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Remarks on the *n*-dimensional geometric measure of compacta

by

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Abstract. By the *n-dimensional geometric measure* of a compactum X lying in the Hilbert space E^ω, one understands the lower bound $\mu_n(X)$ of all positive numbers α such that for every $\varepsilon > 0$ there is an ε-translation $f\colon X \to E^\omega$ such that $f(X)$ lies in a polyhedron $P \subset E^\omega$ for which the n-dimensional measure $|P|_n$ (in the elementary sense) is $\leqslant \alpha$. If dim $P < n$, we assume $|P|_n = 0$, and if $\dim P > n$, we assume $|P|_n = \infty$.

Some relations between geometric measures of two compacta $X, Y \subset E^\omega$ and the pseudo-measures of $X \cup Y$, $X \cap Y$ and $X \times Y$ are studied.

1. Introduction. In the elementary geometry one assigns to every n-dimensional polyhedron P the number $|P|_n$, defined as the sum of the volumes of all n-dimensional simplices belonging to a triangulation of P. If $\dim P < n$, then one assumes that $|P|_n = 0$, and if $\dim P > n$, then $|P|_n = \infty$.

One knows that $|P|_n$ does not depend on the choice of the triangulation of P. Moreover, one sees easily that

(1.1) If $P_1, P_2, \ldots, P_k$ are polyhedra, then $|P_1 \cup \ldots \cup P_k|_n \leqslant \sum_{i=1}^{k} |P_i|_n$.

Let E^ω denote the usual Hilbert space, i.e. the space consisting of all real sequences $(x_1, x_2, \ldots)$, such that $\sum_{i=1}^{\infty} x_i^2 < \infty$, metrized by the formula

$$\varrho\big((x_1, x_2, \ldots)(y_1, y_2, \ldots)\big) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

We may consider the Euclidean m-space E^m as the subset of E^ω consisting of all points $(x_1, x_2, \ldots, x_m, 0, 0, \ldots)$ denoted also by $(x_1, x_2, \ldots, x_m)$.

By the *n-dimensional geometric measure* of a compactum $X \subset E^\omega$, we understand the number $\mu_n(X)$ (finite or ∞) defined as the lower bound of all numbers $\alpha > 0$ such that for every $\varepsilon > 0$, there is an ε-translation $f_\varepsilon\colon X \to E^\omega$ (i.e. a map f_ε satisfying the condition $\varrho(x, f_\varepsilon(X)) < \varepsilon$ for every $x \in X$) such that $f_\varepsilon(X)$ is a subset of a polyhedron $P \subset E^\omega$ with $|P|_n < \alpha$. It is known (see [2]) that:

(1.2) *If* $\dim X < n$, *then* $\mu_n(X) = 0$.
(1.3) *If* $\dim X > n$, *then* $\mu_n(X) = \infty$.
(1.4) *If* $X \subset Y$, *then* $\mu_n(X) \leqslant \mu_n(Y)$.
(1.5) *If* $X \cap Y = \emptyset$, *then* $\mu_n(X \cup Y) = \mu_n(X) + \mu_n(Y)$.
(1.6) *If* P *is a polyhedron, then* $\mu_n(P) = |P|_n$.
(1.7) *If* X *is a continuum, then* $\mu_1(X) \geqslant d(X)$, *where* $d(X)$ *denotes the diameter of* X.
(1.8) *If* L *is an arc, then* $\mu_1(L)$ *is the length* $|L|$ *of* L.

Observe, that (1.2) implies that for every arc L and $n = 2, 3, \ldots$ $\mu_n(L) = 0$. However there exist arcs $L \subset E^\omega$ for which the n-dimensional measure (in the sense of Hausdorff [3]) is positive for $n = 2, 3, \ldots$ In fact, it is well known that there exists in E^ω a set C homeomorphic to the Cantor discontinuum, for which the n-dimensional Hausdorff measure is positive, for every $n = 1, 2, \ldots$ On the other hand, there is an arc $L \subset E^\omega$ containing C. It follows that the n-dimensional Hausdorff measure of L is positive for every $n = 1, 2, \ldots$ However $\mu_n(L) = 0$ for $n = 2, 3, \ldots$, because of (1.2).

2. Geometric measure of n-dimensional compacta. First let us prove the following

(2.1) LEMMA. *For any natural number m and for every positive number ε, there is a positive real number $\delta = \delta(\varepsilon, m)$ such that for every n-dimensional polyhedron R lying in E^m, with $|R|_n < \delta$, there is an ε-translation $g\colon R \to E^m$ such that $g(R)$ is contained in an $(n-1)$-dimensional polyhedron.*

Proof. Let η be a positive real number such that the diameter of the m-cube $\langle 0, \eta\rangle^m$ is less than ε. Let $\delta = \delta(\varepsilon, m)$ be a positive real number such that

$$\delta \cdot \binom{m}{k} < \eta^k \quad \text{for} \quad k = 1, 2, \ldots, m,$$

where

$$\binom{m}{k} = \frac{m!}{k!\cdot(m-k)!}.$$

Let R be an n-dimensional polyhedron in E^m with $|R|_n < \delta$. Assume that R is contained in an m-cube $\langle 0, N\cdot\eta\rangle^m$, where N is a natural number. Hence $\langle 0, N\cdot\eta\rangle^m$ consists of all points $(x_1, x_2, \ldots, x_m) \in E^m$ with $0 \leqslant x_i \leqslant N\cdot\eta$, for $i = 1, 2, \ldots, m$.

Let $\mathfrak{F}$ denote the family of all subsets $\mathfrak{A}$ of the set $\{1, 2, \ldots, m\}$ consisting of n elements. Observe that $\mathfrak{F}$ consists of $\binom{m}{n}$ sets $\mathfrak{A}$. Assign to every $\mathfrak{A} \in \mathfrak{F}$ the n-dimensional face $F_{\mathfrak{A}}$ of $\langle 0, N\cdot\eta\rangle^m$ consisting of all points $(x_1, x_2, \ldots, x_m)$ such that

$$0 \leqslant x_i \leqslant N\cdot\eta, \text{ if } i \in \mathfrak{A} \quad \text{and} \quad x_i = 0, \text{ if } i \notin \mathfrak{A}.$$

Denote by $p_{\mathfrak{A}}$ the map assigning to every point $(x_1, x_2, \ldots, x_m)$ of $\langle 0, N\cdot\eta\rangle^m$ the point $(x'_1, x'_2, \ldots, x'_m)$, where $x'_i = x_i$ if $i \in \mathfrak{A}$ and $x'_i = 0$, if $i \notin \mathfrak{A}$. Thus $p_{\mathfrak{A}}$ is an ortogonal projection of the cube $\langle 0, N\cdot\eta\rangle^m$ onto its n-dimensional face $F_{\mathfrak{A}}$.

One sees easily that for every $\mathfrak{A} \in \mathfrak{F}$, the image $p_{\mathfrak{A}}(R)$ of the polyhedron $R \subset \langle 0, N\cdot\eta\rangle^m$ is a polyhedron lying in $F_{\mathfrak{A}}$ with $|p_{\mathfrak{A}}(R)|_n \leqslant |R|_n$. Since $\binom{m}{n}\cdot|R|_n < \binom{m}{n}\cdot\delta < \eta^n$, we infer that

$$\sum_{\mathfrak{A}\in\mathfrak{F}} |p_{\mathfrak{A}}(R)|_n < \eta^n. \tag{2.2}$$

Divide every interval $\langle (i-1)\cdot\eta, i\cdot\eta\rangle$, where i belongs to $\{1, 2, \ldots, N\}$, into s equal intervals. One gets a subdivision of the cube $\langle 0, N\cdot\eta\rangle^m$ into $(N\cdot s)^m$ small m-cubes with the length of edges $\frac{1}{s}\eta$. Denote by $k_{\mathfrak{A}}$ the number of all such small cubes, which lay in $F_{\mathfrak{A}}$ and which intersect the polyhedron $p_{\mathfrak{A}}(R)$. It follows by (2.2) that for s sufficiently large

$$\sum_{\mathfrak{A}\in\mathfrak{F}} k_{\mathfrak{A}} < s^n. \tag{2.3}$$

Observe that there are s^N subsets of $\langle 0, N\cdot\eta\rangle$ of the form

$$A_1 \cup A_2 \ldots \cup A_N, \tag{2.4}$$

where A_i, $i = 1, 2, \ldots, N$, are intervals with the length $\frac{1}{s}\cdot\eta$ in which the interval $\langle (i-1)\cdot\eta, i\cdot\eta\rangle$ is divided.

Moreover, there are s^{N-1} subsets of $\langle 0, N\cdot\eta\rangle$ of the form (2.4) which contain a given one of the intervals with the length $\frac{1}{s}\cdot\eta$ in which the interval $\langle 0, N\cdot\eta\rangle$ is divided.

Let $\mathfrak{A} \in \mathfrak{F}$, and let C_j, for $j = 1, 2, \ldots, m$ be one of the $s\cdot N$ intervals with the length $\frac{1}{s}\cdot\eta$ in which the interval $\langle 0, N\cdot\eta\rangle$ is divided. If B_j, for $j = 1, 2, \ldots, m$, is a subset of $\langle 0, N\cdot\eta\rangle$ of the form (2.4), then the set $p_{\mathfrak{A}}(B_1 \times B_2 \times \ldots \times B_m)$ contains the set $p_{\mathfrak{A}}(C_1 \times C_2 \times \ldots \times C_m)$ if and only if the set B_j contains the set C_j for every $j \in \mathfrak{A}$. Consequently, there are

$$(s^{N-1})^n \cdot (s^N)^{m-n} = s^{-n}\cdot s^{N\cdot m}$$

sets of the form

$$B_1 \times B_2 \times \ldots \times B_m, \tag{2.5}$$

where B_i is, for $i = 1, 2, \ldots, m$, a subset of $\langle 0, N\cdot\eta\rangle$ of the form (2.4) with the images under $p_{\mathfrak{A}}$ containing the small n-cube $p_{\mathfrak{A}}(C_1 \times C_2 \times \ldots \times C_m)$ lying in $F_{\mathfrak{A}}$. It follows that there are at most $k_{\mathfrak{A}}\cdot s^{-n}\cdot s^{N\cdot m}$ sets of the form (2.5) with the images under $p_{\mathfrak{A}}$ intersecting the polyhedron $p_{\mathfrak{A}}(R)$.

There are $s^{N\cdot m}$ sets of the form (2.5). It follows by (2.3) that

$$\sum_{\mathfrak{A}\in\mathfrak{F}} k_{\mathfrak{A}}\cdot s^{-n}\cdot s^{N\cdot m} < s^{N\cdot m}.$$

Consequently there is a set B of the form $B_1\times B_2\times\ldots\times B_m$ such that

$$p_{\mathfrak{A}}(B)\cap p_{\mathfrak{A}}(R)=\emptyset \quad \text{for every } \mathfrak{A}\in\mathfrak{F}.$$

Let $\mathring{B}$ denote the interior of B in E^m. Then the set

$$Y=\langle 0, N\cdot\eta\rangle^m\setminus\bigcup_{\mathfrak{A}\in\mathfrak{F}} p_{\mathfrak{A}}^{-1}(p_{\mathfrak{A}}(\mathring{B}))$$

contains the polyhedron R.

Denote by S_k, for $k=0, 1, \ldots, m$ the union of all k-dimensional faces of m-cubes of the following form

(2.6) $\langle i_1\cdot\eta, (i_1+1)\cdot\eta\rangle\times\langle i_2\cdot\eta, (i_2+1)\cdot\eta\rangle\times\ldots\times\langle i_m\cdot\eta, (i_m+1)\cdot\eta\rangle,$

where $i_j=0, 1, \ldots, N-1$ for $j=1, 2, \ldots, m$. Let

$$Y_k=Y\cap S_k, \quad \text{for} \quad k=1, 2, \ldots, m$$

(in particular, $S_m=\langle 0, N\cdot\eta\rangle^m$ and $Y_m=Y$).

It is easy to see that for each $k=n, n+1, \ldots, m$ there is a retraction $r_k\colon Y_k\to Y_{k-1}$ such that for every k-face K' of any m-cube of the form (2.6)

$$r_k(K'\cap Y_k)=K'\cap Y_{k-1}.$$

Consequently the map

$$r=r_n\cdot r_{n+1}\cdot\ldots\cdot r_m\colon Y\to Y_{n-1}$$

is a retraction such that $r(K\cap Y)\subset K$ for every m-cube K of the form (2.6). Since the diameter of any m-cube of the form (2.6) is less than ε, we infer that $r\colon Y\to Y_{n-1}$ is an ε-translation. Setting

$$g(x)=r(x) \quad \text{for every } x\in R,$$

one gets a map satisfying the required conditions. Thus the proof of Lemma (2.1) is finished.

Denote by $q_m\colon E^\omega\to E^m$ the orthogonal projection

$$q_m(x_1, x_2, \ldots)=(x_1, x_2, \ldots, x_m) \quad \text{for every } (x_1, x_2, \ldots)\in E^\omega.$$

Let us prove the following

(2.7) THEOREM. *For every n-dimensional compactum $X\subset E^\omega$, the n-dimensional geometric measure is positive.*

Proof. If X is a compactum in E^ω with $\mu_n(X)=0$, then for every positive ε, there is an integer m such that

$$\varrho(x, q_m(x))<\varepsilon \quad \text{for every } x\in X.$$

Let $\delta=\delta(\varepsilon, m)$ be the positive real number from Lemma (2.1). Since $\mu_n(X)=0$, there is an ε-translation $f\colon X\to E^\omega$ such that $f(X)$ is a subset of a polyhedron $P\subset E^\omega$ with $\dim P=n$ and $|P|_n<\delta$. Then

$$\varrho(x, q_mf(x))\leqslant\varrho(x, q_m(x))+\varrho(q_m(x), q_mf(x))$$

for every $x\in X$. So $q_mf\colon X\to E^m$ is an 2δ-translation.

Observe that $q_mf(X)$ is a subset of the polyhedron $R=q_mf(P)$ lying in E^n with $|R|_n<|P|_n<\delta$. By Lemma (2.1), there is an ε-translation $g\colon R\to E^m$ such that $g(R)$ is contained in an $(n-1)$-dimensional polyhedron. The map $gq_mf\colon X\to E^m$ is a 3ε-translation. The image $gq_mf(X)$ is contained in an $(n-1)$-dimensional polyhedron.

Thus, for every $\varepsilon>0$, there is a 3ε-translation of X into E^ω with the image contained in an $(n-1)$-dimensional polyhedron. Hence $\dim X<n$, and the proof of Theorem (2.7) is finished.

(2.8) COROLLARY. *For compacta $X\subset E^\omega$ the vanishing of the n-dimensional geometric measure is a topological invariant.*

(2.9) QUESTION. *Is it true that for every $\varepsilon>0$ there is an $\eta_n>0$ such that for every compactum $X\subset E^\omega$ with $\mu_n(X)<\eta_n$ there is an ε-translation $f_\varepsilon\colon X\to E^\omega$ such that $\dim f_\varepsilon(X)<n$?*

3. Geometric measures for Cartesian products. If P is an k-dimensional polyhedron and R is an m-dimensional polyhedron, then $P\times R$ is a $(k+m)$-dimensional polyhedron and we infer by (1.6) that

(3.1) $$\mu_{k+m}(P\times R)=\mu_k(P)\cdot\mu_m(R).$$

Another situation is for arbitrary compacta. As has been shown by L. S. Pontrjagin [4], there exist two 2-dimensional compacta X and Y such that $\dim(X\times Y)=3$. By Theorem (2.7), both numbers $\mu_2(X)$ and $\mu_2(Y)$ are positive, however (1.2) implies that $\mu_4(X\times Y)=0$. Consequently

(3.2) *There exist 2-dimensional compacta X and Y such that $\mu_4(X\times Y)<\mu_2(X)\cdot\mu_2(Y)$.*

However the following theorem holds true:

(3.3) THEOREM. *If X, Y are compacta lying in E^ω and if $\dim X=k$, $\dim Y=m$, then $\mu_{k+m}(X\times Y)\leqslant\mu_k(X)\cdot\mu_m(Y)$.*

Proof. Both numbers $\mu_k(X)$ and $\mu_m(Y)$ are positive. If at least one of them is infinite, then $\mu_{k+m}(X\times Y)\leqslant\mu_k(X)\cdot\mu_m(Y)$. Thus we can assume that both numbers $\mu_k(X)$ and $\mu_m(Y)$ are finite.

By the definition of μ_n, there exist for every $\eta>0$ two polyhedra P and R such that $\dim P=k$, $\dim R=m$, with

$$\mu_k(X)+\eta>|P|_k, \quad \mu_m(Y)+\eta>|R|_m$$

and that for every $\varepsilon > 0$ there exist two ε-translations

$$f\colon X \to P \quad \text{and} \quad g\colon Y \to R.$$

Setting

$$\varphi(x, y) = (f(x), g(y)) \quad \text{for every } x \in X \text{ and } y \in Y,$$

one gets a 2ε-translation $\varphi\colon X \times Y \to P \times R$. Using (3.1) and (3.2), one infers that

$$\mu_{k+m}(X \times Y) \leqslant \mu_{k+m}(P \times R) = \mu_k(P) \cdot \mu_m(R) < (\mu_k(X) + \eta) \cdot (\mu_m(Y) + \eta).$$

Hence

$$\mu_{k+m}(X \times Y) < (\mu_k(X) + \eta) \cdot (\mu_m(Y) + \eta) \quad \text{for every } \eta > 0.$$

Consequently $\mu_{k+m}(X \times Y) \leqslant \mu_k(X) \cdot \mu_m(Y)$ and the proof of Theorem (3.3) is finished.

4. Geometric measures of unions of compacta. First let us prove the following

(4.1) LEMMA. *Let X and A be two compacta lying in E^ω. For every ε-translation f of X with values lying in a polyhedron $P \subset E^\omega$, there exists a 2ε-translation $g\colon X \cup A \to E^\omega$ and a polyhedron $R \subset E^\omega$ with $\dim R \leqslant \dim A$ such that $g(X \cup A) \subset P \cup R$.*

Proof. We may assume that $P \subset E^m$. One sees easily that there exists an extension f' of f to an ε-translation $f'\colon X \cup A \to E^\omega$. Moreover, there is a natural number k so large, that the orthogonal projection $\varphi\colon E^\omega \to E^{m+k}$ assigns to every point $x = (x_1, x_2, \ldots) \in E^\omega$ the point $\varphi(x) = (x_1, x_2, \ldots, x_{m+k}) \in E^{m+k}$, satisfying the condition

$$\varrho(\varphi f'(x), f'(x)) < \tfrac{1}{2}\varepsilon \quad \text{for every point } x \in X \cup A.$$

Then the set $\varphi f'(A)$ lies in a polyhedron $R' \subset E^{m+k}$ and

$$\varphi f'(x) = f'(x) = f(x) \quad \text{for every } x \in X.$$

Consider a triangulation T of the polyhedron $P \cup R'$ such that all simplices $\Delta \in T$ with $\Delta \cap P \neq \emptyset$, together with their faces, constitute a triangulation T_P of P and all simplices $\Delta \in T$ with $\Delta \cap R' \neq \emptyset$ (and their faces) constitute a triangulation $T_{R'}$ of R'.

Let us say that two maps $g, g'\colon X \cup A \to E^\omega$ are T-associated, if for every $x \in X \cup A$ there is a simplex $\Delta \in T$ containing both points $g(x)$ and $g'(x)$.

Let $m = \dim A$ and $g_0 = \varphi f'$. Consider a simplex $\Delta \in T_{R'} \backslash T_P$ with $\dim \Delta = \dim R$. If $\dim \Delta > m$, then there exists a map g' of the set $g_0^{-1}(\Delta)$ into the boundary $\dot{\Delta}$ of Δ such that $g_0(x) = g'(x)$ for every point $x \in g_0^{-1}((P \cup R) \backslash \mathring{\Delta})$ (where $\mathring{\Delta}$ denotes the interior $\Delta \backslash \dot{\Delta}$ of Δ). Applying this procedure step by step

to all simplices $\Delta \in T_{R'} \backslash T_P$ with $\dim \Delta > m$, one gets a map $g\colon X \cup A \to E^\omega$ which is T-associated to the map g_0 and the set $g(A)$ lies in the m-skeleton R of R' (by triangulation $T_{R'}$).

If the mesh of the triangulation T is less than $\frac{1}{2}\varepsilon$, then $\varrho(g_0(x), g(x)) < \frac{1}{2}\varepsilon$ for every $x \in X \cup A$ and g is a 2ε-translation of $X \cup A$ into $P \cup R$.

(4.2) THEOREM. *If X, A are compacta lying in E^ω and if $\dim A < n$, then $\mu_n(X \cup A) = \mu_n(X)$.*

Proof. By (1.4), $\mu_n(X) \leqslant \mu_n(X \cup A)$. It remains to show that

$$\mu_n(X \cup A) \leqslant \mu_n(X).$$

We can assume that $\mu_n(X) < \infty$. Consider a finite number $\alpha > \mu_n(X)$. Then for every $\varepsilon > 0$ there is an ε-translation $f\colon X \to E^\omega$ such that $f(X)$ is a subset of a polyhedron $P \subset E^\omega$ with $\mu_n(P) < \alpha$. By Lemma (4.1), there is a 2ε-translation $g\colon X \cup A \to E^\omega$ and a polyhedron $R \subset E^\omega$ with $\dim R < n$ such that $g(A \cup X) \subset P \cup R$. But $|P \cup R|_n = |P|_n$ and consequently $\mu_n(X \cup A) < \alpha$ and the proof of Theorem (4.2) is finished.

(4.3) THEOREM. *If X_1, X_2 are compacta lying in E^ω and if $\mu_{n-1}(X_1 \cap X_2) = 0$, then $\mu_n(X_1 \cup X_2) \leqslant \mu_n(X_1) + \mu_n(X_2)$.*

Before giving the proof, we established some lemmas:

(4.4) LEMMA. *Let X_1, X_2 and $X_0 = X_1 \cap X_2$ be compacta lying in E^ω. Then for every $\varepsilon > 0$ there exists a natural number m and an ε-translation $f\colon X_1 \cup X_2 \to E^m$ such that $f(X_0) = f(X_1) \cap f(X_2)$ and $f(X_1), f(X_2), f(X_0)$ are polyhedra with $\dim f(X_i) \leqslant \dim X_i$, for $i = 0, 1, 2$.*

Proof. Setting $X = X_1 \cup X_2$, observe that there are a natural number n and a $\frac{1}{4}\varepsilon$-translation $g'\colon X \to E^n \subset E^\omega$ such that $g'(X)$ and $g'(X_0)$ are polyhedra and $\dim g'(X_0) \leqslant \dim X_0$.

It is known (see [1], p. 166) that there exists a $\frac{1}{2}\varepsilon$-translation $g\colon X \to E^n$ such that $\dim(g(X_i \backslash X_0)) \leqslant \dim(X_i \backslash X_0)$ for $i = 1, 2$ and $g(x) = g'(x)$ for $x \in X$.

It follows that $\dim g(X_i) \leqslant \dim X_i$ for $i = 0, 1, 2$.

We can also assume that $g(X_0)$, $g(X_1)$ and $g(X_2)$ are polyhedra.

By the general position argument, there are $\frac{1}{2}\varepsilon$-translations $h_1\colon g(X_1) \to E^{n+1}$ and $h_2\colon g(X_2) \to E^{n+1}$ such that

$$h_1(x) = h_2(x) = x \quad \text{for } x \in X_0$$

and

$$h_1(X_1 \backslash X_0) \cap h_2(X_2 \backslash X_0) = \emptyset.$$

Let $m = n+1$. The ε-translation $f\colon X \to E^\omega$ satisfying the required conditions can be defined by the formula

$$f(x) = h_i g(x) \quad \text{for every } x \in X_i \text{ and } i = 1, 2.$$

This completes the proof.

Assume that z is a real number and define the map

$$\tau_z\colon E^m \to E^{m+1} \subset E^\omega$$

by the formula

$$\tau_z(x_1, \ldots, x_m) = (x_1, \ldots, x_m, z) \quad \text{for every } (x_1, \ldots, x_m) \in E^m.$$

(4.5) Lemma. *If X_1 and X_2 are polyhedra lying in E^m and if $X_0 = X_1 \cap X_2$, then for every $\varepsilon > 0$ there is an ε-translation*

$$f\colon X_1 \cup X_2 \to \tau_{-\varepsilon/4}(X_1) \cup \tau_{\varepsilon/4}(X_2) \cup \bigcup_{|z| \leqslant \varepsilon/4} \tau_z(X_0).$$

Proof. Let T be a triangulation of the polyhedron $X = X_1 \cup X_2$ such that the mesh of T is less than $\frac{1}{2}\varepsilon$ and that the subfamily T_i of T, which consists of all simplices $\Delta \in T$ with $\Delta \cap X_i = \emptyset$ and of their faces is a triangulation of X_i, for $i = 0, 1, 2$.

Consider the polyhedron

$$Y_n = \tau_{-\varepsilon/4}(X_1) \cup \tau_{\varepsilon/4}(X_2) \cup \bigcup_{|z| \leqslant \varepsilon/4} \tau_z(X_0 \cup X^{(n)})$$

where $X^{(n)}$ denotes the n-skeleton of X with respect to T.

For $i = 1, 2$ and for every simplex $\Delta \in T_i$, the set

$$\tau_\gamma(\Delta) \cup \bigcup_{z \leqslant \varepsilon/4} \tau_z(\dot{\Delta}), \quad \text{where} \quad \gamma = \frac{(1)^i}{4} \cdot \varepsilon,$$

is a retract of $\bigcup_{|z| \leqslant \varepsilon/4} \tau_z(\Delta)$.

Therefore for each $n \geqslant 0$ there is a retraction r_n of the set Y_n to Y_{n-1} such that

$$r_n\Big(\bigcup_{|z| \leqslant \varepsilon/4} \tau_z(\Delta)\Big) \subset \bigcup_{|z| \leqslant \varepsilon/4} \tau_z(\dot{\Delta}) \cup \tau_\gamma(\Delta),$$

where $\Delta \in (T_i \setminus T_0)$ for $i = 1, 2$ is an n-dimensional simplex.

Hence the map $f\colon X \to Y_{-1}$ which is the restriction of

$$r_0 \ldots r_s\colon \bigcup_{|z| \leqslant \varepsilon/4} \tau_z(X) \to Y_{-1}, \quad \text{where} \quad s = \dim X,$$

is an ε-translation. Thus the proof of Lemma (4.5) is finished.

Proof of Theorem (4.3). Suppose that

$$\mu_n(X_i) \leqslant \alpha_i < \infty \quad \text{for} \quad i = 1, 2.$$

Consider $\frac{1}{5}\varepsilon$-translations

$$f_1\colon X_1 \to E^\omega \quad \text{and} \quad f_2\colon X_2 \to E^\omega$$

such that $f_i(X_i)$ is a polyhedron with $|f_i(X_i)|_n \leqslant \alpha_i$ for $i = 1, 2$.

For $i = 1, 2$ there exists a neighborhood U_i of X_i and a $\frac{2}{5}\varepsilon$-translation $\hat{f}_i\colon U_i \to E^\omega$ such that $\hat{f}_i(U_i) = f_i(X_i)$ and $\hat{f}_i$ is an extension of f_i.

Lemmas (4.4) and (4.5) imply that for every $\delta > 0$ there exist δ-translations

$$g\colon X_1 \cup X_2 = X \to g(X) \subset E^\omega \quad \text{and} \quad g_i\colon X_i \to E^\omega \text{ for } i = 1, 2$$

such that $g_1(X_1)$, $g_2(X_2)$ and $g(X) \supset g_1(X_1) \cup g_2(X_2)$ are polyhedra with

$$\dim\big(\overline{g(X) \setminus (g_1(X_1) \cup g_2(X_2))}\big) \leqslant n-1 \quad \text{and} \quad g_1(X_1) \cap g_2(X_2) = \emptyset.$$

Assume that $\delta > 0$ is so small, that $g(X) \subset U_1 \cup U_2$ and $\delta < \frac{1}{5}\varepsilon$. Setting

$$\hat{f}(x) = \hat{f}_i(x) \quad \text{for every } x \in g_i(X_i), \text{ for } i = 1, 2,$$

we obtain a $\frac{2}{5}\varepsilon$-translation $\hat{f}\colon g_1(X_1) \cup g_2(X_2) \to E^\omega$.

Using Lemma (4.1) for the case when $X = g_1(X_1) \cup g_2(X_2)$, A is the closure of the set $g(X) \setminus (g_1(X_1) \cup g_2(X_2))$ and $f = \hat{f}$, we infer that there is a $\frac{4}{5}\varepsilon$-translation $f'\colon g(X) \to E^\omega$ such that the set $Y_0 = f'\big(\overline{g(X) \setminus (g_1(X_1) \cup g_2(X_2))}\big)$ is a polyhedron with $\dim Y_0 \leqslant n-1$.

Since $\frac{1}{5}\varepsilon > \delta$, the composition $h = f'g\colon X_1 \cup X_2 \to E^\omega$ is an ε-translation. It is clear that $h(X_1 \cup X_2) \subset f_1(X_1) \cup f_2(X_2) \cup Y_0$ and

$$|f_1(X_1) \cup f_2(X_2) \cup Y_0|_n \leqslant |f_1(X_1)|_n + |f_2(X_2)|_n + |Y_0|_n \leqslant \alpha_1 + \alpha_2.$$

Thus the proof of Theorem (4.3) is finished.

5. A lemma. In Sections 7 and 8 we shall construct some compacta showing that without the hypothesis $\dim(X \cap Y) < n-1$, no simple relation between $\mu_n(X)$, $\mu_n(Y)$, $\mu_n(X \cap Y)$ and $\mu_n(X \cup Y)$ holds true. We start by the following

(5.1) Lemma. *If Z is a compactum (lying in E^ω) such that for every $\varepsilon > 0$ there exists a decomposition of Z into two disjoint compacta Z_1, Z_2 such that the diameter of every component of Z_1 is less than ε and that Z_2 can be mapped by an ε-translation onto a subset of a polyhedron P with $|P|_n < \alpha$, then $\mu_n(Z) < \alpha$.*

Proof. Consider two neighborhoods U_1 of Z_1 and U_2 of Z_2 with

$$\bar{U}_1 \cap \bar{U}_2 = \emptyset.$$

By our hypothesis, there exists for every component C of Z_1 an open neighborhood $V \subset U_1$ with diameter $d(V) < \varepsilon$ and $(\bar{V} \setminus V) \cap Z_1 = \emptyset$. Since Z_1 is compact, there is a finite system $V_1, V_2, \ldots, V_k$ of such open sets with $Z_1 \subset V_1 \cup \ldots \cup V_k$. Setting

$$W_i = V_i \setminus \bigcup_{j<i} \bar{V}_j \quad \text{for every } i = 1, 2, \ldots, k,$$

one gets open sets $W_1, W_2, \ldots, W_k$ with diameters $<\varepsilon$, covering Z_1. If $W_i \neq \emptyset$, then we select a point $b_i \in W_i$. Setting

$$\varphi(z) = b_i \quad \text{for every } z \in Z_1 \cap W_i,\ i = 1, 2, \ldots, k,$$

one gets an ε-translation φ of Z_1 onto a finite set consisting of all points b_i. Moreover, there exists an ε-translation ψ mapping Z_2 into a polyhedron P with $|P|_n < \alpha$. The map f defined by the formulas:

$$f(z) = \varphi(z) \text{ for } z \in Z_1 \cap W_i \quad \text{and} \quad f(z) = \psi(z) \text{ for } z \in Z_2$$

is an ε-translation of Z into the polyhedron $R = P \cup \{b_1, \ldots, b_k\}$. Then $|R|_n < \alpha$ and we infer that $\mu_n(Z) < \alpha$.

6. A preliminary construction. Consider a sequence $\lambda_0 > \lambda_1 > \ldots$ of positive numbers such that

$$\sum_{m=0}^{\infty} 2^m \cdot \lambda_m \leqslant 1.$$

We have

$$1 - \sum_{m=0}^{k} 2^m \cdot \lambda_m > \lambda_{k+1} \cdot 2^{k+1} \quad \text{for every } k = 0, 1, \ldots$$

Assign to every integer $k = 0, 1, \ldots$ a system σ_k consisting of 2^k disjoint closed intervals $A_{k,i}$, where $i = 1, \ldots, 2^k$, with lengths $|A_{k,i}| = \lambda_k$, lying in the open interval $I = (0, 1)$. We define the system σ_k by the induction:

σ_0 consist of only one closed interval $A_{0,1} = \left\langle \frac{1-\lambda_0}{2}, \frac{1+\lambda_0}{2} \right\rangle$. Assume that $\sigma_0, \ldots, \sigma_k$ are already defined and that $\sigma_0 \cup \ldots \cup \sigma_k$ consists of $1+2+\ldots+2^k = 2^{k+1}-1$ closed intervals $A_{m,i}$, $m = 0, 1, \ldots, k$, $i = 1, 2, \ldots, 2^k$ disjoint one to another, with lengths $|A_{m,i}| = \lambda_m$, and that the set $I \setminus \bigcup_{m=0}^{k} \bigcup_{i=1}^{2^m} A_{m,i}$ is the union of 2^{k+1} open intervals $I_{k,j}$, $j = 1, 2, \ldots, 2^{k+1}$ disjoint and equal one to another. Then the length of $I_{k,j}$ is equal to

$$\frac{1}{2^{k+1}}\Big(1 - \sum_{m=0}^{k} 2^m \cdot \lambda_m\Big) > \lambda_{k+1}.$$

Consequently there exists for every $j = 1, 2, \ldots, 2^{k+1}$ a closed interval $A_{k+1,j}$ with the length $|A_{k+1,j}| = \lambda_{k+1}$ and with the center the same as the center of $I_{k,j}$. The system σ_{k+1} consisting of all intervals $A_{k+1,j}$, $j = 1, 2, \ldots, 2^{k+1}$ satisfies our conditions.

The intervals belonging to σ_k are said to be of *order* k. Observe, that the measure (in the elementary sense of Lebesgue) of the closure $\bar{A}$ of the set

$$A = \bigcup_{k=0}^{\infty} \bigcup_{j=1}^{2^k} A_{k,j}$$

is equal to $\sum_{m=0}^{\infty} 2^m \cdot \lambda_m$.

Moreover, one infers by our construction, that the closure $\overline{I \setminus A}$ of the set $I \setminus A$ (with respect to $\langle 0, 1\rangle$) is a 0-dimensional compactum and its measure is equal to $1 - \sum_{m=0}^{\infty} 2^m \cdot \lambda_m$.

7. Construction of two compacta X, Y such that $\dim(X \cap Y) = 0$ and that $\mu_n(X) + \mu_n(Y) > \mu_n(X \cup Y)$. We preserve the notations of Section 6. In order to construct compacta X, Y satisfying the required conditions, consider in the space E^n the n-dimensional unit cube Q^n consisting of all points $(x_1, \ldots, x_n) \in E^n$ with $0 \leqslant x_i \leqslant 1$ for $i = 1, 2, \ldots, n$.

Let $X = Q^n$. The construction of Y is more complicated:

For every $k = 0, 1, \ldots$ and for every $i = 1, 2, \ldots, 2^k$, let us denote by $a_{k,i}$ and $b_{k,i}$ the end points of $A_{k,i}$ (with $a_{k,i} < b_{k,i}$) and let $c_{k,i}$ denote the center of $A_{k,i}$.

Consider also the set B_k, $k = 0, 1, \ldots$, consisting of all points $(x_1, \ldots, x_n) \in Q^n$ such that x_i belongs to the closure of the set

$$I \setminus \bigcup_{m=0}^{k} \bigcup_{i=1}^{2^m} A_{m,i} \quad \text{for every } i = 1, \ldots, n.$$

First we define (by the induction) a sequence $Y_0, Y_1, \ldots$ of polyhedra in $E^{n+1} \supset E^n$. Setting $Y_0 = X = Q^n$, assume that $Y_0, \ldots, Y_k$ are already constructed. The space Y_l is homeomorphic to X and $X \cap Y_l = B_{l-1}$ for $l = 1, \ldots, k$. For every $l = 1, \ldots, k-1$ every point $x = (x_1, \ldots, x_n, x_{n+1}) \in E^{n+1}$, where $x_{n+1} > \lambda_{l-1}$, belongs to Y_l if and only if $x \in Y_{l-1}$.

Let us denote by $H_{i,j}$, for all natural indices $j \leqslant n$ and for $i \leqslant 2^k$, the set consisting of all points $(x_1, \ldots, x_n, x_{n+1}) \in E^{n+1}$ such that $0 \leqslant x_l \leqslant 1$ for every $l \leqslant n$ and $x_j = c_{k,i}$ and $x_{n+1} = \lambda_k$.

Denote by $W_{i,j}$ (respectively by $Z_{i,j}$) the union of all closed intervals with endpoints $(x_1, \ldots, x_n, x_{n+1}) \in X \subset E^{n+1}$ and $(y_1, \ldots, y_{n+1}) \in H_{i,j}$, where $x_l = y_l$ for every $l \neq j, n+1$ and $x_j = a_{k,i}$, or $x_j = b_{k,i}$ (respectively the set consisting of all points $(x_1, \ldots, x_n, x_{n+1}) \in X$ such that $a_{k,i} \leqslant x_j \leqslant b_{k,i}$). Setting

$$\varphi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n),$$

one obtains the orthogonal projection $\varphi\colon E^{n+1} \to E^n$. Every polyhedron W lying in E^{n+1} is mapped by φ onto a polyhedron $\varphi(W) \subset E^n$ such that $|\varphi(W)|_m \leqslant |W|_m$, $m = 0, 1, \ldots$

Let

$$U_{k+1} = \Big(Y_k \setminus \bigcup_{j=i}^{n} \bigcup_{i=1}^{2^k} Z_{i,j}\Big) \cup \Big(\bigcup_{j=1}^{n} \bigcup_{i=1}^{2^k} W_{i,j}\Big).$$

Let us observe that U_{k+1} is a polyhedron which is mapped by φ onto X and that there exists a subpolyhedron Y_{k+1} of U_{k+1} such that φ restricted to Y_{k+1} is an embedding onto X and $X \cap Y_{k+1} = B_k$. A point x

$= (x_1, \ldots, x_n, x_{n+1}) \in E^{n+1}$ with $x_{n+1} > \lambda_k$, belongs to Y_{k+1} if and only if $x \in Y_k$.

Consider the set B consisting of all points $(x_1, \ldots, x_n) \in Q^n$ such that x_i belongs to the closure of the set $I \backslash A$ for every $i = 1, 2, \ldots, n$. Observe that $B = \bigcap_{k=0}^{\infty} B_k$.

Since B is homeomorphic to the Cartesian product $(\overline{I \backslash A}) \times \ldots \times (\overline{I \backslash A})$ and $\dim \overline{I \backslash A} = 0$, we infer that $\dim B = 0$.

It is clear that the sequence $\{\lambda_k\}_{k=0}^{\infty}$ may be chosen so that $|Y_k|_n < 1+\alpha < 2$ and that $|B|_n > \beta$, where $\beta > \alpha > 0$ and $k = 0, 1, \ldots$

Let M be the set consisting of all points $(x_1, x_2, \ldots, x_{n+1}) \in E^{n+1}$ with $|x_l| \leqslant 2$ for $l = 1, 2, \ldots, n+1$.

The polyhedra $Y_0, Y_1, \ldots$ can be considered as points of the space 2^M of all non-empty subcompacta of M (with the Hausdorff metric). In the space 2^M the sequence of polyhedra Y_m converges to a compactum Y. Assign to every $x = (x_1, \ldots, x_n, x_{n+1}) \in Y$ a point $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_{n+1})$ of Y_k such that $x_l = \hat{x}_l$ for every $l = 1, 2, \ldots, n$.

Now let us define the maps $p_k\colon Y \to Y_k$ and $q_k\colon X \cup Y \to X \cup Y_k$ by the formulas:

$$p_k(x) = \hat{x} \quad \text{for} \quad x \in Y$$

and

$$q_k(x) = \begin{cases} x & \text{for} \quad x \in X, \\ \hat{x} & \text{for} \quad x \in Y. \end{cases}$$

It is clear that p_k is a homeomorphism for every k, and that for every $\varepsilon > 0$ there exists an index k such that p_k and q_k are ε-translations. Therefore

$$\mu_n(Y) \leqslant 1+\alpha$$

and

$$\mu_n(X \cup Y) \leqslant |X \cup Y_k|_n = |X|_n + |Y_k|_n - |X \cap Y_k|_n \leqslant 2+\alpha-\beta < 2.$$

In order to finish the proof that $\mu_n(X \cup Y) \neq \mu_n(X)_n + \mu_n(Y)$ it suffices to show that $\mu_n(Y) \geqslant 1$.

Let us assume that there is a positive γ such that

$$\mu_n(Y) < \gamma < 1 \tag{7.1}$$

and let $\varepsilon > 0$.

Consider a positive number δ such that if x and y are points of M and $\varrho(x, y) < \delta$, then $\varrho(\varphi(x), \varphi(y)) < \varepsilon$. It follows by (7.1) that there exists an δ-translation $f_\delta\colon Y \to M$ such that $f_\delta(Y)$ is a polyhedron with $|f_\delta(Y)|_n \leqslant \gamma < 1$. Setting

$$g(x) = \varphi f_\delta \varphi^{-1}(x) \quad \text{for} \quad x \in X,$$

one gets an ε-translation $g\colon X \to E^{n+1}$ such that

$$|g(X)|_n \leqslant |f_\delta(Y)|_n < \gamma < 1.$$

Thus for every $\varepsilon > 0$ there exists an ε-translation $g\colon X \to E^{n+1}$ such that $g(X)$ is a polyhedron and $|g(X)|_n < \gamma$. Hence $\mu_n(X) \leqslant \gamma < 1$. This contradicts (1.6). Therefore $\mu_n(Y) > 1$.

Since $X \cap Y = B$ and $\dim B = 0$, we conclude that X and Y satisfy the required conditions.

8. Construction of compacta X, Y **such that** $\dim(X \cap Y) = 0$ and $\mu_1(X \cup Y) > \mu_1(X) + \mu_1(Y)$.

Preserving the notations of Section 6, let us set $\lambda_n = 6^{-(n+1)}$ and let us call the interval $A_{k,i}$ *even*, if k is an even number, and *odd* – if k is odd. By X we denote the closure of the union of all even intervals, and by Y – the closure of the union of all odd intervals. Observe that every component of the compactum $\overline{I \backslash Y}$ is either an even interval, or a singleton. For every positive ε there exists only a finite collection of intervals $A_1, \ldots, A_n$ with diameters $\geqslant \varepsilon$. Moreover, for every even interval A there is an ε-neighborhood which is an open interval with both endpoints belonging to odd intervals. It follows, that there exist open intervals $U_1, \ldots, U_n$ disjoint one to another with endpoints belonging to odd intervals such that U_i is an ε-neighborhood of A_i for $i = 1, 2, \ldots$ Consequently, there exists an ε-translation f of the set $U_1 \cup \ldots \cup U_n$ onto the polyhedron $A_1 \cup \ldots \cup A_n$. Since $\mu_1(A_1 \cup \ldots \cup A_n)$ is less than $\sum_{k=0}^{\infty} \sum_{i=1}^{2^k} |A_{k,i}|$, we infer by Lemma (5.1) that $\mu_1(X) < \frac{1}{5}$. By an analogous argument, one shows that $\mu_1(Y) < \frac{1}{5}$.

Since $X \cup Y = \bar{I}$, we infer that $\mu_1(X) + \mu_1(Y) < \mu_1(X \cup Y)$. Moreover, all components of the set $X \cap Y$ are singletons and consequently $\dim(X \cap Y) = 0$.

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