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Notes on topological games

by

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Abstract. The topological game $G(\boldsymbol{K}, X)$ is in the sense of R. Telgársky. Let $\boldsymbol{K}$ and $\boldsymbol{K}'$ be classes of spaces with $\boldsymbol{K} \subset \boldsymbol{K}'$. It is studied in this paper when Player I has a winning strategy in $G(\boldsymbol{K}, X)$ if he has one in $G(\boldsymbol{K}', X)$. We will discuss three questions of this kind.

§ 1. Each space considered here means a topological space and no separation axioms are assumed unless otherwise stated. Throughout this paper, $\boldsymbol{K}$ denotes a class of spaces which are hereditary with respect to closed subspaces. We need no other assumptions of $\boldsymbol{K}$. When we consider such two classes of spaces, they are denoted by $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$.

The topological game $G(\boldsymbol{K}, X)$ is introduced and studied by R. Telgársky [4]. The detail is seen in it. For a class $\boldsymbol{K}$, $I(\boldsymbol{K})$ denotes the class of all spaces X for which Player I has a winning strategy in $G(\boldsymbol{K}, X)$, and the class $I(I(\boldsymbol{K}))$ is abbreviated to $I^2(\boldsymbol{K})$. Moreover, $\boldsymbol{DK}$, $\boldsymbol{LK}$ and $\boldsymbol{SK}$ denote the classes of all spaces being discrete unions of spaces from $\boldsymbol{K}$, locally $\boldsymbol{K}$ and $\boldsymbol{K}$-scattered, respectively.

The purpose of this paper is to study the following three questions:

(A) What kind of a space X, does $X \in I^2(\boldsymbol{K})$ imply $X \in I(\boldsymbol{K})$ for?

(B) What kind of a space X, does $X \in I(\boldsymbol{SK})$ imply $X \in I(\boldsymbol{DK})$ for?

(C) What kind of a space X, does $X \in I(\boldsymbol{K}_1) \cap I(\boldsymbol{K}_2)$ imply $X \in I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$ for?

In § 2, § 3 and § 4, the questions (A), (B) and (C) are answered, respectively. Though the question (B) has been already studied by R. Telgársky [6], we give here another result and the improvement of his ones.

Concerning the topological game $G(\boldsymbol{K}, X)$, we use the notations in [7] rather than in [4]. Here, we do not restate them except the following. Let s be a strategy of Player I in $G(\boldsymbol{K}, X)$. A finite sequence $\langle F_0, F_1, \ldots, F_n\rangle$ of closed sets in X is said to be an *admissible choice of Player* II *for* s *in* $G(\boldsymbol{K}, X)$ (ad. ch. for II $(s, \boldsymbol{K}, X)$) if $F_0 = X$ and the sequence $\langle E_1, F_1, \ldots, E_n, F_n\rangle$, such that $E_i = s(F_0, F_1, \ldots, F_{i-1})$ for $1 \leqslant i \leqslant n$, is admissible for $G(\boldsymbol{K}, X)$. If s is a winning strategy of Player I in $G(\boldsymbol{K}, X)$, then it should be noted that each infinite sequence $\langle F_0, F_1, \ldots\rangle$ of closed sets in X, such that $\langle F_0, F_1, \ldots, F_n\rangle$ is an ad. ch. for II$(s, \boldsymbol{K}, X)$ for each $n \geqslant 1$, has the empty intersection. Since $K_1 \subset K_2$ implies $I(K_1) \subset I(K_2)$, note $I(K) \subset I^2(K)$, $I(DK) \subset I(LK) \subset I(SK)$ and $I(K_1 \cap K_2) \subset I(K_1) \cap I(K_2)$. For a class K, $\tilde{K}$ denotes the class of the empty set $\emptyset$ and all finite closed unions of members of K (i.e., $Y = \bigcup\{Y_i: 1 \leqslant i \leqslant n\}$, where each Y_i is a closed set in X with $Y_i \in K$). Then one can check $I(\tilde{K}) = I(K)$ (cf. [4, Theorem 4.1]). Thus, when discussing a class $I(K)$, we shall use $I(\tilde{K})$ without special mention.

Non-negative integers are denoted by n, m, i, j and k. We denote by N^* the set of the empty set $\emptyset$ and all finite sequences of positive integers. For each $e = \langle n_1, \ldots, n_k\rangle \in N^*$, let $\Sigma e = n_1 + \ldots + n_k$ and $e_{-1} = \langle n_1, \ldots, n_{k-1}\rangle$. Moreover, for each $n \geqslant 1$, let $e \oplus n = \langle n_1, \ldots, n_k, n\rangle$ and $\langle n\rangle_{-1} = \emptyset$. If $e = \emptyset$, then let $\Sigma e = 0$ and $e \oplus n = \langle n\rangle$ for each $n \geqslant 1$.

For a space X, 2^X denotes the collection of all closed sets in X. Note that $X \in K$ implies $2^X \subset K$.

§ 2. The concept of P-spaces was introduced by K. Morita [3]. However, we use here a certain characterization of P-spaces by R. Telgársky [5].

Definition. A space X is said to be a *P-space* if there exists a function $p\colon \bigcup\{(2^X)^n: n \geqslant 1\} \to 2^X$ such that

(i) for each $(F_0, \ldots, F_n) \in (2^X)^{n+1}$ and $n \geqslant 0$,

$$p(F_0, \ldots, F_n) \cap \bigcap\{F_i: i \leqslant n\} = \emptyset,$$

(ii) for each $(F_0, F_1, \ldots) \in (2^X)^\omega$ with $\bigcap\{F_n: n \geqslant 0\} = \emptyset$,

$$\bigcup\{p(F_0, \ldots, F_n): n \geqslant 0\} = X.$$

We call such a function p *P-function.*

The following result is an answer to the question (A) in § 1. It is also a partial answer to [4, Question 4.10].

Theorem 2.1. *Let X be a P-space. Then $X \in I^2(K)$ iff $X \in I(K)$.*

Proof. Let p be a P-function of X. Assume $X \in I^2(K)$. Let s be a winning strategy of Player I in $G(I(K), X)$ and s_E one in $G(K, E)$ for each $E \in I(K) \cap 2^X$. We use below the following notation; for each $e = \langle n_1, \ldots, n_k\rangle \in N^*$, let $\Delta(e) = \langle \emptyset, d_1, \ldots, d_k\rangle$, where $d_i = \langle n_1, \ldots, n_i\rangle$ for $1 \leqslant i \leqslant k$, and $\Delta(\emptyset) = \langle \emptyset\rangle$.

Now, assume that we have already constructed an admissible sequence $\langle E_1, F_1, \ldots, E_m, F_m\rangle$ in $G(K, X)$ such that $E_{i+1} = t(F_0, F_1, \ldots, F_i)$ for $0 \leqslant i \leqslant m-1$, where $F_0 = X$, and such that there exist two collections

$$\{E(e): e \in N^* \text{ with } \Sigma e \leqslant m\} \quad \text{and} \quad \{F(e): e \in N^* \text{ with } \Sigma e \leqslant m\}$$

of closed sets in X, satisfying the following conditions (1)–(4):

(1) For each $e \in N^*$ with $\Sigma e \leqslant m$, $\langle F(d): d \in \Delta(e)\rangle$ is an ad. ch. for II$(s, I(K), X)$ such that $E(e) = s(\langle F(d): d \in \Delta(e)\rangle)$.

(2) For each $e \in N^*$ and $n \geqslant 0$ with $\Sigma e + n \leqslant m$,

$$\mathcal{F}(e, n) = \langle E(e) \cap F_{\Sigma e + i}: 0 \leqslant i \leqslant n\rangle$$

is an ad. ch. for II $(s_e, K, E(e))$, where s_e denotes $s_{E(e)}$.

(3) For $1 \leqslant k \leqslant m$,

$$E_k = \bigcup\{s_e(\mathcal{F}(e, n)): e \in N^* \text{ and } n \geqslant 0 \text{ with } \Sigma e + n = k-1\}.$$

(4) For each $e \in N^*$ with $\Sigma e \leqslant m$,

$$F(e) = F(e_{-1}) \cap F_{\Sigma e} \cap p(\mathcal{F}(e_{-1}, n_e)), \quad \text{where} \quad e = e_{-1} \oplus n_e.$$

For each $e \in N^*$ and $n \geqslant 0$ with $\Sigma e + n = m$, it follows from the assumption (2) that Player I can choose $s_e(\mathcal{F}(e, n))$ in $G(K, E(e))$. So, we can define $t(F_0, \ldots, F_m) = E_{m+1} \in K$ according to the condition (3). Next, Player II chooses an arbitrary closed set F_{m+1} in X such that $F_{m+1} \subset F_m$ and $E_{m+1} \cap F_{m+1} = \emptyset$. Pick any $e \in N^*$ with $\Sigma e = m+1$. Let $e = e_{-1} \oplus n_e$. Put

$$\mathcal{F}(e_{-1}, n_e) = \langle E(e_{-1}) \cap F_i: \Sigma e_{-1} \leqslant i \leqslant m+1\rangle.$$

It is a finite decreasing sequence of closed sets in X. We can define $F(e)$ as the condition (4). Moreover, by the definition of p and this condition, we have

$$E(e_{-1}) \cap F(e) = E(e_{-1}) \cap F_{m+1} \cap p(\mathcal{F}(e_{-1}, n_e)) = \emptyset.$$

Hence it follows from the inductive assumption (1) and from $E(e_{-1}) \cap F(e) = \emptyset$ that $\langle F(d): d \in \Delta(e)\rangle$ is an ad. ch. for II$(s, I(K), X)$. The condition (1) is satisfied. Put $E(e) = s(\langle F(d): d \in \Delta(e)\rangle)$. By $E(e) \subset F(e) \subset F_{\Sigma e}$, we have $\mathcal{F}(e, 0) = \langle E(e)\rangle$. Pick any $e \in N^*$ and $n \geqslant 1$ with $\Sigma e + n = m+1$. In order to show that $\mathcal{F}(e, n)$ is an ad. ch. for II$(s_e, K, E(e))$, it suffices to verify from the inductive assumption (2) that $s_e(\mathcal{F}(e, n-1))$ and $E(e) \cap F_{m+1}$ are disjoint. This is checked by the condition (3) and by $E_{m+1} \cap F_{m+1} = \emptyset$. Hence the condition (2) is satisfied. By induction, we can construct the above for each $m \geqslant 1$.

An infinite sequence $\langle E_1, F_1, E_2, F_2, \ldots\rangle$ of closed sets in X, which is obtained above, is a play in $G(K, X)$. In order to show that t is a winning strategy of Player I in $G(K, X)$, it suffices to show from the choice of each F_m that $\{F_m: m \geqslant 1\}$ has the empty intersection. Assume $x_0 \in \bigcap\{F_m: m \geqslant 1\}$. Moreover, assume $x_0 \in F(e)$. By (2), for each $m \geqslant \Sigma e$, $\mathcal{F}(e, m - \Sigma e)$ is an ad. ch. for II$(s_e, K, E(e))$. It follows from the definition of s_e that $\langle E(e) \cap F_{\Sigma e + k}: k \geqslant 1\rangle$ is a decreasing sequence with the empty intersection. Hence $\{p(\mathcal{F}(e, k)): k \geqslant 1\}$ is a cover of X. Choose some $k_0 \geqslant 1$ such that $x_0 \in p(\mathcal{F}(e, k_0))$. By (4), we have $x_0 \in F(e, k_0)$. Note $x_0 \in F(\emptyset) = X$. We can inductively choose an infinite sequence $\sigma = \langle n_1, n_2, \ldots\rangle$ of natural numbers such that $x_0 \in \bigcap\{F(\sigma_k): k \geqslant 1\}$, where $\sigma_k = \langle n_1, \ldots, n_k\rangle$ for each $k \geqslant 1$. On the other hand, it follows from (1) that $\langle F(\emptyset), F(\sigma_1), \ldots, F(\sigma_k)\rangle$ is an ad. ch.

for $\mathrm{II}(s, I(\boldsymbol{K}), X)$ for each $k \geqslant 1$. By the definition of s, we have $\bigcap \{F(\sigma_k): k \geqslant 1\} = \emptyset$. This contradicts to the above. So we have shown $X \in I(\boldsymbol{K})$. The converse is obvious. The proof is complete.

Remark. Without any assumptions of X, Theorem 2.1 is not true. Indeed, assuming CH there exists a regular Lindelöf space which is in $I^2(\boldsymbol{l})$ but not in $I(\boldsymbol{l})$, where $\boldsymbol{l}$ denotes the class of all one-point spaces. Such a space is seen in [7, Example 5.1], which was pointed out by R. Telgársky.

§ 3. Concerning the question (B) in § 1, R. Telgársky [6] has proved the following results:

(a) For a paracompact space X, $X \in I(\boldsymbol{SK})$ iff $X \in I(\boldsymbol{LK})$, where $\boldsymbol{K}$ is a class of spaces which are invariant with respect to finite closed unions.

(b) For a paracompact space X, $X \in I(\boldsymbol{SC})$ iff $X \in I(\boldsymbol{DC})$, where $\boldsymbol{C}$ is the class of all compact spaces.

In this section, we consider this question for the generalizations of paracompact spaces; submetacompact ($= \theta$-refinable) spaces and subparacompact spaces.

LEMMA 3.1. $\boldsymbol{D}(I(\boldsymbol{DK})) = I(\boldsymbol{DK})$.

LEMMA 3.2. *Let X be a regular submetacompact space. If X is locally K, then $X \in I(\boldsymbol{DK})$.*

Lemma 3.1 is easily checked. The proof of Lemma 3.2 is quite standard and the content of it is a special case of Theorem 3.3 below. So, their proofs are omitted.

LEMMA 3.3. *Let X be a regular submetacompact P-space. If X is K-scattered, then $X \in I(\boldsymbol{DK})$.*

Proof. Let α be the least ordinal such that $X^{(\alpha)} = \emptyset$. We prove by transfinite induction over α. Case 1: $\alpha = \beta + 1$. Then $X^{(\beta)}$ is locally $\boldsymbol{K}$ and $X^{(\beta)} \in 2^X$. By Lemma 3.2, $X^{(\beta)} \in I(\boldsymbol{DK})$. It follows from the inductive assumption that each closed set F in X disjoint from $X^{(\beta)}$ is in $I(\boldsymbol{DK})$. Hence X is in $I^2(\boldsymbol{DK})$. By Theorem 2.1, we have $X \in I(\boldsymbol{DK})$. Case 2: α is a limit ordinal. Since $\{X \setminus X^{(\gamma)}: \gamma < \alpha\}$ is an open cover of X, it follows from the inductive assumption and the regularity of X that X is locally $I(\boldsymbol{DK})$. By Lemmas 3.1 and 3.2, we have $X \in I(\boldsymbol{D}(I(\boldsymbol{DK}))) = I^2(\boldsymbol{DK})$. Hence, by Theorem 2.1 $X \in I(\boldsymbol{DK})$. The proof is complete.

From Theorem 2.1 and Lemma 3.3, we get

THEOREM 3.1. *Let X be a regular submetacompact P-space. Then $X \in I(\boldsymbol{SK})$ iff $X \in I(\boldsymbol{DK})$.*

The game $G^{\#}(\boldsymbol{K}, X)$ has been introduced in [6] to prove the above (a). We consider another game which is a modification of it. Let $G_{\#}(\boldsymbol{K}, X)$ be the game which is obtained by replacing "locally finite" in the definition of $G^{\#}(\boldsymbol{K}, X)$ with "σ-discrete". That is, it is as follows: Player I chooses a closed cover $\{X(t): t \in T\}$ of X such that $\{X(t): t \in T_k\}$ is discrete in X for each $k \geqslant 1$, where $T = \bigcup \{T_k: k \geqslant 1\}$ and $T_k \cap T_{k'} = \emptyset$ if $k \neq k'$. Next, Player II chooses a t_1 in T. After that, Player I chooses a closed subset $Y(t_1)$ of $X(t_1)$ with $Y(t_1) \in \boldsymbol{K}$. Next, Player II chooses a closed subset $Z(t_1)$ of $X(t_1)$ disjoint from $Y(t_1)$. Again, Player I chooses a closed cover $\{X(t_1, t): t \in T\}$ of $Z(t_1)$ such that $\{X(t_1, t): t \in T_k\}$ is discrete in it for each $k \geqslant 1$. Hereafter, repeat the above. Player I wins the play if $\{X(t_1, \ldots, t_n): n \geqslant 1\}$ has the empty intersection and otherwise Player II wins.

LEMMA 3.4. *Let X be a subparacompact space. If $X \in I(\boldsymbol{SK})$, then Player I has a winning strategy in $G_{\#}(\boldsymbol{K}, X)$.*

The proof is quite parallel to that of [6, Proposition 2.2]. The detail of it is left to the reader.

LEMMA 3.5. *If Player I has a winning strategy in $G_{\#}(\boldsymbol{K}, X)$, then $X \in I(\boldsymbol{DK})$.*

Proof. Let $s_{\#}$ be a winning strategy of Player I in $G_{\#}(\boldsymbol{K}, X)$. For each $e = \langle n_1, \ldots, n_k \rangle \in N^*$, let $T(e) = T_{n_1} \times \ldots \times T_{n_k}$.

Now, assume that we have already constructed an admissible sequence $\langle E_1, F_1, \ldots, E_m, F_m \rangle$ in $G(\boldsymbol{DK}, X)$ such that $E_{i+1} = s(F_0, F_1, \ldots, F_i)$ for $0 \leqslant i \leqslant m-1$, where $F_0 = X$, and such that there exist two collections

$$\{X(u, t): (u, t) \in T(e) \times T\} \quad \text{and} \quad \{Y(u, t): (u, t) \in T(e) \times T\}$$

of closed sets in X for each $e \in N^*$ with $\Sigma e \leqslant m-1$ and a collection $\{Z(u): u \in T(e)\}$ of closed sets in X for each $e \in N^*$ with $\Sigma e \leqslant m$, satisfying the following conditions (1)–(3):

(1) For each $e \in N^*$ with $\Sigma e \leqslant m-1$, $u = (t_1, \ldots, t_k) \in T(e)$ and $t_{k+1} \in T$,

$$\{X(t_1, \ldots, t_i): 1 \leqslant i \leqslant k+1\} \quad \text{and} \quad \{Y(t_1, \ldots, t_i): 1 \leqslant i \leqslant k+1\}$$

are the choices of Player I, according to $s_{\#}$, in $G_{\#}(\boldsymbol{K}, X)$.

(2) For each $e \in N^*$ with $\Sigma e \leqslant m$ and $u = (t_1, \ldots, t_k) \in T(e)$, $\{Z(t_1, \ldots, t_i): 1 \leqslant i \leqslant k\}$ is the choice of Player II in $G_{\#}(\boldsymbol{K}, X)$.

(3) For each $e \in N^*$ with $\Sigma e \leqslant m$ and $u \in T(e)$, $Z(u) = X(u) \cap F_{\Sigma e}$.

For each $e \in N^*$ with $\Sigma e = m$ and $u \in T(e)$, it follows from the inductive assumptions that Player I can choose $\{X(u, t): t \in T\}$ and $\{Y(u, t): t \in T\}$ in $G_{\#}(\boldsymbol{K}, X)$, according to $s_{\#}$. Here, we can put

$$E_{m+1} = s(F_0, \ldots, F_m) = \bigcup \{Y(u) \cap F_m: u \in T(d) \text{ and } \Sigma d = m+1\}.$$

Then $E_{m+1} \in \boldsymbol{DK}$. Next, Player II chooses an arbitrary closed set F_{m+1} in X such that $F_{m+1} \subset F_m$ and $E_{m+1} \cap F_{m+1} = \emptyset$. Moreover, for each $d \in N^*$ with $\Sigma d = m+1$ and $u \in T(d)$ we define $Z(u)$ as the condition (3) is satisfied. Then it is easily verified that $Y(u)$ and $Z(u)$ are disjoint. Hence the condition (2) is satisfied. By induction, we can construot the above for each $m \geqslant 1$.

In order to show that s is a winning strategy of Player I in $G(\boldsymbol{DK}, X)$, it suffices to show $\bigcap \{F_m: m \geqslant 1\} = \emptyset$. Assume $x_0 \in \bigcap \{F_m: m \geqslant 1\}$. It is seen by (3) that we can inductively choose an infinite sequence $\langle n_1, n_2, \ldots\rangle$ of natural numbers and some $(t_1, t_2, \ldots) \in T_{n_1} \times T_{n_2} \times \ldots$ such that $x_0 \in X(t_1, \ldots, t_i)$ for each $i \geqslant 1$. On the other hand, we have $\bigcap\{X(t_1, \ldots, t_i)\colon i \geqslant 1\} = \emptyset$ by the definition of $s_{\#}$. This is a contradiction. The proof is complete.

As an immediate consequence of Lemmas 3.4 and 3.5, we have the following result which is an improvement of [6, Theorem 2.5]. However, the ideas of its proof have been greatly dependent on R. Telgársky's ones.

THEOREM 3.2. *Let X be a subparacompact space. Then $X \in I(\boldsymbol{SK})$ iff $X \in I(\boldsymbol{DK})$.*

Remark. According to Theorems 3.1 and 3.2, all $I(\boldsymbol{DC}, X)$ in [7, Theorem 2.1, 3.1 and 4.1] can be replaced by $I(\boldsymbol{SC}, X)$.

Moreover, we consider the following analogous question to (B):

(B′) What kind of a space X, does $X \in I(\boldsymbol{LK})$ imply $X \in I(\boldsymbol{DK})$ for?

LEMMA 3.6. *Let X be a regular submetacompact space. If E is a locally $\boldsymbol{K}$, closed set in X, then there exist two sequences $\{U_n\colon n \geqslant 1\}$ and $\{D_n\colon n \geqslant 1\}$ of subsets of X, satisfying the following conditions:*

(i) *each U_n is open in X and contains E,*

(ii) *each D_n is closed in U_n,*

(iii) $\bigcap\{U_n \backslash D_n\colon n \geqslant 1\} = \emptyset$,

(iv) *$D_1 \cap E \in 2^X \cap \boldsymbol{DK}$, and $D_n \cap E \cap H \in 2^X \cap \boldsymbol{DK}$ whenever $n \geqslant 2$ and H is a closed set in X disjoint from $D_{n-1} \cap E$.*

Proof. By the assumptions of X and E, there exists a sequence $\{\mathscr{W}_n\colon n \geqslant 1\}$ of open covers of X such that for each $n \geqslant 1$ and $W \in \mathscr{W}_n$, $\overline{W} \cap E$ is in $\boldsymbol{K}$ and for each $x \in X$ we can take some $n \geqslant 1$ so that $\mathscr{W}_n$ is point-finite at x. Let $\mathscr{V}_n = \{V \in \mathscr{W}_n\colon V \cap E \neq \emptyset\}$ and $V_n = \bigcup\{V\colon V \in \mathscr{V}_n\}$ for each $n \geqslant 1$. Here, we put $U_n = \bigcap\{V_i\colon i \leqslant n\}$ for each $n \geqslant 1$. For each $i, j \geqslant 1$, let

$$F_{ij} = \{x \in V_i\colon x \text{ is in at most } j \text{ members of } \mathscr{V}_i\}.$$

Then each F_{ij} is closed in V_i. We arrange the sets F_{ij} in the following order:

$$F_{11}, F_{21}, F_{12}, F_{31}, F_{22}, F_{13}, F_{41}, F_{32}, \ldots$$

We represent this sequence by $\langle F_n\colon n \geqslant 1\rangle$. For each $n \geqslant 1$, we put $D_n = \bigcup\{F_i\colon i \leqslant n\} \cap U_n$. Then it is a routine to check that these sequences $\{U_n\colon n \geqslant 1\}$ and $\{D_n\colon n \geqslant 1\}$ satisfy the conditions (i)–(iv).

THEOREM 3.3. *Let X be a regular submetacompact space. Then $X \in I(\boldsymbol{LK})$ iff $X \in I(\boldsymbol{DK})$.*

Proof. Assume $X \in I(\boldsymbol{LK})$. Let s be a winning strategy of Player I in $G(\boldsymbol{LK}, X)$. For each $e \in N^*$, $\Delta(e)$ is defined as in the proof of Theorem 2.1.

Now, assume that we have already constructed an admissible sequence $\langle E_1, F_1, \ldots, E_m, F_m\rangle$ in $G(\boldsymbol{DK}, X)$ such that $E_{i+1} = t(F_0, F_1, \ldots, F_i)$ for $0 \leqslant i \leqslant m-1$, where $F_0 = X$, and such that there exist four collections

$$\{E(e)\colon e \in N^* \text{ with } \Sigma e \leqslant m\}, \quad \{F(e)\colon e \in N^* \text{ with } \Sigma e \leqslant m\},$$

$$\{U(e \oplus n)\colon e \in N^* \text{ with } \Sigma e \leqslant m \text{ and } n \geqslant 1\}$$

and

$$\{D(e \oplus n)\colon e \in N^* \text{ with } \Sigma e \leqslant m \text{ and } n \geqslant 1\}$$

of subsets of X, satisfying the following conditions; for each $e \in N^*$ with $\Sigma e \leqslant m$ and $n \geqslant 1$,

(1) $\langle F(d)\colon d \in \Delta(e)\rangle$ is an ad. ch. for $\mathrm{II}(s, \boldsymbol{LK}, X)$ such that $E(e) = s(\langle F(d)\colon d \in \Delta(e)\rangle)$,

(2) $U(e \oplus n)$ is open in X and contains $E(e)$,

(3) $D(e \oplus n)$ is closed in $U(e \oplus n)$,

(4) $\bigcap\{U(e \oplus n) \backslash D(e \oplus n)\colon n \geqslant 1\} = \emptyset$,

(5) $E(e) \cap D(e \oplus 1) \in 2^X \cap \boldsymbol{DK}$, and $E(e) \cap D(e \oplus n) \cap H \in 2^X \cap \boldsymbol{DK}$ whenever $n \geqslant 2$ and H is a closed set in X disjoint from $E(e) \cap D(e \oplus n-1)$,

(6) $F(e) = F_{\Sigma e} \cap F(e_{-1}) \cap (U(e) \backslash D(e))$,

(7) $E_k = \bigcup\{E(e_{-1}) \cap D(e) \cap F_{k-1}\colon \Sigma e = k\}$ for each $k \leqslant m$.

First, we can define $t(F_0, \ldots, F_m) = E_{m+1}$ according to the condition (7). Then it is easily seen by the assumptions (2), (3), (5) and (7) that E_{m+1} is in $2^X \cap \boldsymbol{DK}$. Next, Player II chooses an arbitrary closed set F_{m+1} in X such that $F_{m+1} \subset F_m$ and $E_{m+1} \cap F_{m+1} = \emptyset$. Pick any $e \in N^*$ with $\Sigma e = m+1$. We can define $F(e)$ as the condition (6) is satisfied. Then $F(e)$ is a closed set in X such that $F(e) \subset F(e_{-1})$. Moreover, it is verified from the assumptions (2) and (7) that $F(e)$ and $E(e_{-1})$ are disjoint. Hence it follows from the inductive assumption that $\langle F(d)\colon d \in \Delta(e)\rangle$ is an ad. ch. for $\mathrm{II}(s, \boldsymbol{LK}, X)$. The condition (1) is satisfied. Player I can choose $E(e) = s(\langle F(d)\colon d \in \Delta(e)\rangle) \in \boldsymbol{LK}$. For the $E(e)$, it follows from Lemma 3.6 that there exist two sequences $\{U(e \oplus n)\colon n \geqslant 1\}$ and $\{D(e \oplus n)\colon n \geqslant 1\}$ of subsets of X, satisfying the conditions (2)–(5). By induction, we can construct the above for each $m \geqslant 1$.

In order to show that t is a winning strategy of Player I in $G(\boldsymbol{DK}, X)$, it suffices to show $\bigcap\{F_m\colon m \geqslant 1\} = \emptyset$. Assume $x_0 \in \bigcap\{F_m\colon m \geqslant 1\}$. Moreover, assume $x_0 \in F(e)$. By (4), we can take some $n_0 \geqslant 1$ such that $x_0 \notin U(e \oplus n_0) \backslash D(e \oplus n_0)$. By (6), we have $x_0 \in F(e \oplus n_0)$. We can inductively choose an infinite sequence $\sigma = \langle n_1, n_2, \ldots\rangle$ of natural numbers such that $x_0 \in \bigcap\{F(\sigma_k)\colon k \geqslant 1\}$, where $\sigma_k = \langle n_1, \ldots, n_k\rangle$ for each $k \geqslant 1$. On the other hand, we obtain $\bigcap\{F(\sigma_k)\colon k \geqslant 1\} = \emptyset$ from (1). This is a contradiction. Hence X is in $I(\boldsymbol{DK})$. The converse is obvious. The proof is complete.

In connection with Theorems 3.1, 3.2 and 3.3, we raise the following natural question which remains open.

QUESTION. Let X be a regular submetacompact space. Does $X \in I(SK)$ imply $X \in I(DK)$?

In the end of this section, we shall notice that the hereditary paracompactness in [4, Theorems 11.1–11.4] can be generalized to the hereditary subparacompactness. These proofs are quite parallel to the original ones.

§ 4. The concept of subnormality was first introduced by T. R. Kramer [2]. Here, we use a certain characterization of it given by J. Chaber [1].

DEFINITION. A space X is said to be *subnormal* if for each disjoint closed sets A, B in X there exist disjoint G_δ-sets C, D in X such that $A \subset C$ and $B \subset D$.

Note that normal spaces, (countably) subparacompact spaces and perfect spaces are subnormal. Moreover, it is easily verified that the Tychonoff plank $\omega_1 \times \omega \setminus \{(\omega_1, \omega)\}$ is a subnormal space which is neither normal, subparacompact nor perfect. Now, we give two answers to the question (C) in § 1.

THEOREM 4.1. *Let X be a subnormal space. Then $X \in I(\boldsymbol{K}_1) \cap I(\boldsymbol{K}_2)$ iff $X \in I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$.*

Proof([1]). Assume $X \in I(\boldsymbol{K}_1) \cap I(\boldsymbol{K}_2)$. Let s_1 and s_2 be winning strategies of Player I in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, X)$, respectively. For each $e = \langle n_1, \ldots, n_k \rangle \in N^*$, we use below the following notations; let $\{i_1, \ldots, i_q\} = \{i \leqslant k: n_i \text{ is odd}\}$ and $\{j_1, \ldots, j_r\} = \{j \leqslant k: n_j \text{ is even}\}$, where assume $i_1 < \ldots < i_q$ and $j_1 < \ldots < j_r$. Here we put $\Delta_1(e) = \langle \emptyset, d_{i_1}, \ldots, d_{i_q} \rangle$ and $\Delta_2(e) = \langle \emptyset, d_{j_1}, \ldots, d_{j_r} \rangle$, where $d_i = \langle n_1, \ldots, n_i \rangle$ for $1 \leqslant i \leqslant k$. It should be noted that $\Delta_1(e) = \langle \emptyset \rangle$ or $\Delta_2(e) = \langle \emptyset \rangle$ may be occured even if $e \neq \emptyset$. Of course, $\Delta_1(\emptyset) = \Delta_2(\emptyset) = \langle \emptyset \rangle$.

Now, assume that we have already constructed an admissible sequence $\langle E_1, F_1, \ldots, E_m, F_m \rangle$ in $G(\boldsymbol{K}_1 \cap \boldsymbol{K}_2, X)$ such that $E_n = s(F_0, F_1, \ldots, F_{n-1})$ for $1 \leqslant n \leqslant m$, where $F_0 = X$, and such that there exists a collection $\{F(e): e \in N^* \text{ with } \Sigma e_{-1} \leqslant m-1\}$ of closed sets in X, satisfying the following conditions (1)–(3):

(1) For each $e \in N^*$ with $\Sigma e_{-1} \leqslant m-1$, $\langle F(d): d \in \Delta_1(e) \rangle$ is an ad. ch. for $\mathrm{II}(s_1, \boldsymbol{K}_1, X)$.

(2) For each $e \in N^*$ with $\Sigma e_{-1} \leqslant m-1$, $\langle F(d): d \in \Delta_2(e) \rangle$ is an ad. ch. for $\mathrm{II}(s_2, \boldsymbol{K}_2, X)$.

(3) For each $e \in N^*$ with $\Sigma e \leqslant m-1$, $F(e) \cap F_{\Sigma e+1}$ is the union of $\{F(e \oplus n): n \geqslant 1\}$.

Pick any $e \in N^*$ with $\Sigma e = m$. By $\Sigma e_{-1} \leqslant m-1$, it follows from the inductive assumptions that Player I in $G(\boldsymbol{K}_1, X)$ and $G(\boldsymbol{K}_2, X)$ can choose

$$A(e) = s_1(\langle F(d): d \in \Delta_1(e) \rangle) \quad \text{and} \quad B(e) = s_2(\langle F(d): d \in \Delta_2(e) \rangle),$$

respectively. Here, running such e, we set

$$E_{m+1} = s(F_0, F_1, \ldots, F_m) = \bigcup \{A(e) \cap B(e): e \in N^* \text{ with } \Sigma e = m\} \cap F_m.$$

Then $E_{m+1} \in \boldsymbol{K}_1 \cap \boldsymbol{K}_2$ and $E_{m+1} \subset F_m$. Next, Player II chooses an arbitrary closed set F_{m+1} in X such that $F_{m+1} \subset F_m$ and $E_{m+1} \cap F_{m+1} = \emptyset$. Again, pick any $e \in N^*$ with $\Sigma e = m$. Since $A(e) \cap F(e) \cap F_{m+1}$ and $B(e) \cap F(e) \cap F_{m+1}$ are disjoint closed sets in $F(e) \cap F_{m+1}$ which is subnormal, there exists a countable collection $\{F(e \oplus n): n \geqslant 1\}$ of closed sets in X such that

(i) $\bigcup \{F(e \oplus n): n \text{ is odd}\} \cap A(e) = \emptyset$,

(ii) $\bigcup \{F(e \oplus n): n \text{ is even}\} \cap B(e) = \emptyset$,

(iii) $\bigcup \{F(e \oplus n): n \geqslant 1\} = F(e) \cap F_{m+1}$.

Here, we have constructed E_{m+1}, F_{m+1} and $\{F(e): e \in N^* \text{ with } \Sigma e_{-1} = m\}$. It is easily verified from the inductive assumptions and the above (i)–(iii) that the conditions (1)–(3) are satisfied. By induction, we can obtain the above for each $m \geqslant 1$.

In order to show that s is a winning strategy of Player I in $G(\boldsymbol{K}_1 \cap \boldsymbol{K}_2, X)$, it suffices to show $\bigcap \{F_m: m \geqslant 1\} = \emptyset$. Assume $x_0 \in \bigcap \{F_m: m \geqslant 1\}$. Note $x_0 \in F(\emptyset) = X$. By (3), we can inductively choose an infinite sequence $\sigma = \langle n_1, n_2, \ldots \rangle$ of natural numbers such that $x_0 \in F(n_1, \ldots, n_k)$ for each $k \geqslant 1$. Let

$$N_1 = \{k \geqslant 1: n_k \text{ is odd}\} \quad \text{and} \quad N_2 = \{k \geqslant 1: n_k \text{ is even}\}.$$

Then N_1 or N_2 is infinite. Assume $N_1 = \{k_1, k_2, \ldots\}$ and $k_1 < k_2 < \ldots$ Let $\sigma^i = \langle n_1, \ldots, n_{k_i} \rangle$ for each $i \geqslant 1$. Since $\Delta_1(\sigma^i) = \langle \emptyset, \sigma^1, \ldots, \sigma^i \rangle$, it follows from (1) that $\langle F(\emptyset), F(\sigma^1), \ldots, F(\sigma^i) \rangle$ is an ad. ch. for II $(s_1, \boldsymbol{K}_1, X)$ for each $i \geqslant 1$. By the definition of s_1, we have $\bigcap \{F(\sigma^i): i \geqslant 1\} = \emptyset$. On the other hand, $x_0 \in \bigcap \{F(\sigma^i): i \geqslant 1\}$. This is a contradiction. In case of N_2 being infinite, the similar argument also yields a contradiction from (2). Hence we have shown $X \in I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$. The converse is obvious. The proof is complete.

THEOREM 4.2. *Let X be a P-space. Then $X \in I(\boldsymbol{K}_1) \cap I(\boldsymbol{K}_2)$ iff $X \in I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$.*

Proof. Assume $X \in I(\boldsymbol{K}_1) \cap I(\boldsymbol{K}_2)$. Let s be a winning strategy of Player I in $G(\boldsymbol{K}_1, X)$. Since $2^X \subset I(\boldsymbol{K}_2)$, s is also a winning strategy of Player I in $G(\boldsymbol{K}_1 \cap I(\boldsymbol{K}_2), X)$. Note $\boldsymbol{K}_1 \cap I(\boldsymbol{K}_2) \subset I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$. Hence s is one in $G(I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2), X)$. We have $X \in I^2(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$. By Theorem 2.1, X is in $I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$.

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Fixed point index for open sets in euclidean spaces*

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Abstract. Using chain approximations of multi-valued mappings a fixed point index for a large class of such mappings of open sets in euclidean spaces is constructed. This fixed point index satisfies all usual properties of fixed point index for single valued maps including commutativity and mod-p property.

Introduction. The aim of the present note is to give an unified approach to fixed point theory for single-valued as well as for certain classes of multi-valued mappings on locally compact polyhedra and in particular on open sets in euclidean spaces. A fixed point index with all usual properties (additivity, homotopy invariance, normalization, commutativity and mod-p property) is constructed. In particular for single-valued maps on open sets in euclidean spaces we obtain the classical theory [5, 9]. The main idea is to use certain chain approximations of a given map and to localize the Lefschetz number of these chain approximations. In the global Lefschetz fixed point theory of multi-valued maps the chain approximations are used in [2, 8, 23, 28–32]. In the case of single-valued maps the fixed point index is defined as local Lefschetz number of chain approximation in [4, 10, 18, 22]. In the case of multi-valued maps on compact polyhedra these two approaches were used in [27] to define a fixed point index with all properties. For a review of results, applications and problems of the fixed point index theory see [12, 13, 15, 25].

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§ 0. Notations. We consider maps $\Phi\colon X \to Y$ for which the sets $\Phi(x)$ are not empty and compact for every $x \in X$. The map Φ is called *upper-semi-continuous* (u.s.c) if for every open set U in Y the set

$$\Phi^{-1}(U) = \{x \in X\colon \Phi(x) \subset U\}$$

is open, [16], ch. 4, p. 32.

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