Remarks on the \( n \)-dimensional geometric measure of compacta

by

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Abstract. By the \( n \)-dimensional geometric measure of a compactum \( X \) lying in the Hilbert space \( E^n \), one understands the lower bound \( \mu(X) \) of all \( n \)-positive numbers \( \alpha \) such that for every \( \varepsilon > 0 \) there is an \( \varepsilon \)-translation \( f_\varepsilon : X \to E^n \) such that \( f_\varepsilon(X) \) lies in a polyhedron \( P \subset E^n \) for which the \( n \)-dimensional measure \( |P|_n \) (in the elementary sense) is \( \alpha \). If \( \dim P < n \), we assume \( |P|_n = 0 \), and if \( \dim P > n \), we assume \( |P|_n = \infty \).

Some relations between geometric measures of two compacta \( X, Y \subset E^n \) and the pseudo-measures of \( X \cup Y, X \cap Y \) and \( X \times Y \) are studied.

1. Introduction. In the elementary geometry one assigns to each \( n \)-dimensional polyhedron \( P \) the number \( |P|_n \) defined as the sum of the volumes of all \( n \)-dimensional simplices belonging to a triangulation of \( P \). If \( \dim P < n \), then one assumes that \( |P|_n = 0 \), and if \( \dim P > n \), then \( |P|_n = \infty \).

One knows that \( |P|_n \) does not depend on the choice of the triangulation of \( P \). Moreover, one sees easily that

\[
\{ P_1, P_2, \ldots, P_k \} \text{ are polyhedra, then } |P_1 \cup \ldots \cup P_k|_n \leq \sum_{i=1}^{k} |P_i|_n.
\]

Let \( E^n \) denote the usual Hilbert space, i.e. the space consisting of all real sequences \((x_1, x_2, \ldots)\), such that \( \sum_{i=1}^{\infty} x_i^2 < \infty \), metrized by the formula

\[
g((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.
\]

We may consider the Euclidean \( m \)-space \( E^m \) as the subset of \( E^n \) consisting of all points \((x_1, x_2, \ldots, x_m, 0, 0, \ldots)\) denoted also by \((x_1, x_2, \ldots, x_m)\).

By the \( n \)-dimensional geometric measure of a compactum \( X \subset E^n \), we understand the number \( \mu(X) \) (finite or infinite) defined as the lower bound of all numbers \( \alpha > 0 \) such that for every \( \varepsilon > 0 \), there is an \( \varepsilon \)-translation \( f_\varepsilon : X \to E^n \) (i.e. a map \( f_\varepsilon \) satisfying the condition \( g(x, f_\varepsilon(x)) < \varepsilon \) for every \( x \in X \)) such that \( f_\varepsilon(X) \) is a subset of a polyhedron \( P \subset E^n \) with \( |P|_n < \alpha \).

It is known (see [2]) that:
If \( \dim X < n \), then \( \mu_n(X) = 0 \).

(1.3) If \( \dim X > n \), then \( \mu_n(X) = \infty \).

(1.4) If \( X \subseteq Y \), then \( \mu_n(X) \leq \mu_n(Y) \).

(1.5) If \( X \cap Y = \emptyset \), then \( \mu_n(X \cup Y) = \mu_n(X) + \mu_n(Y) \).

(1.6) If \( P \) is a polyhedron, then \( \mu_n(P) = |P| \).

(1.7) If \( X \) is a continuum, then \( \mu_n(X) \geq d(X) \), where \( d(X) \) denotes the diameter of \( X \).

(1.8) If \( L \) is an arc, then \( \mu_n(L) \) is the length \( |L| \) of \( L \).

Observe that (1.2) implies that for every arc \( L \) and \( n = 2, 3, \ldots \), \( \mu_n(L) = 0 \). However there exist arcs \( L \subseteq \mathbb{R}^n \) for which the \( n \)-dimensional measure (in the sense of Hausdorff [3]) is positive for \( n = 2, 3, \ldots \). In fact, it is well known that there exists in \( \mathbb{R}^n \) a set \( C \) homeomorphic to the Cantor discontinuum, for which the \( n \)-dimensional Hausdorff measure is positive, for every \( n = 1, 2, \ldots \). On the other hand, there is an arc \( L \subseteq \mathbb{R}^n \) containing \( C \). It follows that the \( n \)-dimensional Hausdorff measure of \( L \) is positive for every \( n = 1, 2, \ldots \). However \( \mu_n(L) = 0 \) for \( n = 2, 3, \ldots \), because of (1.2).

### 2. Geometric measure of \( n \)-dimensional compacts

First let us prove the following

(2.1) **Lemma.** For any natural number \( m \) and for every positive number \( s \), there is a positive real number \( \delta = \delta(s, m) \) such that for every \( n \)-dimensional polyhedron \( R \) lying in \( \mathbb{R}^m \), with \( |R|_s < \delta \), there is an \( s \)-translation \( g: \mathbb{R}^m \to \mathbb{R}^m \) such that \( g(R) \) is contained in an \((n-1)\)-dimensional polyhedron.

**Proof.** Let \( \eta \) be a positive real number such that the diameter of the \( m \)-cube \( (0, \eta)^m \) is less than \( s \). Let \( \delta = \delta(s, m) \) be a positive real number such that

\[
\delta \left( \frac{m!}{k!(m-k)!} \right) < \eta^k \quad \text{for} \quad k = 1, 2, \ldots, m,
\]

where

\[
\binom{m}{k} = \frac{m!}{k!(m-k)!}.
\]

Let \( R \) be an \( n \)-dimensional polyhedron in \( \mathbb{R}^m \) with \( |R|_s < \delta \). Assume that \( R \) is contained in an \( n \)-cube \( (0, N \cdot \eta)^m \), where \( N \) is a natural number. Hence \( (0, N \cdot \eta)^m \) consists of all points \((x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \) with \( 0 \leq x_i \leq N \cdot \eta \) for \( i = 1, 2, \ldots, m \).

Let \( \mathcal{F} \) denote the family of all subsets of \( \mathcal{A} \) of the set \( \{1, 2, \ldots, m\} \) consisting of \( n \) elements. Observe that \( \mathcal{F} \) consists of \( \binom{m}{n} \) sets. Assign to every \( A \in \mathcal{F} \) the \( n \)-dimensional face \( F_A \) of \( (0, N \cdot \eta)^m \) consisting of all points \((x_1, x_2, \ldots, x_m) \) such that

\[
0 \leq x_i \leq N \cdot \eta, \quad \text{if} \quad i \in A, \quad \text{and} \quad x_i = 0, \quad \text{if} \quad i \notin A.
\]

Denote by \( p_A \) the map assigning to every point \((x_1, x_2, \ldots, x_m) \) of \( (0, N \cdot \eta)^m \) the point \((x'_1, x'_2, \ldots, x'_m) \), where \( x'_i = x_i \) if \( i \in A \) and \( x'_i = 0 \), if \( i \notin A \). Thus \( p_A \) is an orthogonal projection of the cube \( (0, N \cdot \eta)^m \) onto its \( n \)-dimensional face \( F_A \).

One sees easily that for every \( A \in \mathcal{F} \), the image \( p_A(R) \) of the polyhedron \( R \subseteq (0, N \cdot \eta)^m \) is a polyhedron lying in \( F_A \) with \( |p_A(R)|_s \leq |R|_s \). Since

\[
\binom{m}{n} \cdot \frac{|R|_s}{|R|_m} = \left( \frac{m}{n} \right) < \eta^n,
\]

we infer that

\[
\sum_{A \in \mathcal{F}} |p_A(R)|_s < \eta^n.
\]

Divide every interval \( \langle (i-1) \cdot \eta, i \cdot \eta \rangle \), where \( i \) belongs to \( \{1, 2, \ldots, N\} \), into \( s \) equal intervals. One gets a subdivision of the cube \( (0, N \cdot \eta)^m \) into \( (N-s)^m \) small \( m \)-cubes with the length of edges \( \frac{1}{N \cdot \eta} \). Denote by \( k_n \) the number of all such small cubes, which lay in \( F_A \) and which intersect the polyhedron \( p_A(R) \). It follows by (2.2) that for \( s \) sufficiently large

\[
\sum_{A \in \mathcal{F}} k_n < s^n.
\]

Observe that there are \( s^n \) subsets of \( (0, N \cdot \eta) \) of the form

\[
A_1 \cup A_2 \cup \ldots \cup A_N,
\]

where \( A_i, i = 1, 2, \ldots, N \), are intervals with the length \( \frac{1}{N \cdot \eta} \) in which the interval \( \langle (i-1) \cdot \eta, i \cdot \eta \rangle \) is divided.

Moreover, there are \( s^{N-1} \) subsets of \( (0, N \cdot \eta) \) of the form (2.4) which contain a given one of the intervals with the length \( \frac{1}{N \cdot \eta} \) in which the interval \( \langle (i-1) \cdot \eta, i \cdot \eta \rangle \) is divided.

Let \( \mathcal{A} \subseteq \mathcal{F} \), and let \( C_j \), for \( j = 1, 2, \ldots, m \), be one of the \( s \cdot N \) intervals with the length \( \frac{1}{s \cdot N \cdot \eta} \) in which the interval \( \langle (i-1) \cdot \eta, i \cdot \eta \rangle \) is divided. If \( B_j \), for \( j = 1, 2, \ldots, m \), is a subset of \( (0, N \cdot \eta) \) of the form (2.4), then the set \( p_B(B_1 \times B_2 \times \ldots \times B_m) \) contains the set \( p_B(C_1 \times C_2 \times \ldots \times C_m) \) if and only if the set \( B_j \) contains the set \( C_j \) for every \( j \in \mathcal{A} \). Consequently, there are

\[
(s^{N-1} \cdot (s^{N-1})^{s^{N-1}}) \cdots = s^{-N} \cdot s^{N^m}
\]

sets of the form

\[
B_1 \times B_2 \times \ldots \times B_m,
\]

where \( B_i \), for \( i = 1, 2, \ldots, m \), a subset of \( (0, N \cdot \eta) \) of the form (2.4) with the images under \( p_B \) containing the small \( n \)-cube \( p_B(C_1 \times C_2 \times \ldots \times C_m) \) lying in \( F_A \). It follows that there are at most \( k_n \cdot s^{-N} \cdot s^{N^m} \) sets of the form (2.5) with the images under \( p_B \) intersecting the polyhedron \( p_B(R) \).
There are $s^{n-m}$ sets of the form (2.5). It follows by (2.3) that
\[
\sum_{n \in \mathbb{N}} k_n s^{-n} s^{n-m} < s^{n-m}.
\]
Consequently there is a set $B$ of the form $B_1 \times B_2 \times \ldots \times B_m$ such that
\[
p_n(B) \cap p_m(R) = \emptyset \quad \text{for every } \forall n \in \mathbb{N}.
\]
Let $\hat{B}$ denote the interior of $B$ in $E^n$. Then the set
\[
Y = \langle 0, N \cdot \eta \rangle^m \setminus \bigcup_{n \in \mathbb{N}} p_n^{-1}(p_n(B))
\]
contains the polyhedron $R$.

Denote by $S_k$ for $k = 0, 1, \ldots, m$ the union of all $k$-dimensional faces of $m$-cubes of the following form
\[
\langle i_1 \cdot \eta, (i_1 + 1) \cdot \eta \rangle \times \langle i_2 \cdot \eta, (i_2 + 1) \cdot \eta \rangle \times \cdots \times \langle i_m \cdot \eta, (i_m + 1) \cdot \eta \rangle,
\]
where $i_j = 0, 1, \ldots, N-1$ for $j = 1, 2, \ldots, m$. Let
\[
Y_k = Y \cap S_k, \quad \text{for } k = 1, 2, \ldots, m
\]
(in particular, $S_0 = \langle 0, N \cdot \eta \rangle^m$ and $Y_0 = Y$).

It is easy to see that for each $k = n, n+1, \ldots, m$ there is a retraction
\[
r_k : Y_n \to Y_{n-1}
\]
such that for every $k$-face $K$ of any $m$-cube of the form (2.6)
\[
r_k(K \cap Y_n) = K' \cap Y_{n-1}
\]
Consequently the map
\[
r = r_m \cdot r_{m-1} \cdot \ldots \cdot r_1 : Y \to Y_{n-1}
\]
is a retraction such that $r(K \cap Y) \subset K$ for every $m$-cube $K$ of the form (2.6). Since the diameter of any $m$-cube of the form (2.6) is less than $\varepsilon$, we infer that $r: Y \to Y_{n-1}$ is an $\varepsilon$-translating. Setting
\[
\varrho(x) = r(x) \quad \text{for every } x \in R,
\]
one gets a map satisfying the required conditions. Thus the proof of Lemma (2.1) is finished.

Denote by $q_m : E^n \to E^m$ the orthogonal projection
\[
q_m(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_m) \quad \text{for every } (x_1, x_2, \ldots) \in E^n.
\]
Let us prove the following

(2.7) Theorem. For every $n$-dimensional compactum $X \subset E^n$, the $n$-dimensional geometric measure is positive.

Proof. If $X$ is a compactum in $E^n$ with $\mu_n(X) = 0$, then for every positive $\varepsilon$, there is an integer $m$ such that
\[
\varrho(x, q_m(x)) < \varepsilon \quad \text{for every } x \in X.
\]
Let $\delta = \delta(n, m)$ be the positive real number from Lemma (2.1). Since $\mu_n(X) = 0$, there is an $\varepsilon$-translation $f : E^n \to E^n$ such that $f(X)$ is a subset of a polyhedron $P \subset E^n$ with $\dim P = n$ and $|P|_n < \delta$. Then
\[
\varrho(x, q_m(f(x)) \leq \varrho(x, q_m(x)) + \varrho(q_m(x), q_m(f(x))
\]
for every $x \in X$. So $q_m : X \to E^n$ is an $\varepsilon$-translation.

Observe that $q_m(f(X)$ is a subset of the polyhedron $R = q_m(P)$ lying in $E^n$ with $|R|_n < |P|_n < \delta$. By Lemma (2.1), there is an $\varepsilon$-translation $g : R \to E^n$ such that $g(R)$ is contained in an $(n-1)$-dimensional polyhedron. The map $gq_m : X \to E^n$ is a $3\varepsilon$-translation. The image $gq_m(x)$ lies in an $(n-1)$-dimensional polyhedron.

Thus, for every $\varepsilon > 0$, there is a $3\varepsilon$-translation of $X$ into $E^n$ with the image contained in an $(n-1)$-dimensional polyhedron. Hence $\dim X < n$, and the proof of Theorem (2.7) is finished.

(2.8) Corollary. For compacta $X \subset E^n$ the vanishing of the $n$-dimensional geometric measure is a topological invariant.

(2.9) Question. Is it true that for every $\varepsilon > 0$ there is an $\eta_\varepsilon > 0$ such that for every compactum $X \subset E^n$ with $\mu_n(X) > \eta_\varepsilon$ there is an $\varepsilon$-translation $f : X \to E^n$ such that $\dim f(X) < n$?

3. Geometric measures for Cartesian products. If $P$ is an $k$-dimensional polyhedron and $R$ is an $m$-dimensional polyhedron, then $P \times R$ is an $(k+m)$-dimensional polyhedron and we infer by (1.6) that
\[
\mu_{k+m}(P \times R) = \mu_k(P) \cdot \mu_m(R).
\]
Another situation is for arbitrary compacta. As has been shown by L. S. Pontrjagin [4], there exist two $2$-dimensional compacta $X$ and $Y$ such that $\dim(X \times Y) = 3$. By Theorem (2.7), both numbers $\mu_2(X)$ and $\mu_2(Y)$ are positive, however (1.2) implies that $\mu_2(X \times Y) = 0$. Consequently
\[
\text{There exist $2$-dimensional compacta $X$ and $Y$ such that } \mu_2(X \times Y) < \mu_2(X) \cdot \mu_2(Y).
\]
However the following theorem holds true:

(3.3) Theorem. If $X$, $Y$ are compacta lying in $E^n$ and if $\dim X = k$, $\dim Y = m$, then $\mu_{k+m}(X \times Y) \leq \mu_k(X) \cdot \mu_m(Y)$.

Proof. Both numbers $\mu_k(X)$ and $\mu_m(Y)$ are positive. If at least one of them is finite, then $\mu_{k+m}(X \times Y) \leq \mu_k(X) \cdot \mu_m(Y)$. Thus we can assume that both numbers $\mu_k(X)$ and $\mu_m(Y)$ are finite.

By the definition of $\mu_n$, there exist for every $\eta > 0$ two polyhedra $P$ and $R$ such that $\dim P = k$, $\dim R = m$, with
\[
\mu_k(X) + \eta > |P|_k, \quad \mu_m(Y) + \eta > |R|_m
\]
and that for every \( \varepsilon > 0 \) there exist two \( \varepsilon \)-translations

\[
f: X \to P \quad \text{and} \quad g: Y \to R.
\]

Setting

\[
\varphi(x, y) = (f(x), g(y)) \quad \text{for every } x \in X \text{ and } y \in Y,
\]

one gets a 2\( \varepsilon \)-translation \( \varphi: X \times Y \to P \times R \). Using (3.1) and (3.2), one infers that

\[
\nu_{k+m}(X \times Y) \leq \nu_{k+m}(P \times R) = \nu_k(P) \cdot \nu_m(R) < (\mu_k(X) + \eta) \cdot (\mu_m(Y) + \eta).
\]

Hence

\[
\nu_{k+m}(X \times Y) < (\mu_k(X) + \eta) \cdot (\mu_m(Y) + \eta) \quad \text{for every } \eta > 0.
\]

Consequently \( \nu_{k+m}(X \times Y) \leq \mu_k(X) \cdot \mu_m(Y) \) and the proof of Theorem (3.3) is finished.

4. Geometric measures of unions of compacta. First let us prove the following

(4.1) Lemma. Let \( X \) and \( A \) be two compacta lying in \( E^n \). For every \( \varepsilon \)-translation \( f \) of \( X \) with values lying in a polyhedron \( P \subset E^n \), there exists a \( 2\varepsilon \)-translation \( g: X \cup A \to E^n \) and a polyhedron \( R \subset E^n \) with \( \dim R \leq \dim A \) such that \( g(A \cup A) \subset P \cup R \).

**Proof.** We may assume that \( P \subset E^n \). One sees easily that there exists an extension \( f^* \) of \( f \) to an \( \varepsilon \)-translation \( f^*: X \cup A \to E^n \). Moreover, there is a natural number \( k \) so large, that the orthogonal projection \( q: E^n \to E^n \) assigns to every point \( x = (x_1, x_2, \ldots) \in E^n \) the point \( \varphi(x) = (x_1, x_2, \ldots, x_{k+1}) \in E^{n+1} \), satisfying the condition

\[
g(\varphi(x), f^*(x)) < \varepsilon \quad \text{for every point } x \in X \cup A.
\]

Then the set \( \varphi(f^*(x)) \) lies in a polyhedron \( R' \subset E^{n+1} \) and

\[
\varphi(f^*(x)) = f^*(x) = f(x) \quad \text{for every } x \in X.
\]

Consider a triangulation \( T \) of the polyhedron \( P \cup R' \) such that all simplices \( \Delta \in T \) with \( \Delta \neq \emptyset \), together with their faces, constitute a triangulation \( T_{P} \) of \( P \) and all simplices \( \Delta \in T \) with \( \Delta \cap R' \neq \emptyset \) (and their faces) constitute a triangulation \( T_{R} \) of \( R' \).

Let us say that two maps \( g, g': X \cup A \to E^n \) are \( T \)-associated, if for every \( x \in X \cup A \) there is a simplex \( \Delta \in T \) containing both points \( g(x) \) and \( g'(x) \).

Let \( m = \dim A \) and \( g_0 = \varphi f^* \). Consider a simplex \( \Delta \in T_{P} \setminus T_{R} \) with \( \dim \Delta = \dim R \). If \( \dim \Delta > m \), then there exists a map \( g' \) of the set \( g_0^{-1}(\Delta) \) into the boundary \( \partial \Delta \) of \( \Delta \) such that \( g_0(x) = g'(x) \) for every point \( x \in g_0^{-1}(P \cup R) \cap \partial \Delta \) (where \( \Delta \) denotes the interior \( \Delta \setminus \partial \Delta \) of \( \Delta \)). Applying this procedure step by step to all simplices \( \Delta \in T_{P} \setminus T_{R} \) with \( \dim \Delta > m \), one gets a map \( g: X \cup A \to E^n \) which is \( T \)-associated to the map \( g_0 \) and the set \( g(A) \) lies in the \( m \)-skeleton \( R' \) of \( R \) (by triangulation \( T_{R} \)).

If the mesh of the triangulation \( T \) is less than \( \varepsilon \), then \( g(\mu_k(X), g(x)) < \varepsilon \) for every \( x \in X \cup A \) and \( g \) is a \( 2\varepsilon \)-translation of \( X \cup A \) into \( P \cup R \).

(4.2) **Theorem.** If \( X, A \) are compacta lying in \( E^n \) and if \( \dim A < n \), then \( \mu_k(X \cup A) = \mu_k(X) \).

**Proof.** By (1.4), \( \mu_k(X) \leq \mu_k(X \cup A) \). It remains to show that

\[
\mu_k(X \cup A) \leq \mu_k(X)
\]

We can assume that \( \mu_k(X) < \infty \). Consider a finite number \( x > \mu_k(X) \). Then for every \( \varepsilon > 0 \) there is an \( \varepsilon \)-translation \( f: X \to E^n \) such that \( f(x) \) is a subset of a polyhedron \( P \subset E^n \) with \( \mu_k(P) < x \). By Lemma (4.1), there is a \( 2\varepsilon \)-translation \( g: X \cup A \to E^n \) and a polyhedron \( R \subset E^n \) with \( \dim R < n \) such that \( g(A \cup A) \subset P \cup R \). But \( \mu_k(P \cup R) = \mu_k(P) \), and consequently \( \mu_k(X \cup A) < x \) and the proof of Theorem (4.2) is finished.

(4.3) **Theorem.** If \( X_1, X_2 \) are compacta lying in \( E^n \) and if \( \mu_{k-1}(X_1 \cup X_2) = 0 \), then \( \mu_{k-1}(X_1 \cup X_2) = \mu_{k-1}(X_1) + \mu_{k-1}(X_2) \).

Before giving the proof, we established some lemmas:

(4.4) **Lemma.** Let \( X_1, X_2 \) and \( X_0 = X_1 \cup X_2 \) be compacta lying in \( E^n \). Then for every \( \varepsilon > 0 \) there exists a natural number \( m \) and an \( \varepsilon \)-translation \( f: X_1 \cup X_2 \to E^n \) such that \( f(X_0) = f(X_1) \cap f(X_2) \) and \( f(X_i), f(X_j) \) are polyhedra with \( \dim f(X_i) < \dim X_i \) for \( i = 0, 1, 2 \).

**Proof.** Setting \( X = X_1 \cup X_2 \), observe that there are a natural number \( n \) and a \( 4\varepsilon \)-translation \( g: X \to E^n \) such that \( g(X) \) and \( g'(X_0) \) are polyhedra and \( \dim g(X_0) < \dim X_0 \).

It is known (see [1], p. 166) that there exists a \( 4\varepsilon \)-translation \( g: X \to E^n \) such that \( \dim g(X_0) \leq \dim X_0 \) for \( i = 1, 2 \) and \( g(x) = g'(x) \) for \( x \in X \).

It follows that \( \dim g(X) \leq \dim X_0 \) for \( i = 0, 1, 2 \).

We can also assume that \( g(X_0), g(X_1) \) and \( g(X_2) \) are polyhedra.

By the general position argument, there are \( 4\varepsilon \)-translations \( h_1: g(X_1) \to E^{n+1} \) and \( h_2: g(X_2) \to E^{n+1} \) such that \( h_1(x) = h_2(x) = x \) for \( x \in X_0 \).

and

\[
h_1(X_1 \setminus X_0) \cup h_2(X_2 \setminus X_0) = \emptyset.
\]

Let \( m = n+1 \). The \( \varepsilon \)-translation \( f: X \to E^n \) satisfying the required conditions can be defined by the formula

\[
f(x) = h_0(x)
\]

for every \( x \in X \) and \( i = 1, 2 \).

This completes the proof.
Assume that $x$ is a real number and define the map

$$
t_x: E^n \to E^{n+1} \subset E^n
$$

by the formula

$$
t_x(x_1, ..., x_n, z) = (x_1, ..., x_n, z)
$$

for every $(x_1, ..., x_n) \in E^n$.

(4.5) Lemma. If $X_1$ and $X_2$ are polyhedra lying in $E^n$ and if $X_0 = X_1 \cap X_2$, then for every $\varepsilon > 0$ there is an $\varepsilon$-translation

$$
f: X_1 \cup X_2 \to \tau_{-\varepsilon}(X_1) \cup \tau_\varepsilon(X_2) \cup \bigcup_{|l| \leq \varepsilon} \tau_l(X_0).
$$

Proof. Let $T$ be a triangulation of the polyhedron $X = X_1 \cup X_2$ such that the mesh of $T$ is less than $\frac{\varepsilon}{4}$ and that the subfamily $T_i$ of $T$, which consists of all simplices $A \in T$ with $A \cap X_i = \emptyset$ and of their faces is a triangulation of $X_i$, for $i = 0, 1, 2$.

Consider the polyhedron

$$
Y = \tau_{-\varepsilon}(X_1) \cup \tau_\varepsilon(X_2) \cup \bigcup_{|l| \leq \varepsilon} \tau_l(X_0 \cup X^{(n)}),
$$

where $X^{(n)}$ denotes the $n$-skeleton of $X$ with respect to $T$.

For $i = 1, 2$ and for every simplex $A \in T_i$, the set

$$
t_i(A) \cup \bigcup_{|l| \leq \varepsilon} \tau_l(A), \quad \gamma = \frac{1 + \varepsilon}{4}
$$

is a retract of

$$
\bigcup_{|l| \leq \varepsilon} \tau_l(A).
$$

Therefore for each $n \geq 0$ there is a retraction $r_0$ of the set $Y_0$ to $Y_{n-1}$ such that

$$
r_0(\bigcup_{|l| \leq \varepsilon} \tau_l(A)) = \bigcup_{|l| \leq \varepsilon} \tau_l(r_0(A)),
$$

where $A \in (T_1, T_2)$ for $i = 1, 2$ is an $n$-dimensional simplex.

Hence the map $f: X \to Y_{n-1}$ which is the restriction of

$$
r_0 \cdots r_s: \bigcup_{|l| \leq \varepsilon} \tau_l(X) \to Y_{n-1}, \quad s = \dim X,
$$

is an $\varepsilon$-translation. Thus the proof of Lemma (4.5) is finished.

Proof of Theorem (4.3). Suppose that

$$
\mu_X(x) \leq \alpha_i < \infty \quad \text{for} \quad i = 1, 2.
$$

Consider $\frac{\varepsilon}{2}$-translations

$$
f_1: X_1 \to E^n \quad \text{and} \quad f_2: X_2 \to E^n
$$

such that $f_i(X_i)$ is a polyhedron with $|f_i(X_i)| \leq \alpha_i$ for $i = 1, 2$.

For $i = 1, 2$ there exists a neighborhood $U_i$ of $X_i$ and a $\frac{\varepsilon}{2}$-translation $\tilde{f}_i: U_i \to E^n$ such that $\tilde{f}_i(U_i) = f_i(X_i)$ and $\tilde{f}_i$ is an extension of $f_i$.

Lemmas (4.4) and (4.5) imply that for every $\delta > 0$ there exist $\delta$-translations

$$
g: X_1 \cup X_2 \to Y \quad \text{and} \quad g_i: X_i \to E^n \quad \text{for} \quad i = 1, 2
$$

such that $g_1(X_1)$, $g_2(X_2)$ and $g(X) = g_1(X_1) \cup g_2(X_2)$ are polyhedra with

$$\dim(g(X)) \leq (g_1(X_1) \cup g_2(X_2)) \leq n - 1 \quad \text{and} \quad g_1(X_1) \cap g_2(X_2) = \emptyset.
$$

Assume that $\delta > 0$ is so small, that $g(X) \subset U_1 \cup U_2$ and $\delta < \frac{\varepsilon}{6}$. Setting

$$
f(x) = f_i(x) \quad \text{for every} \quad x \in g_i(X_i), \quad i = 1, 2,
$$

we obtain a $\frac{\varepsilon}{6}$-translation $\tilde{f}: g_1(X_1) \cup g_2(X_2) \to E^n$.

Using Lemma (4.1) for the case when $X = g_1(X_1) \cup g_2(X_2)$, $A$ is the closure of the set $g(X)$ and $g_1(X_1) \cup g_2(X_2)$ and $f = \tilde{f}$, we infer that there is a $\frac{\varepsilon}{6}$-translation $g: X \to E^n$ such that the set $Y_0 = f'(g(X) \cap g_1(X_1) \cup g_2(X_2))$ is a polyhedron with

$$\dim Y_0 \leq n - 1.
$$

Since $\frac{\varepsilon}{6} > \delta$, the composition $h = g: X_1 \cup X_2 \to E^n$ is an $\varepsilon$-translation.

It is clear that $h(X_1 \cup X_2) = f_1(X_1) \cup f_2(X_2) \cup Y_0$ and

$$
l(h(X_1 \cup X_2)) \leq l(f_1(X_1)) + l(f_2(X_2)) + |Y_0| \leq a_2 + a_2.
$$

Thus the proof of Theorem (4.3) is finished.

5. A lemma. In Sections 7 and 8 we shall construct some compacta showing that without the hypothesis $\dim(X \cap Y) < n - 1$, no simple relation between $\mu_X(X)$, $\mu_Y(Y)$, $\mu(X \cap Y)$ and $\mu(X \cap Y)$ holds true. We start by a following

(5.1) Lemma. If $Z$ is a compactum (lying in $E^n$) such that for every $\varepsilon > 0$ there exists a decomposition of $Z$ into two disjoint compacta $Z_1, Z_2$ such that the diameter of every component of $Z_2$ is less than $\varepsilon$ and that $Z_2$ can be mapped by an $\varepsilon$-translation onto a subset of a polyhedron $P$ with $|P| < a$, then

$$
\mu(Z) < a.
$$

Proof. Consider two neighborhoods $U_1$ of $Z_1$ and $U_2$ of $Z_2$ with

$$
U_1 \cap U_2 = \emptyset.
$$

By our hypothesis, there exists for every component $C$ of $Z_1$ an open neighborhood $V \subset U_1$ with diameter $d(V) < a$ and $(V \setminus V \cap C) = \emptyset$. Since $Z_2$ is compact, there is a finite system $V_1, V_2, ..., V_k$ of such open sets with $Z_2 = V_1 \cup ... \cup V_k$. Setting

$$
W_j = V_j \setminus \bigcup_{i \neq j} V_i \quad \text{for every} \quad i = 1, 2, ..., k,
$$

$$
W_j
$$
one gets open sets \( W_1, W_2, ..., W_k \) with diameters \( < \varepsilon \), covering \( Z_1 \). If \( W_i \neq \emptyset \), then we select a point \( b_i \in W_i \). Setting
\[
\phi(z) = b_i \quad \text{for every } z \in Z_1 \cap W_i, \quad i = 1, 2, ..., k,
\]
one gets an \( \varepsilon \)-translation \( \phi \) of \( Z_1 \) onto a finite set consisting of all points \( b_i \).
Moreover, there exists an \( \varepsilon \)-translation \( \psi \) mapping \( Z_2 \) into a polyhedron \( P \) with \( |P| < \alpha \). The map \( f \) defined by the formulas:
\[
f(z) = \phi(z) \quad \text{for } z \in Z_1 \cap W_i \quad \text{and} \quad f(z) = \psi(z) \quad \text{for } z \in Z_2
\]
is an \( \varepsilon \)-translation of \( Z \) into the polyhedron \( R = P \cup \{b_1, ..., b_k\} \). Then \( |R| < \alpha \) and we infer that \( \mu(X) < \alpha \).

6. A preliminary construction. Consider a sequence \( \lambda_0 > \lambda_1 > ... \) of positive numbers such that
\[
\sum_{n=0}^{\infty} 2^n \lambda_n \leq 1.
\]
We have
\[
1 - \sum_{n=0}^{k} 2^n \lambda_n > \lambda_{k+1} \cdot 2^{k+1}
\]
for every \( k = 0, 1, ... \).

Assign to each integer \( k = 0, 1, ... \) a system \( \sigma_k \) consisting of \( 2^k \) disjoint closed intervals \( A_k, j \) where \( j = 1, ..., 2^k \), with lengths \( |A_k, j| = \lambda_j \), lying in the open interval \( I = (0, 1) \). We define the system \( \sigma_0 \) by the induction:
\[
\sigma_0 \text{ consists of only one closed interval } A_{0,1} = \left[1 - \lambda_0, 1 + \lambda_0\right],
\]
and assume that \( \sigma_0, \sigma_1, \sigma_2, \) are already defined and that \( \sigma_0 \cup \ldots \cup \sigma_k \) consists of \( 1 + 2 + \ldots + 2^k = 2^{k+1} - 1 \) closed intervals \( A_{m, j} \), \( m = 0, 1, ..., k; \) \( j = 1, 2, ..., 2^m \) disjoint one to another, with lengths \( |A_{m, j}| = \lambda_j \), and that the set
\[
I \setminus \bigcup_{m=0}^{k} \bigcup_{j=1}^{2^m} A_{m, j}
\]
is the union of \( 2^m \) open intervals \( I_{k, j}, j = 1, 2, ..., 2^{k+1} \) disjoint and equal one to another. Then the length of \( I_{k, j} \) is equal to
\[
\frac{1}{2^{k+1}} \left(1 - \sum_{n=0}^{k} 2^n \lambda_n\right) > \lambda_{k+1} \cdot 2^{k+1}.
\]
Consequently there exists for every \( j = 1, 2, ..., 2^k + 1 \) a closed interval \( A_{k+1, j} \)
with the length \( |A_{k+1, j}| = \lambda_{k+1} \) and with the center the same as the center of \( I_{k, j} \). The system \( \sigma_{k+1} \) consisting of all intervals \( A_{k+1, j}, j = 1, 2, ..., 2^{k+1} \) satisfies our conditions.

The intervals belonging to \( \sigma_k \) are said to be of order \( k \). Observe, that the measure (in the elementary sense of Lebesgue) of the closure \( \bar{A} \) of the set
\[
A = \bigcup_{k=0}^{\infty} \bigcup_{j=1}^{2^k} A_{k, j}
\]
is equal to \( \sum_{n=0}^{\infty} 2^n \lambda_n \).

Moreover, one infers by our construction, that the closure \( \bar{A} \) of the set \( I \) (with respect to \( \langle 0, 1 \rangle \) is a 0-dimensional compactum and its measure is equal to
\[
1 - \sum_{n=0}^{\infty} 2^n \lambda_n.
\]

7. Construction of two compacta \( X, Y \) such that \( \dim(X \cap Y) = 0 \) and that
\[
\mu(X) + \mu(Y) > \mu(X \cup Y).
\]
We preserve the notations of Section 6. In order to construct compacta \( X, Y \) satisfying the required conditions, consider in the space \( E^* \) the \( n \)-dimensional unit cube \( Q^n \) consisting of all points \( (x_1, ..., x_n) \in E^n \) with \( 0 \leq x_i \leq 1 \) for \( i = 1, 2, ..., n \).

Let \( X = Q^n \). The construction of \( Y \) is more complicated:
For every \( k = 0, 1, ... \) and for every \( i = 1, 2, ..., 2^k \) let us denote by \( \alpha_{ij} \) and \( b_{ij} \) the end points of \( A_{ij} \) (with \( \alpha_{ij} < b_{ij} \)) and let \( c_{ij} \) denote the center of \( A_{ij} \).

Consider also the set \( B_k, k = 0, 1, ... \) consisting of all points \( (x_1, ..., x_n) \in Q^n \) such that \( x_i \) belongs to the closure of the set
\[
I \setminus \bigcup_{m=0}^{2^k} \bigcup_{j=1}^{2^m} A_{m, j}
\]
for every \( i = 1, ..., n \). \( B_k \) is a compact subset of \( Q^n \) for every \( k \).

First we define (by the induction) a sequence \( Y_0, Y_1, ... \) of polyhedra in \( E^{n+1} = E^n \). Setting \( Y_0 = X = Q^n \), assume that \( Y_{i-1} \) and \( Y_i \) are already constructed. The space \( Y_i \) is homeomorphic to \( X \) and \( X \cap Y_i = B_{i-1} \) for \( i = 1, ..., k \). For every \( k = 1, 2, ..., k - 1 \) every point \( x = (x_1, ..., x_n, x_{n+1}) \in E^{n+1} \), where \( x_{n+1} \geq \lambda_{n+1} \), belongs to \( Y_{n+1} \) and only if \( x \in Y_n \).

Let us denote by \( H_{ij} \) for all natural indices \( j \leq n \) and for \( i \leq 2^k \) the set consisting of all points \( (x_1, ..., x_n, x_{n+1}) \in E^{n+1} \) such that \( 0 \leq x_i \leq 1 \) for every \( l \leq n \) and \( x_j = \alpha_{ij} \) and \( x_{n+1} = \lambda_{n+1} \).

Denote by \( W_{ij} \) (respectively by \( Z_{ij} \)) the union of all closed intervals with endpoints \( (x_1, ..., x_n, x_{n+1}) \in E^{n+1} \) and \( (y_1, ..., y_{n+1}) \in H_{ij} \), where \( x_i = y_i \) for every \( i \neq j, n + 1 \) and \( x_j = \alpha_{ij} \) or \( x_j = b_{ij} \) (respectively the set consisting of all points \( (x_1, ..., x_n, x_{n+1}) \in E^{n+1} \) such that \( \alpha_{ij} < x_j < b_{ij} \)).

Setting
\[
\varphi(x_1, ..., x_n, x_{n+1}) = (x_1, ..., x_n),
\]
one obtains the orthogonal projection \( \varphi: E^{n+1} \to E^n \). Every polyhedron \( W \) lying in \( E^{n+1} \) is mapped by \( \varphi \) onto a polyhedron \( \varphi(W) \subset E^n \) such that
\[
|\varphi(W)| < |W|, \quad m = 0, 1, ...
\]

Let
\[
U_{k+1} = (Y_k \cap \bigcup_{j=1}^{2^k} Z_{ij} \cup (\bigcup_{j=1}^{2^{k+1}} W_{ij})),
\]
Let us observe that \( U_{k+1} \) is a polyhedron which is mapped by \( \varphi \) onto \( X \) and that there exists a subpolyhedron \( Y_{k+1} \) of \( U_{k+1} \) such that \( \varphi \) restricted to \( Y_{k+1} \) is an embedding onto \( X \) and \( X \cap Y_{k+1} = B_k \). A point \( x \)
one gets an $\varepsilon$-translation $g: X \to E^{n+1}$ such that 
\[ |g(x)|_u \leq |f_\varepsilon(Y)|_u < \gamma < 1.\]

Thus for every $\varepsilon > 0$ there exists an $\varepsilon$-translation $g: X \to E^{n+1}$ such that
$g(X)$ is a polyhedron and $\|g(X)\|_u < \gamma$. Hence $\mu_k(X) \leq \gamma < 1$. This contradicts (1.6). Therefore $\mu_k(Y) > 1$.

Since $X \cap Y = B$ and $\dim B = 0$, we conclude that $X$ and $Y$ satisfy the required conditions.

8. Construction of compacta $X$, $Y$ such that $\dim(X \cap Y) = 0$ and
$\mu_k(X \cup Y) > \mu_k(X) + \mu_k(Y)$.

Preserving the notations of Section 6, let us set $\lambda = 6^{-n+3}$ and let us call the interval $A_k$ even, if $k$ is an even number, and odd if $k$ is odd. By $X$ we denote the closure of the union of all even intervals, and by $Y$ the closure of the union of all odd intervals. Observe that every component of the compactum $|\bar{Y}|$ is either an even interval, or a singleton.

For every positive $\varepsilon$ there exists only a finite collection of intervals $A_1, \ldots, A_n$ with diameters $\geq \varepsilon$. Moreover, for every even interval $A$ there is an $\varepsilon$-neighborhood which is an open interval with both endpoints belonging to odd intervals. It follows, that there exist open intervals $U_1, \ldots, U_n$ disjoint one to another with endpoints belonging to odd intervals such that $U_i$ is an $\varepsilon$-neighborhood of $A_i$ for $i = 1, 2, \ldots$. Consequently, there exists an $\varepsilon$-translation $f$ of the set $U_1 \cup \cdots \cup U_n$ onto the polyhedron $A_1 \cup \cdots \cup A_n$.

Since $\mu_k(A_1 \cup \cdots \cup A_n) > \mu_k(A_i \cup A_j)$, we infer by Lemma (5.1) that
$\mu_k(X) < \frac{1}{2}$. By an analogous argument, one shows that $\mu_k(Y) < \frac{1}{2}$.

Since $X \cup Y = I$, we infer that $\mu_k(X) + \mu_k(Y) < \mu_k(X \cup Y)$. Moreover, all components of the set $X \cap Y$ are singletons and consequently $\dim(X \cap Y) = 0$.

References


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