[4] T. Jech, *Set Theory*, Academic Press, New York 1978.
[5] K. Kuratowski, *Topology I*, New York–London–Warszawa 1966.
[6] D. A. Martin, *Borel Determinacy*, Ann. of Math. 102 (1975), pp. 363–371.
[7] J. C. Oxtoby, *The Banach–Mazur game and Banach Category theorem*, Contributions to the theory of games iii, Princeton 1957, pp. 159–163.
[8] – *Measure and category*, Springer-Verlag, New York 1971.
[9] K. Schilling, *Properties invariant under infinitary Boolean operations*, Ph. D. Thesis, Berkeley 1981.
[10] – and R. Vaught, *Borel games and the Baire property*, to appear.
[11] R. Sikorski, *Boolean algebras*, Springer-Verlag, New York 1969.

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On Borel-measurable collections of countable-dimensional sets

by

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Abstract. Let B be a Borel set in the product $S\times T$ of compact metrizable spaces, whose vertical sections $B(s)$ are countable-dimensional (i.e. unions of countably many zero-dimensional sets) G_δ-sets in T. It is an open question whether the small transfinite dimension ind of the vertical sections of B is bounded, i.e. if $\sup\{\operatorname{ind} B(s)\colon s\in S\}<\omega_1$. We show that a certain additional assumption about B (an existence of a Borel-measurable, point-finite, sectionwise separation for B, see Definition 3.2) guarantees that this is true.

§ 1. Preliminaries. In this paper we consider only separable metrizable spaces and "compactum" means "compact space". Our terminology concerning analytic sets follows [K] and the terminology related to dimension theory follows [A–P], [E1] and [Na].

1.1. Terminology and notation. A closed set L in a space X *separates* two disjoint sets A and B in X if $X\setminus L=U\cup V$, U and V being disjoint open sets with $A\subset U$ and $B\subset V$. We denote by ω the set of natural numbers, I is the real unit interval and Fin ω is the set of all non-empty finite subsets of ω. We identify the power set $2^{\mathrm{Fin}\,\omega}$ with the Cantor cube $\{0,1\}^{\mathrm{Fin}\,\omega}$, i.e. we identify each subset of Fin ω with its characteristic function and we consider the characteristic functions with pointwise topology. The symbol $|A|$ stands for the cardinality of the set A. A sequence $\langle A_i\colon i\in\omega\rangle$ of subsets of X is *point-finite* if for each $x\in X$ the set $\{i\in\omega\colon x\in A_i\}$ is finite (thus we exclude the possibility that one set occurs in the sequence infinitely many times). Given a set E in the product $S\times T$ we denote by $E(s)$ the vertical section $\{t\in T\colon (s,t)\in E\}$ of the set E at the point $s\in S$.

1.2. Countable-dimensional sets and the small transfinite dimension. A space X is *countable-dimensional* if $X=\bigcup_{i=1}^{\infty}X_i$, X_i being zero-dimensional.

The small transfinite dimension ind is the ordinal-valued function obtained through the extension of the classical Menger–Urysohn inductive dimension by transfinite induction. If the transfinite dimension is not defined for X, we write $\operatorname{ind} X=\infty$; since our spaces have always a countable base, if $\operatorname{ind} X\neq\infty$, then $\operatorname{ind} X<\omega_1$.

By a theorem of Hurewicz, for a completely metrizable space X, $\operatorname{ind} X \neq \infty$ if and only if X is countable-dimensional. We refer the reader for the information about the topic to [A–P; Ch. 10], [Na; Ch. VI], [E2] and [E–EP].

1.3. NAGATA'S THEOREM. *A subspace E of a compactum T is countable-dimensional if and only if there exists a point-finite sequence $\langle L_i\colon i\in\omega\rangle$ of closed subsets of the space E separating in E the pairs of sets with disjoint closures in T (i.e. for each pair of sets A, B in E whose closures in T are disjoint there exists an i such that L_i separates the sets A and B in E).*

More precisely, this is a particular case of Nagata's Theorem [Na; Theorem VI.2] stated in the form convenient for our purposes.

1.4. Brouwer–Kleene order. We shall consider in the set $\operatorname{Fin}\omega$ of all non-empty finite subsets of ω the order $\prec$ inverse to the lexicographic order, i.e. $\sigma\prec\tau$ means that there exists an $n\in\omega$ such that $\sigma\cap\{1,\ldots,n-1\} = \tau\cap\{1,\ldots,n-1\}$ and $n\in\sigma\setminus\tau$.

It is a well-known property of the order $\prec$ that (see [K–M 1; Ch. X, § 7, Corollary 4]): given a decreasing sequence $\sigma_1\succ\sigma_2\succ\ldots$ of elements of $\operatorname{Fin}\omega$, there exists an increasing sequence $j(1)<j(2)<\ldots$ of natural numbers such that for every k there is an n with $\{j(1),\ldots,j(k)\}\subset\sigma_n$.

Let us put:

(1) $$\operatorname{WO}(\operatorname{Fin}\omega) = \{\Gamma\subset\operatorname{Fin}\omega\colon \Gamma \text{ is well-ordered by } \prec\}.$$

The property of order $\prec$ stated above can be reformulated easily in the following way: $\Gamma\in\operatorname{WO}(\operatorname{Fin}\omega)$ iff for each infinite $\tau\subset\omega$ there exists a finite $\nu\subset\tau$ such that no element of Γ contains ν.

For each $\Gamma\in 2^{\operatorname{Fin}\omega}$ we put

(2) $$\operatorname{type}\Gamma = \begin{cases}\text{the order type of } \Gamma, & \text{if } \Gamma\in\operatorname{WO}(\operatorname{Fin}\omega),\\ \infty, & \text{if } \Gamma\notin\operatorname{WO}(\operatorname{Fin}\omega),\end{cases}$$

and let us agree that $\alpha<\infty$ for each ordinal α.

1.5. LUSIN'S COVERING THEOREM [K; § 39, VIII]. *If Λ is an analytic set in $2^{\operatorname{Fin}\omega}$ and $\Lambda\subset\operatorname{WO}(\operatorname{Fin}\omega)$, then $\sup\{\operatorname{type}\Gamma\colon \Gamma\in\Lambda\}<\omega_1$.*

§ 2. The problem. The results of this paper are related to the following open problem:

2.1. PROBLEM. *If B is a Borel set in the product $S\times T$ of compact spaces and each vertical section $B(s)$ of B is a countable-dimensional G_δ-set in T, is it true that $\sup\{\operatorname{ind}B(s)\colon s\in S\}<\omega_1$?*

We shall prove in Section 6 that this problem can be also stated equivalently as follows:

2.2. PROBLEM. *If $\mathscr{K}$ is an upper-semicontinuous decomposition of a compactum T into countable-dimensional compacta, is it true that there exists a countable-dimensional compactum K which contains topologically each member of the collection $\mathscr{K}$?*

We conjecture that the answer to this question is negative (in [P3; § 2] we indicated some difficulties in searching for a counterexample; some other comments about the problem are given in [P2; § 1, sec. 6]).

§ 3. The result. Our main result concerns a certain class of sets in the product $S\times T$ of compact spaces which we describe below.

3.1. DEFINITION. Let B be a set in the product $S\times T$ of compact spaces such that each vertical section $B(s)$ of B is countable-dimensional. Then, by Nagata's Theorem 1.3, for each $s\in S$, there exists a point-finite sequence $\langle L_i(s)\colon i\in\omega\rangle$ of closed subsets of the space $B(s)$ separating in $B(s)$ the pairs of sets with disjoint closures in T. We shall say that the *set B admits a Borel-measurable, point-finite, sectionwise separation* if the sequences $\langle L_i(s)\colon i\in\omega\rangle$ can be chosen in a Borel-measurable way, i.e. for each $i\in\omega$, if $U\subset T$ is open then the set $\{s\in S\colon L_i(s)\cap U\neq\emptyset\}$ is Borel, cf. [K–M 2, p. 405].

3.2. THEOREM. *Let B be a Borel set in the product $S\times T$ of compact spaces such that each vertical section $B(s)$ of B is a countable-dimensional G_δ-set in T. If, moreover, the set B admits a Borel-measurable, point-finite, sectionwise separation (see Definition 3.1), then $\sup\{\operatorname{ind}B(s)\colon s\in S\}<\omega_1$.*

The proof of this result, given in Section 5, is based upon Lusin's Covering Theorem 1.5: we shall assign to each $s\in S$ a set $\Gamma(s)\in\operatorname{WO}(\operatorname{Fin}\omega)$ such that $\operatorname{ind}B(s)\leqslant\operatorname{type}\Gamma(s)$ (see sec. 1.4) and the set $\{\Theta\colon \Theta\subset\Gamma(s)$ for some $s\in S\}\subset\operatorname{WO}(\operatorname{Fin}\omega)$ is analytic. To carry out this task we use a topological invariant, "the transfinite order of a space", which we consider in section 4.

3.3. Remark. Let B be a Borel set in the product $S\times T$ of compact spaces whose vertical sections $B(s)$ are countable-dimensional G_δ-sets in T.

(a) I do not know, whether the set B admits a Borel-measurable point-finite, sectionwise separation (say, under the additional assumption that $\sup\{\operatorname{ind}B(s)\colon s\in S\}<\omega_1$).

(b) If the set B is countable-dimensional, then it satisfies the assumptions of Theorem 3.2 by Nagata's Theorem 1.3. However, the assumptions of Theorem 3.2 do not imply that B is countable-dimensional, even if B is compact and the correspondences $s\to L_i(s)$ in Definition 3.1 are upper-semicontinuous, see Section 7.

(c) If we assume that the vertical sections of the set B are compact, then the assertion of Theorem 3.2 holds true under a weaker assumption that the collections $\langle L_i(s)\colon i\in\omega\rangle$ defined in sec. 3.1 separate only points from closed sets in $B(s)$. The proof of this statement is similar to the proof given in Section 5 (with some essential simplifications possible in this case), see [P3; Remark 1.3]. I do not know, whether this weaker assumption is also sufficient in the general case.

§ 4. The transfinite order of a space. We shall define in this section a topological invariant which will be used in the proof of Theorem 3.2 in § 5 (cf. also sec. 7.2 for some comments about this invariant).

4.1. The transfinite order of a sequence of sets. Given a sequence $\langle A_i\colon i\in\omega\rangle$ of subsets of a set X, let us define (see sec. 1.4, (2)):

(1) $$\operatorname{Ord}\langle A_i\colon i\in\omega\rangle = \operatorname{type}\{\sigma\in\operatorname{Fin}\omega\colon \bigcap_{i\in\sigma} A_i\neq\emptyset\}.$$

It follows, by the properties of Brouwer–Kleene order stated in sec. 1.4, that $\operatorname{Ord}\langle A_i\colon i\in\omega\rangle\neq\infty$ if and only if no infinite subsequence $\langle A_{i_j}\colon j\in\omega\rangle$ has the finite intersection property (cf. the notion of a strongly point-finite family of sets considered by R. Engelking and the author in [E–RP]).

4.2. Definition. Given a separable metrizable space X we define the transfinite order of the space X by the formula (see (1)):

(2) $\operatorname{ord} X = \min\{\operatorname{Ord}\langle \operatorname{Fr} B_i\colon i\in\omega\rangle\colon \langle B_i\colon i\in\omega\rangle$ is an open base in $X\}$,

where $\operatorname{Fr} E$ is the boundary of the set E.

The remark in Section 4.1 and a result proved in [E–RP; Theorem 2] show that, for an arbitrary space X, the conditions $\operatorname{ind} X\neq\infty$ and $\operatorname{ord} X\neq\infty$ are equivalent. Moreover, we have the following fact (cf. also sec. 7.2):

4.3. Lemma. *For any space X, $\operatorname{ind} X\leqslant \operatorname{ord} X$.*

Proof. We proceed by transfinite induction. If $\operatorname{ord} X=0$ then there exists a base $\langle B_i\colon i\in\omega\rangle$ in X such that $\operatorname{Ord}\langle \operatorname{Fr} B_i\colon i\in\omega\rangle=0$, so each $\operatorname{Fr} B_i$ is empty and $\operatorname{ind} X=0$. Assume that $\alpha>0$ is a countable ordinal and, for each space Y,

(3) $$\operatorname{ind} Y\leqslant \operatorname{ord} Y,\quad \text{whenever } \operatorname{ord} Y<\alpha,$$

and let X be a space such that

(4) $$\operatorname{ord} X=\alpha.$$

Let us consider an open base $\langle B_i\colon i\in\omega\rangle$ in X such that

(5) $$\operatorname{Ord}\langle \operatorname{Fr} B_i\colon i\in\omega\rangle=\alpha.$$

We claim that for each $m\in\omega$

(6) $$\operatorname{ind}\operatorname{Fr} B_m<\alpha.$$

Since this is true when $\operatorname{Fr} B_m$ is a discrete space, let us assume that the set Y of non-isolated points of the space $\operatorname{Fr} B_m$ is non-empty. Thus

(7) $$Y\subset \operatorname{Fr} B_m \quad\text{and}\quad \operatorname{ind} Y=\operatorname{ind}\operatorname{Fr} B_m,$$

and let

(8) $$G_i=Y\cap B_{m+i},\quad i\in\omega.$$

Then $\langle G_i\colon i\in\omega\rangle$ is an open base in the space Y and

(9) $$\operatorname{Fr}_Y G_i\subset \operatorname{Fr} B_{m+i},$$

Fr_Y being the boundary operator in the space Y. For each $\sigma\in\operatorname{Fin}\omega$ we put

$$\sigma^*=\{m\}\cup\{m+i\colon i\in\sigma\}.$$

The correspondence $\sigma\to\sigma^*$ preserves the order $\prec$ and, by (9) and (7), if $\bigcap_{i\in\sigma}\operatorname{Fr}_Y G_i\neq\emptyset$, then $\operatorname{Fr} B_m\cap\bigcap_{i\in\sigma}\operatorname{Fr} B_{m+i}\neq\emptyset$. Thus, if we set $\Gamma=\{\sigma\in\operatorname{Fin}\omega\colon \bigcap_{i\in\sigma}\operatorname{Fr}_Y G_i\neq\emptyset\}$ and $\Lambda=\{\sigma\in\operatorname{Fin}\omega\colon \bigcap_{i\in\sigma}\operatorname{Fr} B_i\neq\emptyset\}$, we see that $\{\sigma^*\colon \sigma\in\Gamma\}\subset\Lambda$. Moreover, $\{m\}\in\Lambda$ and $\sigma^*\prec\{m\}$ for each $\sigma\in\Gamma$, so $\operatorname{type}\Gamma+1\leqslant\operatorname{type}\Lambda$. Now (see (1) and (2)), $\operatorname{ord} Y\leqslant\operatorname{type}\Gamma$ and $\operatorname{type}\Lambda=\alpha$, by (5), so $\operatorname{ord} Y<\alpha$, hence by the inductive assumption (3) and by (7) we obtain $\operatorname{ind}\operatorname{Fr} B_m=\operatorname{ind} Y\leqslant\operatorname{ord} Y<\alpha$, just proving (6).

By (6) we have $\operatorname{ind} X\leqslant\alpha$ which completes the induction.

§ 5. Proof of Theorem 3.2. Let $B\subset S\times T$ satisfy the assumptions of Theorem 3.2 and let $\langle L_i(s)\colon i\in\omega\rangle$ be sequences such as in Definition 3.1. By a theorem of Saint-Raymond [S], B being a Borel set with G_δ vertical sections, there exist Borel sets $C_1\subset C_2\subset\ldots$ in $S\times T$ such that each vertical section $C_i(s)$ is compact ($i\in\omega$, $s\in S$) and $(S\times T)\setminus B=\bigcup_{i=1}^{\infty} C_i$.

(I) *Let, for each $\sigma\in\operatorname{Fin}\omega$, $L_\sigma(s)=\bigcap_{i\in\sigma}\overline{L_i(s)}$, the closure being taken in T, and for each $s\in S$, $\Gamma(s)=\{\sigma\in\operatorname{Fin}\omega\colon L_\sigma(s)\neq\emptyset$ and $L_\tau(s)\cap C_{|\tau|}(s)=\emptyset$ for each $\tau\subset\sigma\}$. Then, for each $s\in S$, $\Gamma(s)\subset\operatorname{WO}(\operatorname{Fin}\omega)$ (see sec. 1.4.).*

Proof. Suppose on the contrary that for some $s\in S$ there exists a descending sequence $\sigma_1\succ\sigma_2\succ\ldots$ of members of $\Gamma(s)$. Let $j(1)<j(2)<\ldots$ be an increasing sequence of natural numbers such that $\tau_p=\{j(1),\ldots,j(p)\}\subset\sigma_{i(p)}$ for $p=1,2,\ldots$ (see sec. 1.4). Then, for each p

$$L_{\tau_p}(s)\neq\emptyset\quad\text{and}\quad L_{\tau_p}(s)\cap C_p(s)=\emptyset.$$

Now, $L_{\tau_1}(s)\supset L_{\tau_2}(s)\supset\ldots$, $\bigcap_i L_{\tau_i}(s)\subset B(s)$, and $L_{\tau_p}(s)\cap B(s)=\bigcap\{L_{j(i)}(s)\colon i\leqslant p\}$, so the compactness of T yields that $\bigcap\{L_{j(i)}(s)\colon i\in\omega\}\neq\emptyset$, contrary to the assumption that the sequence $\langle L_i(s)\colon i\in\omega\rangle$ was point-finite.

(II) *Let $\Gamma(s)$ be as in* (I). *Then (see sec.* 1.4 *and* 4.2): $\operatorname{ind} B(s)\leqslant\operatorname{type}\Gamma(s)$, *for each $s\in S$.*

Proof. Fix a point $s\in S$, Let $B_1, B_2,\ldots$ be a collection of open sets in the space T such that

(1) $$B_i \cap C_i(s) = \emptyset, \quad i \in \omega,$$

(2) $\{B_i \cap B(s)\colon i \in \omega\}$ is a base in the space $B(s)$.

Let $V(i, j)$ be open sets in T such that

(3) $$\overline{V(i, k)} \subset V(i, k+1) \subset \overline{V(i, k+1)} \subset B_i, \quad \bigcup_k V(i, k) = B_i,$$

and let

$$A_{ik} = \overline{V(i, k)} \cap B(s), \quad B_{ik} = B(s) \setminus V(i, k+1).$$

For each pair of indices (i, k) choose a set L_{ik} from the collection $L_1, L_2, \ldots$ such that $L_{ik}(s)$ separates in the space $B(s)$ the sets A_{ik} and B_{ik} (note that A_{ik} and B_{ik} have disjoint closures in T). By (3),

(4) $$L_{ik}(s) \cap L_{il}(s) = \emptyset \quad \text{for} \quad k \neq l.$$

Let us choose a minimal set $\omega' \subset \omega$ with the property that for each pair (i, k) there exists exactly one $j \in \omega'$ such that $L_{ik}(s) = L_j(s)$. Now, the sets L_{ik} were chosen in such a way that the sets $L_j(s)$, $j \in \omega'$, separate in $B(s)$ the points from closed sets, and therefore there is an open base $\langle G_j\colon j \in \omega' \rangle$ in $B(s)$ such that $\operatorname{Fr} G_j \subset L_j(s)$ for $j \in \omega'$. Therefore (see sec. 4.2, 4.1 and Lemma 4.3)

$$\operatorname{ind} B(s) \leqslant \operatorname{ord} B(s) \leqslant \operatorname{Ord} \langle L_j(s)\colon j \in \omega' \rangle,$$

and hence it is enough to show that (see sec. 4.1 and 1.4 (2))

(5) $$\{\sigma \in \operatorname{Fin}\omega\colon \sigma \subset \omega' \text{ and } \bigcap_{j \in \sigma} L_j(s) \neq \emptyset\} \subset \Gamma(s).$$

Let $\sigma \subset \omega'$ be finite and let

(6) $$\bigcap_{j \in \sigma} L_j(s) \neq \emptyset.$$

For each $j \in \sigma$ let (i_j, k_j) be such that $L_j(s) = L_{i_j k_j}(s)$. Thus $\overline{L_j(s)} \cap C_{i_j}(s) = \emptyset$ (see (1) and (3)) and since the numbers i_j are distinct by (4), $\max\{i_j\colon j \in \sigma\} \geqslant |\sigma|$, and therefore $L_\sigma(s) = \bigcap_{j \in \sigma} \overline{L_j(s)} \subset B(s) \setminus C_{|\sigma|}(s)$. We have checked this way that (6) implies $L_\sigma(s) \cap C_{|\sigma|}(s) = \emptyset$, and thus (5) holds (cf. (I)) which completes the proof of (II).

(III) *The set $\Lambda = \{\Theta \in 2^{\operatorname{Fin}\omega}\colon \Theta \subset \Gamma(s)$ for some $s \in S\}$ is analytic, $\Gamma(s)$ being as in* (I).

Proof. Let us observe that (cf. (I))

$$L_\sigma = \{(s, t)\colon t \in \bigcap_{i \in \sigma} \overline{L_i(s)}\} \text{ is Borel}, \quad \sigma \in \operatorname{Fin}\omega.$$

Indeed, each set $L_{(i)}$ is Borel (as, for an arbitrary open countable base $\mathscr{V}$ in T, $L_{(i)} = S \times T \setminus \bigcup_{V \in \mathscr{V}} \{s \in S\colon L_i(s) \cap V = \emptyset\} \times V$) and $L_\sigma = \bigcap_{i \in \sigma} L_{(i)}$.

Now, L_σ and $L_\sigma \cap C_{|\sigma|}$ are Borel sets whose vertical sections are compact, and therefore, by Kunugui–Novikov Theorem [K–M 2; p. 471] the following sets are Borel:

$$A_\sigma = \{s \in S\colon L_\sigma(s) \neq \emptyset\} = \operatorname{proj}_S L_\sigma,$$
$$E_\sigma = \{s \in S\colon L_\sigma(s) \cap C_{|\sigma|}(s) = \emptyset\} = S \setminus \operatorname{proj}_S(L_\sigma \cap C_{|\sigma|}),$$
$$S_\sigma = A_\sigma \cap \bigcap \{E_\tau\colon \tau \subset \sigma\},$$

and so are the sets

$$\Sigma_\sigma = \{\Theta\colon \sigma \notin \Theta\} \times S \cup \{\Theta\colon \sigma \in \Theta\} \times S_\sigma \subset 2^{\operatorname{Fin}\omega} \times S.$$

It remains to observe that the set Λ is the projection onto $2^{\operatorname{Fin}\omega}$ of the intersection $\bigcap \{\Sigma_\sigma\colon \sigma \in \operatorname{Fin}\omega\}$.

(IV) We are ready now to complete the proof. Let us consider the analytic set Λ defined in (III). By (I), $\Lambda \subset \operatorname{WO}(\operatorname{Fin}\omega)$ and hence Lusin's Covering Theorem 1.5 and (II) yield:

$$\sup\{\operatorname{ind} B(s)\colon s \in S\} \leqslant \sup\{\operatorname{type} \Gamma(s)\colon s \in S\} = \sup\{\operatorname{type} \Theta\colon \Theta \in \Lambda\} < \omega_1.$$

§ 6. Proof that Problems 2.1 and 2.2 are equivalent.

(I) Let $\mathscr{K}$ be an upper-semicontinuous decomposition of a compactum T into countable-dimensional compacta. The collection $\mathscr{K}$ is an analytic set in the hyperspace 2^T (i.e. in the space of compact subsets of T endowed with the Hausdorff metric), and hence there exists a continuous map $\Phi\colon P \to \mathscr{K}$ of the irrationals $P \subset [0, 1] = S$ onto $\mathscr{K}$. Let $B = \{(s, t)\colon t \in \Phi(s)\}$. Then B is a G_δ-set in the product $S \times T$ and $\{B(s)\colon s \in S\} \in \mathscr{K}$. Therefore, if the solution of Problem 2.1 is positive, then so is the solution of Problem 2.2.

(II) Suppose that the solution of Problem 2.1 is negative, and let B be a Borel set in the product $S \times T$ of compact spaces whose vertical sections are countable-dimensional G_δ-sets and $\sup\{\operatorname{ind} B(s)\colon s \in S\} = \omega_1$.

Since the complement $S \times T \setminus B$ is a Borel set with σ-compact sections, a theorem of Saint-Raymond [S] yields an existence of Borel sets with σ-compact sections B_i such that $S \times T \setminus B = B_1 \cup B_2 \cup \ldots$ Next, by Kunugui–Novikov Theorem [K–M 2; p. 471], each map $s \to B_i(s)$ from S to the hyperspace 2^T of the compactum T (cf. (I)) is Borel-measurable, and therefore the set

$$\mathscr{A} = \{\langle s, B_1(s), B_2(s), \ldots \rangle\colon s \in S\} \subset S \times 2^T \times 2^T \times \ldots$$

is analytic. Thus there exists a continuous mapping $\Phi\colon P \to \mathscr{A}$ of the irrationals $P \subset [0, 1] = I$ onto $\mathscr{A}$, where

$$\Phi(u) = \langle f(u), T_1(u), T_2(u), \ldots \rangle \in \mathscr{A}.$$

Let us consider the following sets in the product $L = I \times S \times T$:

$$H = \{\langle u, f(u), t \rangle\colon u \in P, t \in T\}$$

and

$$L_i = \{\langle u, f(u), t\rangle:\ u\in P,\ t\in T_i(u) = B_i(f(u))\}.$$

These sets are closed in the space $P\times S\times T$, as the functions f and T_i are continuous. Let us put

$$E = H\backslash(L_1\cup L_2\cup \ldots).$$

Then E is a G_δ-set in L and, since $E(u, f(u)) = T\backslash \bigcup_{i=1}^{\infty} B_i(f(u)) = B(f(u))$, one gets also

(1) $$\sup\{\operatorname{ind} E(u):\ u\in P\} = \omega_1.$$

By a theorem of Kuratowski [K2; Théorème 2], E being a G_δ-set in L, there exists a continuous mapping $g\colon L\to M$ onto a compactum M such that the restriction $g|E\colon E\to g(E)$ is a homeomorphism and $g(L\backslash E) = M\backslash g(E)$ is a union of countably many polytopes (and hence – countable-dimensional). Let $p\colon I\times S\times T\to I$ be the projection and let $h = (p, g)\colon L\to I\times M$ be the diagonal mapping. Then the projection $I\times M\to I$ restricted to the compactum $K = h(L)$ is a mapping $q\colon K\to L$ such that $q\circ h = p$. Let us consider the upper-semicontinuous decomposition $\mathscr{K} = \{q^{-1}(u):\ u\in I\}$ of the compactum K. Since $q^{-1}(u) = h(p^{-1}(u))$, it follows that $q^{-1}(u)$ is the union of a set homeomorphic to $E(u)$ and the countable-dimensional set $q^{-1}(u)\backslash h(E)$, hence $q^{-1}(u)$ is countable dimensional, but also $\operatorname{ind} q^{-1}(u) \geqslant \operatorname{ind} E(u)$, and therefore by (1) there exists no countable-dimensional compactum containing topologically all members of the collection $\mathscr{K}$. Thus $\mathscr{K}$ provides a negative solution of Problem 2.2

§ 7. Comments.

7.1. Example. *There exists a compact set B in the product $S\times T$, S being the Cantor set and T the Hilbert cube, such that: each vertical section $B(s)$ is countable-dimensional, there exist upper-semicontinuous correspondences $s\to L_i(s)$ (i.e. $\{s:\ L_i(s)\cap F\neq \emptyset\}$ is closed for any closed $F\subset T$) such that the collections $\langle L_i(s):\ i\in\omega\rangle$ satisfy the assumptions of Theorem 3.2, but the compactum B is not countable-dimensional* (*cf.* [P3; § 5]).

Proof. In [P1] a compactum B in the product $S\times T$ of compact spaces was constructed, such that B is not countable-dimensional, but there exists a function $f\colon S\to T$ of the first Baire class such that $f(s)\in B(s)$ and $B\backslash G(f)$ is countable-dimensional, $G(f)$ being the graph of f (thus $B(s) = \{f(s)\}\cup(B\backslash G(f))$ is countable-dimensional). The space $B\backslash G(f)$ being countable-dimensional, there exists a point-finite sequence $\langle E_i:\ i\in\omega\rangle$ of closed subsets of $B\backslash G(f)$ separating in $B\backslash G(f)$ the pairs of sets with disjoint closures in $S\times T$ (see Nagata's Theorem 1.3). Let $T_1, T_2, \ldots$ be a countable collection of closed subsets of T such that for each pair (C, D) of disjoint closed sets in T there exists an i with $C\subset \operatorname{Int} T_i\subset T_i\subset T\backslash D$. Now, $f^{-1}(T_i)$ being a G_δ-set in the Cantor set S, there exist disjoint compact sets $S_{i1}, S_{i2}, \ldots$ such that $S\backslash f^{-1}(T_i) = S_{i1}\cup S_{i2}\cup\ldots$ Put $L_{ij}(k) = E_k\cap(S_{ij}\times T_i)$ and arrange the collection $\langle L_{ij}(k):\ i\leqslant k,\ j = 1, 2, \ldots\rangle$ into a sequence $\langle L_i:\ i\in\omega\rangle$. Then the sequence $\langle L_i:\ i\in\omega\rangle$ is point-finite, the sets L_i are compact (notice that $L_i\cap G(f) = \emptyset$) and therefore the correspondences $s\to L_i(s)$ are upper-semicontinuous. Let C and D be closed disjoint subsets of $B(s)$. Changing, if necessary, the notation and adding the point $f(s)$ to one of these sets, one can assume that the pair (C, D) is such that $f(s)\in D$. Let i be such that $C\subset \operatorname{Int} T_i\subset T_i\subset T\backslash D$. There are infinitely many indices k for which $E_k(s)$ separates in $B\backslash G(f)$ the sets C and $T\backslash \operatorname{Int} T_i$ (recall that T is the Hilbert cube), so one can choose such an index $k\geqslant i$. Now, $E_k(s)\subset T_i$ and since $f(s)\notin T_i$, there exists a j with $s\in S_{ij}$, so $L_{ij}(k)(s) = E_k(s)$ and, for some m, $L_m = L_{ij}(k)$. Therefore the sequence $\langle L_i(s):\ i\in\omega\rangle$ separates in $B(s)$ the pairs of disjoint closed sets.

Remark. In particular, Theorem 3.2 shows that, for the compactum B considered in the above proof, $\sup\{\operatorname{ind} B(s):\ s\in S\} < \omega_1$. A direct proof of this fact is given in [P3; sec. 5.3 (I)].

7.2. A connection with the "minimal index" of P. S. Novikov. We shall show that, in the realm of the class of compacta, the small transfinite dimension provides essentialy the same classification as a certain topological invariant (similar to $\operatorname{ord} X$, see § 4) which can be interpreted as a "minimal index" of P. S. Novikov [No].

Given a compactum X let (see sec. 4.1 and 4.2)

(1) $$\operatorname{Ord} X = \min\{\operatorname{Ord}\langle \operatorname{Fr} B_i:\ i\in\omega\rangle:\ \langle B_i:\ i\in\omega\rangle \text{ is an open base for closed sets in } X\},$$

where a collection of open sets $\langle B_i:\ i\in\omega\rangle$ is a base for closed sets in X if for each closed set F and an open set U containing F there exists an i with $F\subset B_i\subset U$.

Obviously, $\operatorname{ord} X\leqslant \operatorname{Ord} X$, so Lemma 4.3 yields the inequality $\operatorname{ind} X\leqslant \operatorname{Ord} X$. Actually, there exists a monotone function Φ which maps countable ordinals to countable ordinals and ∞ to ∞, such that for each compactum X,

(2) $$\operatorname{ind} X\leqslant \operatorname{Ord} X\leqslant \Phi(\operatorname{ind} X).$$

To define the function Φ, let us consider for each $\alpha<\omega_1$ a countable-dimensional compactum X_α containing topologically all compacta S with $\operatorname{ind} S\leqslant \alpha$, see [P 2; Corollary 1.3]. The arguments given in sec. 4.2 show that, given a compactum X, $\operatorname{Ord} X\neq\infty$ if and only if X is countable dimensional. Thus, for each $\alpha<\omega_1$, $\Phi(\alpha) = \sup\{\operatorname{Ord} X_\beta:\ \beta\leqslant\alpha\}$ is a countable ordinal and it is easy to check that the function augumented by the condition $\Phi(\infty) = \infty$ satisfies (2).

Let $\underline{H}$ be the hyperspace of the Hilbert cube I^ω, and let $\underline{C} = \{X \in \underline{H}: X$ is countable-dimensional$\}$. Let (A_1, B_1), (A_2, B_2), ... be a sequence of pairs of disjoint closed sets in I^ω such that for each pair of disjoint closed sets in I^ω there exists an i such that $A \subset A_i$ and $B \subset B_i$. Let $\underline{L}_i$ be the set of all partitions in I^ω between the sets A_i and B_i. The space $\underline{L}_i$ being an intersection of a closed and an open set in $\underline{H}$ is topologically complete, and so is the product $\underline{S} = \underline{L}_1 \times \underline{L}_2 \times \ldots$ Let us define a closed Lusin sieve $\mathscr{W} = \{\underline{W}_\sigma: \sigma \in \mathrm{Fin}\,\omega\}$ in the product space $\underline{H} \times \underline{S}$ by letting

$$\underline{W}_\sigma = \{\langle X, L_1, L_2, \ldots\rangle: X \cap \bigcap_{i \in \sigma} L_i \neq \emptyset\}.$$

Let, for $\langle X, L_1, L_2, \ldots\rangle \in \underline{H} \times \underline{S}$,

$$M\langle X, L_1, L_2, \ldots\rangle = \{\sigma \in \mathrm{Fin}\,\omega: \langle X, L_1, L_2, \ldots\rangle \in \underline{W}_\sigma\}.$$

Let us recall that the set $L(\mathscr{W})$ sifted by the sieve $\mathscr{W}$ is given by the formula (cf. sec. 1.4)

$$\langle X, L_1, L_2, \ldots\rangle \in L(\mathscr{W}) \equiv M\langle X, L_1, L_2, \ldots\rangle \in \mathrm{WO}(\mathrm{Fin}\,\omega),$$

and let us recall that

$$\mathrm{Index}\langle X, L_1, L_2, \ldots\rangle = \mathrm{type}\, M\langle X, L_1, L_2, \ldots\rangle$$

is the Lusin–Sierpiński index of the point $\langle X, L_1, L_2, \ldots\rangle$ with respect to $\mathscr{W}$. Now (see sec. 4.1)

$$\mathrm{Index}\langle X, L_1, L_2, \ldots\rangle = \mathrm{Ord}\langle X \cap L_i: i \in \omega\rangle$$

and this easily yields the equality

$$\text{(3)} \qquad \mathrm{Ord}\,X = \min\{\mathrm{Index}\langle X, L_1, L_2, \ldots\rangle: \langle L_1, L_2, \ldots\rangle \in \underline{S}\}.$$

Now, the formula on the right hand side is the definition of a "minimal index" of P. S. Novikov [No; § 2] for the set $\underline{C} = \mathrm{proj}_{\underline{H}}[(\underline{H} \times \underline{S}) \setminus L(\mathscr{W})]$. Thus (2) and (3) show that ind behaves over $\underline{C}$ like a "minimal index" of P. S. Novikov. We do not know, however, whether, actually, $\underline{C}$ is a coanalytic set (we have seen that it is a projection of a coanalytic set) and whether ind behaves like a Lusin–Sierpiński index over $\underline{C}$ (this last question is in fact yet another reformulation of Problem 2.1).

Remark. Formula (2) shows that for each compactum X, $\mathrm{ind}\,X \leqslant \mathrm{ord}\,X \leqslant \Phi(\mathrm{ind}\,X)$. We do not know whether a similar assertion is still valid in the class of all topologically complete spaces, cf. [E1; Problem 5.10].

References

[A–P] P. S. Aleksandrov and B. A. Pasynkov, *Introduction to Dimension Theory* (Russian), Moskva 1973.

[E1] R. Engelking, *Dimension Theory*, Warszawa 1978.

[E2] R. Engelking, *Transfinite dimension*, in: *Surveys in General Topology*, ed. G. M. Reed, New York 1980, pp. 131–161.

[E–EP] – and E. Pol, *Countable-dimensional spaces: a survey*, Diss. Math. 216, Warszawa 1983.

[E–RP] – and R. Pol, *Some characterization of spaces that have transfinite dimension*, Topology and Appl. 15 (1983), pp. 247–253.

[K 1] K. Kuratowski, *Topology*, vol. I and II, Warszawa 1966 and 1968.

[K 2] – *Sur les prolongement des fonctions continues et les transformations en polytopes*, Fund. Math. 24 (1935), pp. 258–268.

[K–M 1] – and A. Mostowski, *Set Theory*, Amsterdam 1968.

[K–M 2] – – *Set theory with an introduction to descriptive set theory*, Warszawa 1976.

[Na] J. Nagata, *Modern Dimension Theory*, Amsterdam 1965.

[No] P. S. Novikov, *Sur la séparabilité des ensembles projectifs de seconde classe*, Fund. Math. 25 (1935), pp. 459–466.

[P1] R. Pol, *A weakly infinite-dimensional compactum which is not countable-dimensional*, Proc. Amer. Math. Soc. 82 (1981), pp. 634–636.

[P2] – *On classification of weakly infinite-dimensional compacta*, Fund. Math. 116 (1983), pp. 169–188.

[P3] – *On analytic-measurable collections of countable-dimensional and weakly infinite-dimensional compacta*, Proc. Fifth. Prague Top. Symp. 1981, Helderman Verlag, 1983.

[S] J. Saint-Raymond, *Boréliens á coupes $\mathscr{K}_\sigma$*, Bull. Soc. Math. France 104 (1976), pp. 389–406.

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