On absolutely $A_{2}^{1}$ operations

by

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Abstract. Every absolutely $A_{2}^{1}$ Boolean operation preserves the Baire property in all topological spaces, and, as a consequence, measurability in all $\sigma$-finite complete measure spaces.

It is a classical theorem that the operation $(\lambda)$ preserves the Baire property in all topological spaces, and measurability in all $\sigma$-finite complete measure spaces.

R. Solovay (unpublished) introduced the class of absolutely $A_{2}^{1}$ sets (to be defined in the next section) in Polish spaces, and proved that they have the Baire property, and are Lebesgue measurable. Solovay's results were rediscovered and extended by Fenstad and Normann [3], who showed that an absolutely $A_{1}^{1}$ set in an analytic space is measurable with respect to any $\sigma$-finite, complete, regular Borel measure.

In order to extend these results further, R. Vaught (unpublished; announced at Wroclaw, 1977) considered the absolutely $A_{2}^{1}$ Boolean operations, and showed that these operations preserve the Baire property in any topological space satisfying the countable chain condition, and measurability with respect to any $\sigma$-finite complete measure.

The main result here, in analogy with and extending the classical theorem cited above, is

Theorem 3.3. All absolutely $A_{2}^{1}$ Boolean operations preserve the Baire property in all topological spaces.

From 3.3, using a theorem in [8], we directly infer the part of Vaught's result dealing with measure.

Now let $\mathfrak{B}$ be an arbitrary $\sigma$-field of sets on a set $X$, and let $I$ be a $\sigma$-ideal on $X$ such that $I \subseteq \mathfrak{B}$. Vaught proved

Theorem 4.1. If the Boolean algebra $\mathfrak{B}/I$ satisfies the countable chain condition, then $\mathfrak{B}$ is invariant under all absolutely $A_{2}^{1}$ Boolean operations.

We cannot infer 4.1 directly from 3.3. However, we do show that, by introducing a simple device, the pattern of our proof of 3.3 carries over into a new proof of 4.1.

Most of the material herein appears in the author's doctoral dissertation [9]. I am grateful to my thesis advisor, Robert Vaught, for his help in all aspects of its preparation.
Section 1. Preliminaries. We assume some familiarity with the terminology of set-theoretic forcing and Boolean-valued models. Our notation is taken from [4].

A set $S \subseteq \mathcal{P}(o)$ is absolutely $\mathcal{A}_4$ if there exist $\mathcal{Z}_2$ formulas $\Phi$ and $\Psi$ and a parameter $\tau \in \mathcal{P}(o)$, such that

$$\forall x \in \mathcal{A}_4 \Phi(x, \tau) \iff \neg \Psi(x, \tau)$$

for all $x \in x$ and all $s \in S$ if and only if $\Phi(x, \tau)$ holds, and if and only if $\neg \Psi(x, \tau)$ holds, and for all complete Boolean algebras $\mathcal{B}$,

$$\forall x \in \mathcal{A}_4 \phi(\mathcal{A}_4, \tau) \equiv \neg \psi(\mathcal{A}_4, \tau)$$

For some basic facts about the class of absolutely $\mathcal{A}_4$ sets, see [3].

Let $S$ be any subset of $\mathcal{P}(o)$. For any non-empty set $X, S$ determines an $\omega$-ary operation $\mathcal{E}_S$ on $\mathcal{P}(X)$ where

$$\mathcal{E}_S(A_n, n \in \omega) = \{x \in X : \forall n \in A_n \in \mathcal{E}_S\}$$

Operations so obtained are called Boolean. Boolean operations have many pleasant and easily verified properties. For example, the reader will easily show

$$\mathcal{E}_S(A_n, n \in \omega) \cup \mathcal{E}_S(B_n, n \in \omega) \subseteq \mathcal{E}_S(A \cup B)$$

Now the key definition: The Boolean operation $\mathcal{E}_S$ is said to be an absolutely $\mathcal{A}_4$ Boolean operation provided $S$ is an absolutely $\mathcal{A}_4$ subset of $\mathcal{P}(o)$.

Examples of absolutely $\mathcal{A}_4$ Boolean operations are the operation (4), the $R$-operations of Kolmogorov, and the Borel game operations [10]. That the Borel game operations are absolutely $\mathcal{A}_4$ is a consequence of D. Martin's Borel determinacy theorem [6]. In [10], R. Vaught and the present author prove that the Borel game operations preserve the Baire property in all topological spaces (a special case of our main theorem) without appealing to the fact that these operations are absolutely $\mathcal{A}_4$.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be structures. We write $\mathcal{A}_2 < \mathcal{A}_1$ to mean that $\mathcal{A}_1$ is an elementary substructure of $\mathcal{A}_1$, in the usual sense of model theory (as in, e.g., [1]). The Löwenheim-Skolem theorem [1] says that for any structure $\mathcal{A}_1$, and any countable subset $A$ of the universe of $\mathcal{A}_1$, there exists a countable structure $\mathcal{A}_2$ such that $\mathcal{A}_2 < \mathcal{A}_1$, and the universe of $\mathcal{A}_2$ includes $A$. The letter $\mathcal{B}$ will always denote the Boolean algebra $(B, \wedge, \lor, -)$, where $\wedge$, $\lor$, and $-$ are $\mathcal{A}_3$.

Let $D$ be an ultrafilter over $\mathcal{B}$, and $S$ a subset of $B$ such that $\mathcal{A}_S$ exists. We say that $D$ preserves the infimum $\mathcal{A}_S$ provided that $\mathcal{A}_S$ is $D$-id if, and only if, $S \subseteq D$. For any set $M$, the ultrafilter $D$ is called $M$-generic if, for all $S \subseteq M$ such that $S \subseteq B$ and $\mathcal{A}_S$ exists, $D$ preserves the infimum $\mathcal{A}_S$. The Rasiowa-Sikorski theorem [11] states that, if $M$ is a countable set, then there exists an ultrafilter $D$ over $\mathcal{B}$ which is $M$-generic.

Section 2. The main lemma. In this section we will prove a lemma which is a combination of the Löwenheim-Skolem and Rasiowa-Sikorski theorems, in a game setting. Our proof is a modification of well-known proof of these two theorems.

Given a Boolean algebra $\mathcal{B}$, a transitive (in the usual sense of set theory) set $\mathcal{W}$, and a countable set $T \subseteq \mathcal{W}$, we will find a countable set $M \subseteq \mathcal{W}$ and a filter $\mathcal{D}$ over $\mathcal{B}$ such that

1. $(\mathcal{M}, \mathcal{E}) < (\mathcal{W}, \mathcal{E})$,
2. $\mathcal{D} \cap \mathcal{M}$ is an ultrafilter over $\mathcal{B} \cap \mathcal{M}$,
3. $(\forall A \in \mathcal{M})$ if $A \subseteq B$ and $\mathcal{A}_B$ exists, then $\mathcal{A}_B \subseteq \mathcal{D} \cap \mathcal{M}$ or $(\mathcal{B} \in \mathcal{A}_B \cap \mathcal{M} \in \mathcal{D})$.

Note that (3) implies that $\mathcal{D} \cap \mathcal{M}$ is $M$-generic over $\mathcal{B} \cap \mathcal{M}$. Below, $p_i$ and $q_i$ are to range over non-zero elements of $\mathcal{B}$, and $a_i$ over arbitrary sets, for $i = 0, 1, 2, \ldots$.

Lemma 2.1. Let $\mathcal{B}$ be a Boolean algebra, $\mathcal{W}$ a transitive set such that $\mathcal{B} \subseteq \mathcal{W}$, $T \subseteq \mathcal{W}$ a countable set, and $d \subseteq B$. Then

$$\forall p_0 < d \exists q_0 \leq p_0 \exists a_0 \in \mathcal{W} \forall p_1 \leq q_0 \exists a_{1} \leq p_1 \exists a_{2} \in \mathcal{W} \ldots$$

$$((1), (2) and (3) above hold, for $M = T \cup \{p_i, q_i, a_i : i \in \omega\}$ and $D = \{b \in B : q_b \leq b$ for some $n \in \omega\}$. (The notation of (4) means that player II (the 3-player, or $(q, a)$-player) has a winning strategy in the indicated game.)

Proof. Fix $\mathcal{B}, \mathcal{W}, T$ and $d$ as in the hypothesis of 2.1. We use $a = (p_0, q_0, a_0, \ldots, p_n, q_n, a_n, \ldots)$ to range over completed legal runs of the game in (4). Put $a_0 = (p_0, \ldots, q_0)$ and $a_0 = (p_0, \ldots, p_n)$ (i.e., the first $3n+1$ and $3n+1$ entries in $\alpha$, respectively). We put $M(a) \in T \cup \{p_0, q_0, a_0 : a_0 \in \omega\}$, and $D(a) = \{b \in B : q_b \leq b$ for some $n \in \omega\}.

Let $Z(E)$ be the first-order language with equality and the binary relation symbol $\in$. Let $\varphi$ be an existential $Z(E)$-formula with free variables among $x_0, \ldots, x_\xi$; say $\varphi = 3x_0 \exists x_1 \ldots \exists x_\xi$. Let $x_0, \ldots, x_\xi \in \mathcal{W}$. We say that $I$ has dealt with the pair $(\varphi, (x_0, \ldots, x_\xi))$ at $a_\xi$ if either $W = \varphi[x_0, \ldots, x_\xi]$, or $W = \varphi[x_0, \ldots, x_\xi]_{a_\xi}$.

(*1) For all $a, n, \varphi$ and all $x_0, \ldots, x_n \in \mathcal{W}$, there exist $q$ and $a$ such that $I$ has dealt with the pair $(\varphi, (x_0, \ldots, x_n))$ at $a_\xi$.

(Here $\cdot$ denotes concatenation of sequences.)

(**1) If for all $\varphi$, and all $x_0, \ldots, x_n \in \mathcal{M}(a)$ there exists $n$ such that $I$ has dealt with $(\varphi, (x_0, \ldots, x_n))$ at $a_n$, then (1) holds for $M = \mathcal{M}(a)$.

(**1) follows immediately from a well-known characterization of the elementary substructure relation [1, p. 108].
Now suppose $b \in B$. We say $H$ has dealt with $b$ at $\alpha_n$ if either $q_n \leq b$ or $q_n \leq b$.

(2) For all $a_n \in o$, and $b \in B$, there exist $q$ and $a$ such that $H$ has dealt with $b$ at $a_n$.

Indeed, if $p_n \leq b$, let $(q, a) = (p_n, 0)$. Otherwise, let $(q, a) = (p_n - b, 0)$.

(2') If for all $b \in M(a) \cap B$, there exists $n$ such that $H$ has dealt with $b$ at $a_n$ then (3) holds for $D = D(a)$ and $M = M(a)$.

(2) is easily checked.

Finally, if $A \subseteq B$, we say that $H$ has dealt with $A$ at $\alpha_n$ if either $\Pi^A A$ does not exist, or $q_n \leq \Pi^A A$ and $a_n \notin \Pi^A A$, or for some $b \in A$, $q_n \leq b$ and $a_n = b$.

(3) For all $a_n \in o$ and $A \subseteq B$, there exist $q$ and $a$ such that $H$ has dealt with $A$ at $a_n$.

Indeed, if for all $b \in A$, $p_n \leq b$, then $p_n \leq \Pi^A A$. Let $(q, a) = (p_n, \Pi^A A)$. Otherwise, there exists $b \in A$ such that $p_n \leq b$. Let $(q, a) = (p_n - b, b)$.

(3') If for all $a_n \in M(a)$ such that $A \subseteq B$, there exists $n$ such that $H$ has dealt with $A$ at $a_n$, then (3) holds for $M = M(a)$ and $D = D(a)$.

(3) follows at once from our definitions.

Player II's winning strategy in (4) is now apparent. Let the variable $\chi$ range over things to be "dealt with", which, that is, pairs $(\chi, (x, \ldots, x_i))$ with $x, \ldots, x_i \in B$. Elements of $B$, and subsets $A$ of $B$. Say $(\chi, (x, \ldots, x_i))$, $b$, or $A$ arise in $V \subseteq W$ if $x, \ldots, x_i \in V$, or $b \in V$, or $A \subseteq V$, respectively.

Before play begins, II assigns to each $\chi$ which arises in $T$ a natural number, by creating an injection from the set of all such $\chi$ into $\omega$, in such a way that the range of $f$ misses an infinite set of natural numbers; $f(\chi) = n$ is to be interpreted as meaning that II promises to deal with $\chi$ on his $n$th move.

Now suppose play has gone $(p_0, q_0, a_0, \ldots, p_n)$. If $f(\chi) = n$ for some $\chi$, then by (1), (2), or (3), there exist $q_n$ and $a_n$ such that player II has dealt with $A$ at $(p_0, q_0, a_0, \ldots, p_n, q_n, a_n)$. Player II plays $(q_n, a_n)$, otherwise he plays arbitrarily. He then assigns a number $f(\chi) > n$ for each $\chi$, which arises from $T \cup \{(p_0, q_0, a_0, \ldots, p_n, q_n, a_n)\}$ for which $f(\chi)$ has not yet been defined, in such a way that it is still an injection whose range misses an infinite set of natural numbers.

It is clear that if $\alpha = (p_0, q_0, a_0, \ldots)$ is a play in which II follows this strategy, every $\chi$ which arises in $M(\omega)$ is dealt with at some $a_n$. Thus by (1), (2), and (3), respectively, (1), (2), and (3) hold with $M(\omega)$ for $M$ and $D(\omega)$ for $D$. This completes the proof.

Section 3. Absolute $4^1$ operations and the Baire property. Let $X$ be an arbitrary topological space, fixed throughout this section. We use $U$ and $V$ (often subscripted) to range over non-empty open sets in $X$. For $A, B \subseteq X$, we write $A \subseteq B$ to mean that $A \sim B$ is meager (first category), and $A \sim B$ to mean $A \subseteq B$ and $B \subseteq A$. $A$ is said to have the Baire property if $A = \emptyset$ for some open set $\emptyset$.

Let $I$ be the ideal of meager sets in $X$, and $BP$ the class of Baire property sets in $X$. Then $I$ is a $\sigma$-ideal in $X$, and $BP$ is a $\sigma$-field of subsets of $X$ [11]. We consider the Boolean algebra $\mathcal{P}(X)/I$, and denote its elements $A/I$, for $A \subseteq X$. An extremely important subalgebra is $BP/I$. The Birkhoff–Ulman theorem [11] says that $BP/I$ is a complete Boolean algebra.

As a final preliminary, we recall the Banach–Mazur theorem [7].

THEOREM (Banach–Mazur). Let $A \subseteq X$. For all non-empty open sets $U$, $A \uplus U$, if and only if

$$V \subseteq U \iff V \subseteq U_0 \quad \forall V \subseteq U_0 \quad \forall V \subseteq U_0 \quad \forall V \subseteq A$$

We are now ready to state and prove a theorem from which our main theorem, 3.3, will follow at once.

THEOREM 3.2. Suppose $A = \langle x : x \in I \rangle$ is a sequence of open sets in $X$. Further suppose that $S \subseteq \mathcal{P}(o)$ is a $\Sigma^1_2$ set; say $\mathcal{P}(x)$, and $\mathcal{F}$ is a $\mathcal{F}^x$ formula such that,

$$\forall x \in o, x \in S \iff \exists r \in \mathcal{F}(x), \mathcal{F}$$

for all $x \in o$, $x \in S$ if, and only if, $\mathcal{F}(x, r)$ holds.

Let $r$ be the $BP/I$-Boolean valued subsets of $\mathcal{F}$ such that,

$$\forall x \in o, r(x) = A_x/I$$

Then

$$\mathcal{F}^x(A_x/I) \geq \mathcal{F}^x(r, I)$$

PROOF. For short, set $\mathcal{B} = (\mathcal{B}, \wedge, \vee, \alpha, \beta, \leq, 0, 1) = BP/I$, $b_n = A_n/I$ for $n = 0, 1, \ldots$, and $d = \mathcal{F}^x(r, I)$.

We must discuss a technical point. It would be convenient to have available an elementary substructure of the universe. Of course, no such structure need exist, but, for any formula $\theta$ of the language of set theory, it is a theorem, the "reflection principle" (see [4]), that for any set $x$ there exists a transitive set $W \subseteq x$ such that

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SUBCLAIM.

$D'$ is a $N$-generic ultrafilter over $\mathcal{B}$. 

Proof of subclaim. That $D'$ is an ultrafilter over $\mathcal{B}$ follows easily from (2). To prove genericity, suppose $C \subseteq N$ and $C \subseteq D'$. We show that $\Pi^0_\alpha C \subseteq D'$. Indeed, since $C \subseteq N$, $C = A'$ for some $A \subseteq M$. Since $A' \subseteq D'$, $A' \subseteq B'$ and so by isomorphism, (1) and (5), $A \subseteq B$. Since $\mathcal{B}$ is complete, $\Pi^0_\alpha A$ exists. Thus by (3)

$$\forall \alpha A \in B \cap M \quad \exists (\exists b \in \mathcal{B} \cap M) \in D'$$

Suppose, for contradiction, that the latter holds. Then $b' \in A'$, so $b' \in D'$, so $b \in D$. This is impossible, since $b \in D$ and $D$ is a filter.

Therefore $\Pi^0_\alpha A \in B \cap M$, so $(\Pi^0_\alpha A) \subseteq D'$. A short argument shows that $(\Pi^0_\alpha A') = \Pi^0_\alpha C$. Thus $\Pi^0_\alpha C \subseteq D'$, and the subclaim is established.

Hence $N[D']$ is a generic extension of $N$.

Recall that we put $||\Phi(r, \ell)||_{\omega} = d$. As it is a basic fact about forcing that "limited forcing is definable", and it is easily checked that $r, \ell, \mathcal{B}$ and $d \in M$ by (5), (1) and the fact that $\sim$ is an isomorphism, we have

$$||\Phi(r', \ell')||_{\omega} = d'.$$

Now $d' \in D'$ (indeed $d \supseteq V_\omega I$ and so $d \in D$), so by the "truth lemma" for forcing

$$N[D'] = \Phi(s, r')$$

where $s = \{\exists n \in \omega : b' \in D\}$. Recall that $\Phi$ was assumed to be a $\Sigma_1$ formula, and so by Mostowski's absoluteness theorem, $\Phi(s, r')$ is true, i.e.

$$(7) 
\exists s = \{\exists n \in \omega : b' \in D\} \in S.$$

Now, unscrambling definitions,

$$b' \in D' \quad \text{if and only if} \quad b' \in D,$$

and

$$b' \in D' \quad \text{if and only if} \quad b' \in D.$$
Proof of claim. Let \((U_n, V_n, \ldots)\) be a play of the game in (8) in which player II has followed the strategy \(G\). Suppose \(x \in \bigcap V_n\).

Fix \(n \in \omega\). If \(x \in A_n\), then \(V_n \subseteq A_n\), so certainly \(3mV_n \subseteq A_n\).

Conversely, suppose \(3mV_n \subseteq A_n\). If \(V_n \cap A_n = \emptyset\), one sees easily that one of the sets \(V_n, V_{n+1}\) is meager, contradicting the fact that \(U\) has no non-empty open meager subsets. Thus \(V_n \subseteq A_n\), and so \(x \in A_n\).

Hence for all \(n \in \omega\), \(x \in A_n\) if and only if \(3mV_n \subseteq A_n\). Now since \(G\) was a winning strategy for (6), we have \(\{n : x \in A_n\} \in S\), that is, \(x \in \mathcal{G}^+(A)\). This completes the proof of the claim.

Thus by (8) and the Banach–Mazur theorem, \(\mathcal{G}^+(A) \supseteq U\). Recalling that \(U/I = \Vert \Phi(r, \tilde{t})\Vert_\omega\), we have \(\mathcal{G}^+(A)/I \supseteq \Vert \Phi(r, \tilde{t})\Vert_\omega\).

and the proof of 3.2 is complete.

As promised, from 3.2 we obtain a short proof of 3.3.

Theorem 3.3. Absolutely \(A_1\) Boolean operations preserve the Baire property.

To determine, let \(A = (\mathcal{A}_n : n \in \omega)\) be a sequence of subsets of \(X\) with the Baire property. Suppose \(S \in \mathcal{P}(\omega)\) is an absolutely \(A_1\) set; say \(\Phi\) and \(\Psi\) are \(\Sigma^1_1\) formulas and \(t \subseteq \omega\) such that

\[
\text{for all } x \subseteq \omega, \ x \in S \text{ if and only if } \forall t_0 \exists t_1 \left( \Phi(x, t_0) \iff \Psi(x, t_1) \right),
\]

and

\[
\text{for all complete Boolean algebras } \mathcal{B}, \ \|\forall x \subseteq \omega \Phi(x, t)\|_{\mathcal{B}} = 1.
\]

Then

1. \(\mathcal{G}^+(A)\) has the Baire property.
2. If \(r\) is the Boolean \(BP/I\)-valued subset of \(\mathcal{O}\) such that, for all \(n \in \omega, r(\tilde{b}) = A_n/I\), then \(\mathcal{G}^+(A)/I = \|\Phi(r, \tilde{t})\|_{\mathcal{B}^{\mathcal{I}}_1}\).

Proof. First, note that 2) implies 1). Next, observe that, using (1.1), 2) reduces to the case where all of the sets \(A_n\) are open. With this reduction, all of the hypotheses of 3.2 are satisfied, and we have

\[
\mathcal{G}^+(A)/I \supseteq \|\Phi(r, \tilde{t})\|_{\mathcal{B}^{\mathcal{I}}_1}.
\]

Likewise, the hypotheses of 3.2 are satisfied with \(\Psi\) for \(\Phi\) and \(\sim S\) for \(S\). Thus

\[
\mathcal{G}^+(A)/I \supseteq \|\Psi(r, \tilde{t})\|_{\mathcal{B}^{\mathcal{I}}_1}.
\]

It is routine to check that \(\mathcal{G}^+(A) = \sim \mathcal{G}^+(A)\). Also, by (9), \(\|\Psi(r, \tilde{t})\|_{\mathcal{B}^{\mathcal{I}}_1} = \|\Phi(r, \tilde{t})\|_{\mathcal{B}^{\mathcal{I}}_1}\). Thus, taking complements in (11) we have

\[
\|\Phi(r, \tilde{t})\|_{\mathcal{B}^{\mathcal{I}}_1} \supseteq \mathcal{G}^+(A)/I.
\]

(10) and (12) yield the conclusion of 2), and we are done.

Remarks. 1) It is clear from the proof of 3.3 that in (9) the phrase "for all complete Boolean algebras \(\mathcal{B}\)" may be weakened to "for \(\mathcal{B} = BP/I\)".

2) Let \((X, \mathcal{M}, \mu)\) be an arbitrary \(\sigma\)-finite complete measure space. By a theorem in \([6]\), there exists a topology \(\mathcal{I}\) for \(X\) with the property

\[
\"(X, \mathcal{M}, \mu)\text{-measurable } = (X, \mathcal{J})\text{-Baire property}\"
\]

and

\[
\"(X, \mathcal{M}, \mu)\text{-measure zero } = (X, \mathcal{I})\text{-meager}\"
\]

Using the topological space \((X, \mathcal{J})\), 3.3 yields at once 3.3', which is 3.3 with "\(\sigma\)-finite measure space" replacing "topological space", "measurability" replacing "the Baire property", \(\mathcal{M}\) replacing \(BP\), and letting \(I\) be the ideal of sets of measure zero. In particular, all absolutely \(A_1\) Boolean operations preserve measurability.

In the next section we shall, by another method, establish a generalization of 3.3' (but not of 3.3) originally due to Vaught.

3) Let \(X\) be the space of infinite subsets of \(\omega\) with the Ellentuck topology (see \([2]\)) 3.3, along with the results of \([2]\), yield at once the result that absolutely \(A_1\) sets are Ramsey. This extends the result of Silver and Mathias that analytic sets are Ramsey.

Section 4. Other fields of sets. Let \(X\) be an arbitrary non-empty set, \(\mathcal{I}\) a \(\sigma\)-field of subsets of \(X\), and \(I\) a \(\sigma\)-ideal on \(X\) such that \(I \subseteq \mathcal{I}\).

Our main object in this section is to prove the following theorem, due to R. Vaught:

Theorem 4.1. Suppose the Boolean algebra \(\mathcal{B}/I\) satisfies the countable chain condition. Then \(\mathcal{I}\) is invariant under all absolutely \(A_1\) Boolean operations.

4.1 overlaps with but does not include 3.3. However, 4.1 does include 3.3', as the Boolean algebra "measurable/measure zero" satisfies the countable chain condition for any \(\sigma\)-finite measure space \(X\).

Our proof of 4.1 is accomplished by assigning new (non-topological) meanings to the topological terminology used in Theorems 3.2 and 3.3 (in particular "open", "meager", and "Baire property") in such a way that our proofs of 3.2 and 3.3 are valid for these new meanings. The reinterpretation 3.3 will immediately imply 4.1.

For the remainder of this section, fix arbitrary \(X, \mathcal{I}\) and \(I\) as above. As usual, we form the Boolean algebra \(\mathcal{B}(X)/I\), and subalgebra \(\mathcal{B} = \mathcal{B}/I\). We also add the hypothesis

(13) \(\mathcal{B}\) satisfies the countable chain condition.

Let \(A, B \subseteq X\). We say \(A\) is open if \(A \in \mathcal{I} - I\) or if \(A = \emptyset\). We say \(A\) is meager if \(A \subseteq I\).

(Aside: The space \(X\) consisting of \(X\) along with its open sets as just defined, is not in general a topological space, though as we shall see, \(X\)
satisfies a few key theorems of topology, suitably interpreted. It is true, that if
one applies one of the usual definitions of ‘meager’ (in terms of ‘open’) to our
X, our notion of ‘meager’ is obtained (even without assuming (13); see [9]), but
we will not need this fact).
Say A ⊆ B if A ⊆ B is meager. We say A ≈ B if A ⊆ B and B ⊆ A. A
has the Baire property if A ≈ C for some open set C.
Let SP be the class of sets with the Baire property. Then one easily sees
that
4.2. BP = S, so BP/I = S/I (≈ S).
We shall need a few simple facts.
4.3. Every collection of disjoint open sets is countable.
Indeed, 4.3 is equivalent (13), as an easy argument shows. As a conse-
quence of 4.3, we obtain 4.4.
4.4. The class of open sets is closed under disjoint unions.
4.5. S is a complete Boolean algebra.
4.5 follows from a general theorem about Boolean algebras: A σ-
complete Boolean algebra satisfying the countable chain condition is com-
plete (see [11]). Alternatively, 4.5 can be obtained directly from 4.4, without
using (13).
Finally, a version of the Banach–Mazur theorem holds. Let U, V be
usually subscripted, range over non-empty open sets.

**Theorem 4.6.** Let A ⊆ X. For all non-empty open sets U, A ⊇ U if and only if
\[ \forall U \subseteq V \subseteq U \ni U \cap A \subseteq V \subseteq A. \]  

Proof. Suppose A ⊇ U, i.e., U ⊆ A ∈ I. Then, in particular,
\[ U \cap A = U \sim (U \sim A) \subseteq A. \]
Say player 1 plays \( U \subseteq U \). We claim that \( U \cap A \) is nonempty and
open. Indeed, \( U \cap A = U \cap (U \cap A) \subseteq A \). Also \( U \cap A \subseteq I \), for otherwise \( U \cap A = U \) \( \cap (U \cap A) \subseteq (U \cap A) \subseteq A \) \( \cap I \), contradicting the fact that \( U \)
open. Thus \( U \cap A \subseteq \emptyset \). Otherwise, player 2 plays \( V \cap A \subseteq U \cap A \). Then, no matter how play continues, \( \forall V \subseteq A \). This proves (14).

Our proof of the converse copies the proof in [7] for topological spaces.
Suppose P is a winning strategy for player II in (14). For a partial P-play,
we mean a sequence \( z = (U_0, V_0, \ldots, U_n, V_n) \) such that \( U_0 \subseteq \cdots \subseteq V_n \) and II has played according to P. Another partial P-play \( z' = (U_0, V_0, \ldots, U_n, V_n) \) is said to be disjoint from \( z \) if \( V_n \cap V'_n = \emptyset \).
Let \( S_0 \) be a maximal disjoint set of partial P-plays \( (U_0, V_0) \), and let
\( T_0 = \bigcup \{ V_0 : \text{for some } U_0, (U_0, V_0) \in S_0 \} \). Then by 4.4, \( T_0 \) is open. Thus \( U \sim T_0 \subseteq \emptyset \). Actually, \( U \sim T_0 \subseteq I \), for otherwise \( U \sim T_0 \) is open and
open, so for some \( V \) we could add \( (U \sim T_0, V) \) to \( S_0 \), contradicting the
maximality of \( S_0 \).
Now let \( S_1 \) be a maximal disjoint set of partial P-plays \( (U_0, V_0, U_1, V_1) \) such that \( (U_0, V_0) \in S_0 \), and let
\[ \begin{align*}
T_1 &= \bigcup \{ V_1 : \text{for some } U_0, V_0, U_1, (U_0, V_0, U_1, V_1) \in S_1 \}. 
\end{align*}
\]
Again by 4.4, \( T_1 \) is open. We claim that \( U \sim T_1 \subseteq I \). It suffices to show that \( T_0 \sim T_1 \subseteq I \), for then \( U \sim T_1 \subseteq \left( U \sim T_0 \right) \cup \left( T_0 \sim T_1 \right) \subseteq I \).
Indeed, suppose for contradiction that \( T_0 \sim T_1 \) is non-empty and open.
Now \( T_0 \sim T_1 = \bigcup \{ V_0 \sim T_1 : \text{for some } U_0, V_0, U_1, V_1 \in S_1 \} \); by 4.3, this is a
countable union, so for some \( (U_0, V_0) \in S_0 \), \( V_0 \sim T_1 \) is non-empty and
open. But then we could add \( (U_0, V_0, U_1, V_1, T_1, V) \) to \( S_1 \), for some \( V \), contradicting
the maximality of \( S_1 \). This proves the claim.

Continuing in this manner, obtain a sequence of partial P-plays, \( S_0, S_1, \ldots \), and a sequence of open sets, \( T_0, T_1, \ldots \), such that, for all \( i \), \( U \sim T_i \subseteq I \). Thus \( U \sim \bigcap T_i \subseteq I \). We complete the proof by showing that
\[ \bigcap T_i = A. \]
Let \( x \in \bigcap T_i \). Since \( x \in T_0 \), there is a unique \( (U_0, V_0) \in S_0 \) such that \( x \in V_0 \).
Since \( x \in T_1 \), this is a unique \( (U_0, V_0, U_1, V_1) \in S_1 \) such that \( x \in V_1 \). But then
\( (U_0, V_0, U_1, V_1) \in S_1 \) by construction, and so \( x \in V_1 \); thus by uniqueness, \( U_0, V_0 \)
and \( U_1, V_1 \). Continuing in this way, obtain a play \( (U_0, V_0, U_1, V_1, \ldots) \) of (14)
according to the winning strategy \( F \). Therefore \( \forall \forall V_i \subseteq A, \) so \( x \in A \).

Thus \( \forall i \in \emptyset V_i \subseteq A \), so since \( U \sim \bigcap T_i \subseteq I \), \( U \sim A \subseteq I \), that is, \( A \supseteq U \), as
desired.

The reader may now read the statements and proofs of Theorems 3.2 and
3.3 using the notions of open, Baire property, meager, BP, I, and \( \leqslant \) of
this section. The only needed change is the reference to 3.1 just before (6).
In our current context, we may take \( U \) to be any open set such that \( d(U, I) = 1 \),
as there are no non-empty open meager sets. Indeed, if \( U \) is non-empty and
open then, by definition, \( U \notin I \), so \( U \) is not meager.
Recalling 4.2, then, the first line of 3.3 as now interpreted, implies that 3 is
invariant under all absolutely \( d_1 \) Boolean operations. Thus we have 4.1.

References
On Borel-measurable collections of countable-dimensional sets

by

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Abstract. Let $B$ be a Borel set in the product $S \times T$ of compact metrizable spaces, whose vertical sections $B(s) \subseteq T$ are countable-dimensional (i.e., unions of countably many zero-dimensional sets $G_\alpha$-sets in $T$). It is an open question whether the small transfinite dimension $\text{ind} \ B(s)$ of the vertical sections of $B$ is bounded, i.e., if $\sup \ \{ \text{ind} \ B(s) : s \in S \} < \omega_1$. We show that a certain additional assumption about $B$ (an existence of a Borel-measurable, point-finite, sectionwise separation for $B$, see Definition 3.2) guarantees that this is true.

§ 1. Preliminaries. In this paper we consider only separable metrizable spaces and "compactum" means "compact space". Our terminology concerning analytic sets follows [K] and the terminology related to dimension theory follows [A P], [E1] and [Na].

1.1. Terminology and notation. A closed set $L$ in a space $X$ separates two disjoint sets $A$ and $B$ in $X$ if $X \setminus L = U \cup V$, $U$ and $V$ being disjoint open sets with $A \subseteq U$ and $B \subseteq V$. We denote by $\omega$ the set of natural numbers, $I$ is the real unit interval and $\text{Fin} \ \omega$ is the set of all non-empty finite subsets of $\omega$. We identify the power set $2^{\text{Fin} \ \omega}$ with the Cantor cube $[0,1]^{\text{Fin} \ \omega}$, i.e., we identify each subset of $\text{Fin} \ \omega$ with its characteristic function and we consider the characteristic functions with pointwise topology. The symbol $|A|$ stands for the cardinality of the set $A$. A sequence $\langle A_i : i \in \omega \rangle$ of sets $A$ is point-finite if for each $x \in X$ the set $\{ i \in \omega : x \in A_i \}$ is finite (thus we exclude the possibility that one set occurs in the sequence infinitely many times). Given a set $E$ in the product $S \times T$ we denote by $E(s)$ the vertical section $\{ t \in T : (s, t) \in E \}$ of the set $E$ at the point $s \in S$.

1.2. Countable-dimensional sets and the small transfinite dimension. A space $X$ is countable-dimensional if $X = \bigcup_1^\omega X_i$, $X_i$ being zero-dimensional.

The small transfinite dimension $\text{ind}$ is the ordinal-valued function obtained through the extension of the classical Menger–Urysohn inductive dimension by transfinite induction. If the transfinite dimension is not defined for $X$, we write $\text{ind} X = \infty$; since our spaces have always a countable base, if $\text{ind} X \neq \infty$, then $\text{ind} X < \omega_1$. [4] T. Jech, Set Theory, Academic Press, New York 1978.


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