued. Similarly we see that each intermediate simplicial multifunction φ'' which is fixed point free on all p-simplexes ($p<m$) satisfies the assumption $\gamma(\varphi'')>4\mu(K)$ of Lemma 5. It follows that

$$\bar{d}(\varphi, \psi)\leqslant\bar{d}(\varphi, \varphi')+\bar{d}(\varphi', \psi)<\tfrac{1}{4}\varepsilon+m\cdot\frac{\varepsilon}{2m}<\varepsilon.$$

The verification that ψ satisfies i) and ii) is analogous to the one in [2], pp. 118–119.

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A universal metacompact developable T_1-space of weight $\mathfrak{m}$

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Abstract. For each cardinal number $\mathfrak{m}$ we construct a metacompact developable T_1-space $T(\mathfrak{m})$. If $\mathfrak{m}$ is infinite, then $T(\mathfrak{m})^{\aleph_0}$ is universal for all metacompact developable T_1-spaces of weight $\mathfrak{m}$. The space $T(0)$ is the set of irrational numbers with a weaker topology and $T(0)^{\aleph_0}$ is universal for all perfect T_1-spaces of countable weight. Each $T(\mathfrak{m})$ is built of $\mathfrak{m}$ copies of $T(0)$. Moreover, each mapping of a closed subset of a perfect space X into $T(0)$ can be extended to a mapping of X into $T(0)$.

In [3] we have introduced a method of constructing mappings into metacompact developable T_1-spaces. More precisely, we have constructed, for a point-finite open cover $\mathscr{U}$ of a perfect space X, a continuous mapping p of X onto a metacompact developable T_1-space Z such that each element of $\mathscr{U}$ is an inverse image of an open subset of Z.

The examination of this construction shows that the space Z can be regarded as a subspace of a metacompact developable T_1-space T which depends only on the cardinality of $\mathscr{U}$.

In the first section of this paper we give a modification of the construction from [3]. The ideas of the first section are used in the second section to construct, for each cardinal number $\mathfrak{m}$, a metacompact developable T_1-space $T(\mathfrak{m})$ and a point-finite collection $\mathscr{G}$ of open subsets of $T(\mathfrak{m})$ such that, for any perfect space X and any point-finite collection $\mathscr{U}$ of cardinality $\mathfrak{m}$ consisting of open subsets of X, there exists a mapping $f\colon X\to T(\mathfrak{m})$ satisfying $\mathscr{U}=\{f^{-1}(G)\colon G\in\mathscr{G}\}$.

The weight of $T(\mathfrak{m})$ is $\mathfrak{m}+\aleph_0$ and it follows that, for infinite $\mathfrak{m}$, $T(\mathfrak{m})^{\aleph_0}$ is universal for all metacompact developable T_1-spaces of weight $\mathfrak{m}$.

A space with properties similar to the properties of $T(0)$ is constructed in [6]. Our construction of $T(0)$ is more direct and can be regarded as a simplification of the construction in [6]. We prove an extension theorem for mappings into $T(\mathfrak{m})$ (Theorem 2) and obtain a number of corollaries showing that $T(0)$ can be considered to be a D-line (see Remarks 3 and 4).

We shall use the terminology and notation from [5]. By a mapping we always mean a continuous function. Metacompact spaces are not necessarily Hausdorff but all spaces we consider are T_1-spaces. If $\mathscr{D}$ is a family of subsets