Subopen multifunctions and selections

by

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Abstract. Let $m: X \to Y$ be a subopen (e.g., open graph) multifunction with infinitely connected values. If (X, A) is a relative CW complex it is proved that any continuous partial selection $g: A \to Y$ for m can be extended to a continuous selection $f: X \to Y$ for m. A fixed point corollary is also given.

Let X and Y be topological spaces. A set valued function $m: X \to Y$ will be called a *multifunction* if $m(x) \neq \emptyset$, all x in X. $G(m) = \{(x, y) |\ y \in m(x)\} \subset X \times Y$ is the graph of m. Let $p: G(m) \to X$ be the projection.

DEFINITION. m is a *subopen multifunction* if for each x in X there is an open nbhd U, a Serre fibration $q: T \to U$, and an embedding $e: p^{-1}(U) \to T$ such that $e(p^{-1}(U))$ is open in T and $qe = p$.

Recall that m is an open-graph multifunction if $G(m)$ is an open subset of $X \times Y$. Use $U = X$ and $T = X \times Y$ to see that *every open graph multifunction is a subopen multifunction.*

The selection theorem referred to in the title is the following.

1.1. THEOREM. *Let (X, A) be a relative* CW *complex and $m: X \to Y$ a subopen multifunction with $m(x)$ infinitely connected for all x in X. Then any continuous partial selection $g: A \to Y$ for m can be extended to a continuous selection $f: X \to Y$ for m. Furthermore, any two such extensions are homotopic by a homotopy $\{f_t\}$ with each f_t a selection for m extending g.*

This will be deduced from the following main result:

1.2. THEOREM. *Let $T \to B$ be a Serre fibration and E open in T. Assume that each $E(b)$ is N-connected and non-empty. Then $E \to B$ is an N-fibration.*

Theorem 1.1 generalizes the result of [2] and [3] in that there are no metric hypotheses on the domain, there are no conditions on the codomain at all, and "open-graph" is replaced by "sub-open".

The method of proof of 1.1 is to first use Theorem 1.2 to prove that $G(m) \to X$ is a Serre fibration and then apply obstruction theory.

Theorem 1.2 is related to results of Wong [5], however, the metric hypotheses used in [5] are avoided here. As a corollary to 1.2 (or 1.1) we obtain a lifting theorem which can be viewed as a generalization of [5, Th. 2.1, Cor. 2.2]. In Section 2 an ANR version of Theorem 1.1 is proved and a multifunction fixed point theorem is given which generalizes slightly a result of [3].

If it is not assumed that m has contractible values then more elaborate hypotheses are required for selection theorems. This will be discussed in a separate paper.

1. Notation, proof of Theorem 1.1 using Theorem 1.2. A single valued (continuous) function $f\colon X \to Y$ is a (continuous) selection for $m\colon X \to Y$ if $f(x) \in m(x)$, all x in X. If $A \subset X$ and $g\colon A \to Y$ is a single valued function such that $g(x) \in m(x)$, all x in A, then g is called a *partial selection* for m (A may be empty).

A map is a continuous function. I is the unit interval. If $c\colon C \to B$ and $d\colon D \to B$ are given maps then a map $f\colon C \to D$ is *over* B if $df = c$. Let $p\colon B \times Z \to B$ and $r\colon B \times Z \to Z$ be the projections. If $j\colon E \subset B \times Z$ then pj and rj will be written as p and r. If $q\colon T \to B$ is a map and E open in T the composite $E \subset T \to B$ will be denoted by $p\colon E \to B$. If $p\colon E \to B$ and $D \subset B$, let $E_D = E(D) = p^{-1}(D)$. $E_b = E(b)$ is the fiber over b. For a multifunction $m\colon X \to Y$ let $G(m) \subset X \times Y$ be the graph and $p\colon G(m) \to X$ the projection. Note that $p^{-1}(x) = x \times m(x) \equiv m(x)$.

A space is called *N-connected* if every map $\partial I^{j+1} \to X$ can be extended to a map $I^{j+1} \to X$, $0 \leqslant j \leqslant N$. X is *infinitely connected* if it is N-connected for all N.

Proof of Theorem 1.1 assuming $G(m) \to X$ is a Serre fibration. Consider

$$\begin{array}{ccc} A & \xrightarrow{g'} & G(m) \\ \cap & & \cap \\ X & = & X \end{array}$$

where $g'(a) = (a, g(a))$. $G(m) \to X$ is a surjective Serre fibration with fiber $m(x)$ which is infinitely connected – so standard obstruction theory (e.g. [4, p. 404, Th. 22, p. 416, Th. 9]) gives an extension $f'\colon X \to G(m)$ of g', over X, and shows that two such extentions are homotopic, rel(A), over X. Let $f = rf'$. Then f is the extension of g required in the theorem and if f_1 and f_2 are two such and $H\colon f_1' \sim f_2'$ the given homotopy then $rH\colon f_1 \sim f_2$ is the homotopy required in the theorem. ∎

We are left with the problem of proving that $G(m) \to X$ is a Serre fibration. Let us say that a map $E \to B$ is an N-fibration if it has the homotopy lifting property for CW complexes of dimension $\leqslant N$. A Serre fibration is map which is an N-fibration for all N. To prove $E \to B$ an N-fibration it suffices [4, p. 375, p. 416] to prove that it has the homotopy lifting property for cubes of dimension $\leqslant N$. By breaking a cube into smaller cubes it is easily shown that $E \to B$ is an N-fibration if B has an open cover $\{U\}$ such that each $E(U) \to U$ is an N-fibration. In our case $G(m) \to X$ is locally an open subset of a Serre fibration so that the theorem will follow from Theorem 1.2 below.

1.2. Theorem. *Let $T \to B$ be a Serre fibration and E open in T. Assume each $E(b)$ is N-connected and non-empty. Then $E \to B$ is an N-fibration.*

In proving 1.2 the following lemma will be useful:

1.3. Lemma. *Suppose $T \to B$ a Serre fibration, E open in T. Let (X, A) be a relative* CW *complex, $f\colon X \to E$ a map, $G\colon A \times I \to E$ a homotopy with $G_0 = f|A$ and $pG(a, t) = pf(a)$, all a, t. Then G extends to a homotopy $H\colon X \times I \to E$ with $H_0 = f$ and $pH(x, t) = pf(x)$, all x, t.*

Proof of Lemma 1.3. Let $Q = X \times O \cup A \times I \subset X \times I$ and $G' = f \cup G\colon Q \to E$. So we must find a map $H\colon X \times I \to E$ extending G' and over B (where $X \times I \to B$ is $pf\pi$, π the projection). Let $j\colon E \to T$ be the inclusion. Because $T \to B$ is a Serre fibration jG' has an extension to $g\colon X \times I \to T$ over B [4, p. 416, proof of Th. 9]. Since E is open $V = g^{-1}(E)$ is an open nbhd of Q and if F is the restriction of g to V then $F\colon V \to E$ is an extension of G' over B. Because A is closed in X which is normal there is a map $M\colon X \times I \to V$ with M the identity on Q and $\pi M = \pi$ (I is compact, so get $A \times I \subset U \times I \subset V$, U open. Apply the Urysohn lemma to A and $X - U$ to get $e\colon X \to I$, $e(x) = 1$ if x in A, $e(x) = 0$, if x in $X - U$. Define $M(x, t) = (x, e(x)t)$). Now let $H = FM$ and H is the desired map.

2. Proof of Theorem 1.2 and corollaries. The proof is by induction on N. First take $N = 0$. Consider:

$$\begin{array}{ccc} 0 & \xrightarrow{g} & E \subset T \\ {\scriptstyle i}\cap & & \downarrow{\scriptstyle p} \;\swarrow{\scriptstyle q} \\ I & \xrightarrow{f} & B \end{array}$$

Given that $pg = fi$ it is necessary to find an extension $F\colon I \to E$ of g over B. For $t \in I$ choose $e(t) \in p^{-1}f(t) \subset E$. Because $T \to B$ is a 0-fibration there is an $h\colon I \to T$, over B, with $h(t) = e(t) \in E$. $h^{-1}(E)$ is open in I so t has an open nbhd W in I and a map $u\colon W \to E$ over B (u is a restriction of h). Since I is compact there are $0 = a_0 < a_1 < \ldots < a_n = 1$ with $F_i\colon [a_{i-1}, a_i] \to E$, $pF_i = f$.

It is now necessary to modify each F_i to F_i' so that F_1' extends g and $F_i'(a_i) = F_{i+1}'(a_i)$. Then the F_i' fit together to give the desired F. Let $F_1(0) = c$, $g(0) = d$. Since $E(f(0))$ is path connected there is a homotopy $G\colon 0 \times I \to E$ of $F_1|0$ which is over B from c to d. By Lemma 1.3 (with $X = [0, a_1]$, $A = 0$) the homotopy extends to a homotopy of F_1 which is over B. Let F_1' be the other end of the homotopy so that $pF_1' = f$, $F_1'(0) = g(0)$. Similarly F_2 is modified to F_2' so that $pF_2' = f$, $F_2'(a_1) = F_1'(a_1)$, etc.

Now assume 1.2 for $0 \leqslant M < N$ and prove it for N. Consider

$$\begin{array}{ccc} I^N \times 0 & \longrightarrow & E \\ \cap & \nearrow & \downarrow \\ I^N \times I & \longrightarrow & B \end{array} \qquad \text{or} \qquad \begin{array}{ccc} I^{N-1} \times 0 & \longrightarrow & E^I \\ \cap & \nearrow & \downarrow \\ I^{N-1} \times I & \longrightarrow & B^I \end{array}$$

To prove $E \to B$ an N-fibration it is necessary to show the dashed arrow exists in the first diagram. Now let E^I by the space of maps I to E with the compact-open topology. By adjointness (replacing maps $P \times Q \to R$ by maps $P \to R^Q$) this is equivalent to finding the dashed arrow in the second diagram, i.e., showing $E^I \to B^I$ is an $(N-1)$-fibration. Now E is open in T so E^I is open in T^I and also, $T^I \to B^I$ is a Serre fibration. By the induction hypothesis it will suffice to prove $K(u)$ is $(N-1)$-connected and non-empty for each $u \in B^I$ where $K(u) = \{h\colon I \to E|\ ph = u\}$. The fact that $K(u)$ is non-empty follows easily from the fact that $E \to B$ is a 0-fibration. Consider

$$\begin{array}{ccc} \partial I^{j+1} & \longrightarrow & K(u) \\ \cap & \nearrow & \\ I^{j+1} & & \end{array} \qquad \begin{array}{ccc} I \times \partial I^{j+1} & \xrightarrow{f} & E \\ \cap & \nearrow & \\ I \times I^{j+1} & & \Big\downarrow p \\ \downarrow \pi & & \\ I & \xrightarrow{u} & B \end{array}$$

$0 \leqslant j \leqslant N-1$

It suffices to find the dashed arrow in the first diagram – but, again by adjointness, this is equivalent to finding F in the second diagram. Taking the part of the second diagram over t and $u(t)$ gives

$$\begin{array}{ccc} t \times \partial(I^{j+1}) & \xrightarrow{f_t} & E(u(t)) \\ \cap & \nearrow_{e_t} & \\ t \times I^{j+1} & & \end{array}$$

and e_t exists and is unique up to homotopy rel (∂I^{j+1}) since $E(u(t))$ is N-connected. Let $Q(t) = I \times \partial(I^{j+1}) \cup t \times I^{j+1}$ and $f_t = f \cup e_t\colon Q(t) \to E$. Because $T \to B$ is a Serre fibration jf_t has an extension $H_t\colon I \times I^{j+1} \to T$ over B. $H_t^{-1}(E)$ is an open nbhd of $Q(t)$ and by restriction we obtain a map $F_t\colon W \times I^{j+1} \to E$ extending f_t over B where W is a nbhd of t in I. Since I is compact there are $0 = a_0 < a_1 < \ldots < a_n = 1$ and extensions $F_i\colon [a_{i-1}, a_i] \times \times I^{j+1} \to E$ of f over B.

Now the F_i must be modified. Let $F'_1 = F_1$ and $F'_1|a_1 \times I^{j+1} = c$, $F_2|a_1 \times I^{j+1} = d$. Because $E(u(a_1))$ is N-connected d and c are homotopic rel$(a_1 \times \partial I^{j+1})$ as maps to $E(u(a_1)) \subset E$. Now Lemma 1.3 can be applied (with $X = [a_1, a_2] \times I^{j+1}$, $A = a_1 \times I^{j+1}$) to give F'_2 extending f and over B with $F'_2|a_1 \times I^{j+1} = c$. The other F_i are modified by the same procedure and they then fit together to give the required F. ∎

Notes. (1) The proof shows that if $T \to B$ is an $(N+1)$-fibration then $E \to B$ is an N-fibration.

(2) If $E(b)$ in 1.2 is not N-connected then more elaborate hypotheses are required and this will be studied in a separate paper. The most natural generalization would be to assume the $E(b)$'s are N-equivalent – but the following example shows that this is not sufficient.

Let $B = I$, $P = (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$, $Z = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $E = (I \times P) \cup \cup ((\frac{1}{2}, 1] \times Z) \subset I \times Z$. Then E is open in $I \times Z$. Let $p\colon E \to I$ be the composition $E \subset I \times Z \to I$. All of the fibers $E(t)$ are homotopically equivalent (to the discrete space with two points) but $E \to I$ is not a 0-fibration since the identity path: $I \to I$ in the base can not be lifted with given end point $(1, \frac{3}{4})$.

Now consider the following situation

$$\begin{array}{ccc} A & \xrightarrow{g} & E \subset T \\ \cap & & \downarrow \quad \swarrow \\ X & \xrightarrow{f} & B \end{array}$$

2.1. Corollary. *Suppose $T \to B$ a Serre fibration, E open in T, each $E(b)$ non-empty and infinitely connected, (X, A) a relative* CW *complex. Then there is a map $F\colon X \to E$ extending g over B and any two such extensions are homotopic.*

Proof. By 1.2 $E \to B$ is a Serre fibration and the result follows by standard obstruction theory, e.g. [4, p. 404, Th. 22, p. 416, Th. 9].

2.2. Corollary (to the proof). *Suppose X is a compact finite dimensional* ANR *and $m\colon X \to Y$ a subopen multifunction with each $m(x)$ infinitely connected. Then m has a continuous selection and any two such are homotopic by a homotopy $\{f_t\}$ with each f_t a selection for m.*

Proof. Theorem 1.2 and the paragraph preceeding it show that $G(m) \to X$ is a Serre fibration. By [1, p. 122] X is a retract of a finite dimensional polyhedron, say $i\colon X \to P$, $u\colon P \to X$, $ui = id$. As in the first paragraph of the proof of Theorem 1.1 there is a lifting $f'\colon P \to G(m)$, $pf' = u$, and any two such liftings are homotopic. Let $f = rf'i$. Then f is the desired selection and the needed homotopy is similarly defined.

2.3. Corollary. *Suppose X a compact finite dimensional contractible* ANR *and $m\colon X \to X$ a subopen multifunction with contractible values. Then m has a fixed point (i.e., $x \in m(x)$).*

Proof. Corollary 2.2 gives a selection and the Brouwer fixed point theorem applies to the selection to give a fixed point.

Notes. (1) Corollary 2.1 can also be deduced directly from Theorem 1.1. Corollary 2.1 generalizes [5, 2.1, 2.2].

(2) Corollary 2.3 generalizes the fixed point theorem of [3]. The results of the present paper will be used in a separate paper to obtain a more general fixed point theorem of the Lefschetz type.

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Notes on topological games

by

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Abstract. The topological game $G(\boldsymbol{K}, X)$ is in the sense of R. Telgársky. Let $\boldsymbol{K}$ and $\boldsymbol{K}'$ be classes of spaces with $\boldsymbol{K} \subset \boldsymbol{K}'$. It is studied in this paper when Player I has a winning strategy in $G(\boldsymbol{K}, X)$ if he has one in $G(\boldsymbol{K}', X)$. We will discuss three questions of this kind.

§ 1. Each space considered here means a topological space and no separation axioms are assumed unless otherwise stated. Throughout this paper, $\boldsymbol{K}$ denotes a class of spaces which are hereditary with respect to closed subspaces. We need no other assumptions of $\boldsymbol{K}$. When we consider such two classes of spaces, they are denoted by $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$.

The topological game $G(\boldsymbol{K}, X)$ is introduced and studied by R. Telgársky [4]. The detail is seen in it. For a class $\boldsymbol{K}$, $I(\boldsymbol{K})$ denotes the class of all spaces X for which Player I has a winning strategy in $G(\boldsymbol{K}, X)$, and the class $I(I(\boldsymbol{K}))$ is abbreviated to $I^2(\boldsymbol{K})$. Moreover, $\boldsymbol{DK}$, $\boldsymbol{LK}$ and $\boldsymbol{SK}$ denote the classes of all spaces being discrete unions of spaces from $\boldsymbol{K}$, locally $\boldsymbol{K}$ and $\boldsymbol{K}$-scattered, respectively.

The purpose of this paper is to study the following three questions:

(A) What kind of a space X, does $X \in I^2(\boldsymbol{K})$ imply $X \in I(\boldsymbol{K})$ for?

(B) What kind of a space X, does $X \in I(\boldsymbol{SK})$ imply $X \in I(\boldsymbol{DK})$ for?

(C) What kind of a space X, does $X \in I(\boldsymbol{K}_1) \cap I(\boldsymbol{K}_2)$ imply $X \in I(\boldsymbol{K}_1 \cap \boldsymbol{K}_2)$ for?

In § 2, § 3 and § 4, the questions (A), (B) and (C) are answered, respectively. Though the question (B) has been already studied by R. Telgársky [6], we give here another result and the improvement of his ones.

Concerning the topological game $G(\boldsymbol{K}, X)$, we use the notations in [7] rather than in [4]. Here, we do not restate them except the following. Let s be a strategy of Player I in $G(\boldsymbol{K}, X)$. A finite sequence $\langle F_0, F_1, \ldots, F_n\rangle$ of closed sets in X is said to be an *admissible choice of Player* II *for s in* $G(\boldsymbol{K}, X)$ (ad. ch. for II $(s, \boldsymbol{K}, X)$) if $F_0 = X$ and the sequence $\langle E_1, F_1, \ldots, E_n, F_n\rangle$, such that $E_i = s(F_0, F_1, \ldots, F_{i-1})$ for $1 \leqslant i \leqslant n$, is admissible for $G(\boldsymbol{K}, X)$. If s is a winning strategy of Player I in $G(\boldsymbol{K}, X)$, then it should be noted that each infinite sequence $\langle F_0, F_1, \ldots\rangle$ of closed sets in X, such that $\langle F_0, F_1, \ldots, F_n\rangle$ is an ad. ch. for II$(s, \boldsymbol{K}, X)$ for each $n \geqslant 1$, has