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On absolutely Δ_2^1 operations

by

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Abstract. Every absolutely Δ_2^1 Boolean operation preserves the Baire property in all topological spaces, and, as a consequence, measurability in all σ-finite complete measure spaces.

It is a classical theorem that the operation (A) preserves the Baire property in all topological spaces, and measurability in all σ-finite complete measure spaces.

R. Solovay (unpublished) introduced the class of *absolutely Δ_2^1 sets* (to be defined in the next section) in Polish spaces, and proved that they have the Baire property, and are Lebesgue measurable. Solovay's results were rediscovered and extended by Fenstad and Normann [3], who showed that an absolutely Δ_2^1 set in an analytic space is measurable with respect to any σ-finite, complete, regular Borel measure.

In order to extend these results further, R. Vaught (unpublished; announced at Wrocław, 1977) considered the *absolutely Δ_2^1 Boolean operations*, and showed that these operations preserve the Baire property in any topological space satisfying the countable chain condition, and measurability with respect to any σ-finite complete measure.

The main result here, in analogy with and extending the classical theorem cited above, is

THEOREM 3.3. *All absolutely Δ_2^1 Boolean operations preserve the Baire property in all topological spaces.*

From 3.3, using a theorem in [8], we directly infer the part of Vaught's result dealing with measure.

Now let $\mathfrak{I}$ be an arbitrary σ-field of sets on a set X, and let I be a σ-ideal on X such that $I \subset \mathfrak{I}$. Vaught proved

THEOREM 4.1. *If the Boolean algebra $\mathfrak{I}/I$ satisfies the countable chain condition, then $\mathfrak{I}$ is invariant under all absolutely Δ_2^1 Boolean operations.*

We cannot infer 4.1 directly from 3.3. However, we do show that, by introducing a simple device, the pattern of our proof of 3.3 carries over into a new proof of 4.1.

Most of the material herein appears in the author's doctoral dissertation [9]. I am grateful to my thesis advisor, Robert Vaught, for his help in all aspects of its preparation.

Section 1. Preliminaries. We assume some familiarity with the terminology of set-theoretic forcing and Boolean-valued models. Our notation is taken from [4].

A set $S \subseteq \mathscr{P}(\omega)$ is *absolutely* Δ_2^1 if there exist Σ_2^1 formulas Φ and Ψ and a parameter $t \in \mathscr{P}(\omega)$, such that

for all $x \subseteq \omega$, $x \in S$ if and only if $\Phi(x, t)$ holds, if and only if $\sim\Psi(x, t)$ holds, and for all complete Boolean algebras $\mathfrak{B}$,

$$\|\forall x \subseteq \check{\omega}(\Phi(x, \check{t}) \leftrightarrow \sim\Psi(x, \check{t}))\|_{\mathfrak{B}} = 1.$$

(For some basic facts about the class of absolutely Δ_2^1 sets, see [3].)

Let S be any subset of $\mathscr{P}(\omega)$. For any non-empty set X, S determines an ω-ary operation $\mathcal{O}_S^X$ on $\mathscr{P}(X)$ where

$$\mathcal{O}_S^X((A_n\colon n \in \omega)) = \{x \in X\colon \{n\colon x \in A_n\} \in S\}.$$

Operations so obtained are called Boolean. Boolean operations have many pleasant and easily verified properties. For example, the reader will easily show

(1.1) $$\mathcal{O}_S^X((A_n\colon n \in \omega)) \Delta \mathcal{O}_S^X((B_n\colon n \in \omega)) \subseteq \bigcup_{n \in \omega} (A_n \Delta B_n).$$

Now the key definition: The Boolean operation $\mathcal{O}_S^X$ is said to be an *absolutely* Δ_2^1 *Boolean operation* provided S is an absolutely Δ_2^1 subset of $\mathscr{P}(\omega)$.

Examples of absolutely Δ_2^1 Boolean operations are the operation (A), the R-operations of Kolmogorov, and the Borel game operations [10]. That the Borel game operations are absolutely Δ_2^1 is a consequence of D. Martin's Borel determinacy theorem [6]. In [10], R. Vaught and the present author prove that the Borel game operations preserve the Baire property in all topological spaces (a special case of our main theorem) without appealing to the fact that these operations are absolutely Δ_2^1.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures. We write $\mathfrak{A}_2 \prec \mathfrak{A}_1$ to mean that $\mathfrak{A}_2$ is an elementary substructure of $\mathfrak{A}_1$ in the usual sense of model theory (as in, e.g., [1]). The Löwenheim–Skolem theorem [1] says that for any structure $\mathfrak{A}_1$, and any countable subset A of the universe of $\mathfrak{A}_1$, there exists a countable structure $\mathfrak{A}_2$ such that $\mathfrak{A}_2 \prec \mathfrak{A}_1$ and the universe of $\mathfrak{A}_2$ includes A. The letter $\mathfrak{B}$ will always denote the Boolean algebra $(B, \wedge, \vee, -, \leqslant, 0, 1)$.

Let D be an ultrafilter over $\mathfrak{B}$, and S a subset of B such that $\Pi^{\mathfrak{B}}S$ exists. We say that D *preserves the infimum* $\Pi^{\mathfrak{B}}S$ provided that $\Pi^{\mathfrak{B}}S \in D$ if, and only if, $S \subseteq D$. For any set M, the ultrafilter D is called *M-generic* if, for all $S \in M$ such that $S \subseteq B$ and $\Pi^{\mathfrak{B}}S$ exists, D preserves the infimum $\Pi^{\mathfrak{B}}S$. The Rasiowa–Sikorski theorem [11] states that, if M is a countable set, then there exists an ultrafilter D over $\mathfrak{B}$ which is M-generic.

Section 2. The main lemma. In this section we will prove a lemma which is a combination of the Löwenheim–Skolem and Rasiowa–Sikorski theorems, in a game setting. Our proof is a modification of well-known proof of these two theorems.

Given a Boolean algebra $\mathfrak{B}$, a transitive (in the usual sense of set theory) set W and a countable set $T \subseteq W$, we will find a countable set $M \supseteq T$ and a filter D over $\mathfrak{B}$ such that

(1) $(M, \in) \prec (W, \in)$,

(2) $D \cap M$ is an ultrafilter over $\mathfrak{B}/B \cap M$,

(3) $(\forall A \in M)$(if $A \subseteq B$ and $\Pi^{\mathfrak{B}}A$ exists, then $\Pi^{\mathfrak{B}}A \in D \cap M$ or $(\exists b \in A \cap M)\bar{b} \in D$).

Note that (3) implies that $D \cap M$ is M-generic over $\mathfrak{B}/B \cap M$. Below, p_i and q_i are to range over non-zero elements of B, and a_i over arbitrary sets, for $i = 0, 1, 2, \ldots$

LEMMA 2.1. *Let $\mathfrak{B}$ be a Boolean algebra, W a transitive set such that $\mathfrak{B} \in W$, $T \subseteq W$ a countable set, and $d \in B$. Then*

(4) $$(\forall p_0 \leqslant d\, \exists q_0 \leqslant p_0\, \exists a_0 \in W\, \forall p_1 \leqslant q_0\, \exists q_1 \leqslant p_1\, \exists a_1 \in W \ldots)$$

((1), (2) *and* (3) *above hold, for* $M = T \cup \{p_i, q_i, a_i\colon i \in \omega\}$ *and* $D = \{b \in B\colon q_n \leqslant b$ *for some* $n\}$).

(The notation of (4) means that player II (the $\exists$-player, or (q, a)-player) has a winning strategy in the indicated game.)

Proof. Fix $\mathfrak{B}$, W, T and d as in the hypothesis of 2.1. We use $\alpha = (p_0, q_0, a_0, \ldots, p_n, q_n, a_n, \ldots)$ to range over completed legal runs of the game in (4). Put $\alpha_n = (p_0, \ldots, a_n)$ and $\alpha_n^- = (p_0, \ldots, p_n)$ (i.e., the first $3n+3$ and $3n+1$ entries in α, respectively). We put $M(\alpha) \in T \cup \{p_n, q_n, a_n\colon n \in \omega\}$, and $D(\alpha) = \{b \in B\colon q_n \leqslant b$ for some $n \in \omega\}$.

Let $\mathscr{L}(\in)$ be the first-order language with equality and the binary relation symbol $\in$. Let φ be an existential $\mathscr{L}(\in)$-formula with free variables among $v_0, \ldots, v_k$; say $\varphi = \exists v_i \Psi$. Let $x_0, \ldots, x_k \in W$. We say that *II has dealt with the pair* $(\varphi, (x_0, \ldots, x_k))$ at α_n if either $W \models \neg\varphi[x_0, \ldots, x_k]$, or $W \models \psi[x_0, \ldots, x_k]\begin{bmatrix} v_i \\ a_n \end{bmatrix}$.

(*1) For all α, n, φ and all $x_0, \ldots, x_k \in W$, there exist q and a such that II has dealt with the pair $(\varphi, (x_0, \ldots, x_k))$ at $\alpha_n^- {}^\frown (q, a)$. (Here $^\frown$ denotes concatenation of sequences.)

(**1) If for all φ, and all $x_0, \ldots, x_k \in M(\alpha)$ there exists n such that II has dealt with $(\varphi, (x_0, \ldots, x_k))$ at α_n, then (1) holds for $M = M(\alpha)$.

(**1) follows immediately from a well-known characterization of the elementary substructure relation [1, p. 108].

Now suppose $b \in B$. We say *II has dealt with b at* α_n if either $q_n \leqslant b$ or $q_n \leqslant \bar{b}$.

(*2) For all a, $n \in \omega$, and $b \in B$, there exist q and a such that II has dealt with b at $\alpha_n^{-\frown}(q, a)$.

Indeed, if $p_n \leqslant b$, let $(q, a) = (p_n, \emptyset)$. Otherwise, let $(q, a) = (p_n - b, \emptyset)$.

(**2) If for all $b \in M(\alpha) \cap B$, there exists n such that II has dealt with b at α_n, then (2) holds for $D = D(\alpha)$ and $M = M(\alpha)$.

(**2) is easily checked.

Finally, if $A \subseteq B$, we say that *II has dealt with A at* α_n if either $\Pi^{\mathfrak{B}} A$ does not exist, or $q_n \leqslant \Pi^{\mathfrak{B}} A$ and $a_n = \Pi^{\mathfrak{B}} A$, or for some $b \in A$, $q_n \leqslant \bar{b}$ and $a_n = b$.

(*3) For all α, $n \in \omega$ and $A \subseteq B$, there exist q and a such that II has dealt with A at $\alpha_n^{-\frown}(q, a)$.

Indeed, if for all $b \in A$, $p_n \leqslant b$, then $p_n \leqslant \Pi^{\mathfrak{B}} A$. Let $(q, a) = (p_n, \Pi^{\mathfrak{B}} A)$. Otherwise, there exists $b \in A$ such that $p_n \nleqslant b$. Let $(q, a) = (p_n - b, b)$.

(**3) If for all $A \in M(\alpha)$ such that $A \subseteq B$, there exists n such that II has dealt with A at α_n, then (3) holds for $M = M(\alpha)$ and $D = D(\alpha)$.

(**3) follows at once from our definitions.

Player II's winning strategy in (4) is now apparent. Let the variable χ range over things to be "dealt with", that is, pairs $(\varphi, (x_0, \ldots, x_k))$ with $x_0, \ldots, x_k \in W$, elements b of B, and subsets A of B. Say $(\varphi, (x_0, \ldots, x_k))$, b, or A *arise in* $V \subseteq W$ if, $x_0, \ldots, x_k \in V$, or $b \in V$, or $A \in V$, respectively.

Before play begins, II assigns to each χ which arises in T a natural number, by creating an injection from the set of all such χ into ω, in such a way that the range of f misses an infinite set of natural numbers; $f(\chi) = n$ is to be interpreted as meaning that II promises to deal with χ on his nth move.

Now suppose play to date has gone $(p_0, q_0, a_0, \ldots, p_n)$. If $f(\chi) = n$ for some χ, then by (*1), (*2), or (*3), there exist q_n and a_n such that player II has dealt with χ at $(p_0, q_0, a_0, \ldots, p_n, q_n, a_n)$; player II plays (q_n, a_n). Otherwise he plays arbitrarily. He then assigns a number $f(\chi) > n$ to each χ which arises from $T \cup \{p_0, q_0, a_0, \ldots, p_n, q_n, a_n\}$ for which $f(\chi)$ has not yet been defined, in such a way that f is still an injection whose range misses an infinite set of natural numbers.

It is clear that if $\alpha = (p_0, q_0, a_0, \ldots)$ is a play in which player II follows this strategy, every χ which arises in $M(\alpha)$ is dealt with at some α_n. Thus by (**1), (**2), and (**3), respectively, (1), (2), and (3) hold with $M(\alpha)$ for M and $D(\alpha)$ for D. This completes the proof.

Section 3. Absolute Δ_2^1 operations and the Baire property. Let X be an arbitrary topological space, fixed throughout this section. We use U and V (often subscripted) to range over non-empty open sets in X. For $A, B \subseteq X$, we write $A \leqslant B$ to mean that $A \sim B$ is meager (first category), and $A \equiv B$ to mean $A \leqslant B$ and $B \leqslant A$. A is said to have the Baire property if $A \equiv \mathcal{O}$ for some open set $\mathcal{O}$.

Let I be the ideal of meager sets in X, and BP the class of Baire property sets in X. Then I is a σ-ideal in X, and BP is a σ-field of subsets of X [11]. We form the Boolean algebra $\mathcal{P}(X)/I$, and denote its elements A/I, for $A \subseteq X$. An extremely important subalgebra is BP/I. The Birkhoff–Ulam theorem [11] says that BP/I is a complete Boolean algebra.

X is a Baire space if no non-empty open set is meager. Open meager sets will be a nuisance, so we use a device from [5] to avoid them. Let E be the union of all open meager sets in X, and let $D = X \sim E$. Then [5] D is comeager, and is regular open (the closure of an open set $\mathcal{O}$).

3.1. *If $b \in BP/I$, and $b \neq 0$, then there exists an open set U such that $b = U/I$, and U has no non-empty open meager subsets (i.e., U is a Baire space).*

Indeed, by definition, there exists an open V such that $b = V/I$. If D and $\mathcal{O}$ are as above, then $U = V \cap \mathcal{O}$ has the desired properties.

As a final preliminary, we recall the Banach–Mazur theorem [7].

THEOREM (Banach–Mazur). *Let $A \subseteq X$. For all non-empty open sets U, $A \geqslant U$, if and only if*

$$\forall U_0 \subseteq U\ \exists V_0 \subseteq U_0\ \forall U_1 \subseteq V_0 \ldots \bigcap_{i \in \omega} V_i \subseteq A.$$

We are now ready to state and prove a theorem from which our main theorem, 3.3, will follow at once.

THEOREM 3.2. *Suppose $A = (A_n\colon n \in \omega)$ is a sequence of open sets in X. Further suppose that $S \subseteq \mathcal{P}(\omega)$ is a Σ_2^1 set; say $t \in \mathcal{P}(\omega)$, and Φ is a Σ_2^1 formula such that,*

for all $x \subseteq \omega$, $x \in S$ if, and only if, $\Phi(x, t)$ holds.

Let r be the BP/I-Boolean valued subsets of $\check{\omega}$ such that,

for all $n \in \omega$, $r(\check{n}) = A_n/I$.

Then

$$\mathcal{O}_S^X(A)/I \geqslant \|\Phi(r, \check{t})\|_{BP/I}.$$

Proof. For short, set $\mathfrak{B} = (B, \wedge, \vee, -, \leqslant, 0, 1) = BP/I$, $b_n = A_n/I$ for $n = 0, 1, \ldots$, and $d = \|\Phi(r, \check{t})\|_{\mathfrak{B}}$.

We must discuss a technical point. It would be convenient to have available an elementary substructure of the universe. Of course, no such structure need exists, but, for any formula θ of the language of set theory, it is a theorem, the "reflection principle" (see [4]), that for any set z there exists a transitive set $W \supseteq z$ such that

$(5)_\theta$ for all $x_0, x_1, \ldots, x_n \in W$, $\theta(x_0, x_1, \ldots, x_n)$ holds in W if and only if $\theta(x_0, x_1, \ldots, x_n)$ holds (in the universe).

Accordingly, let θ be the conjunction of all the (finitely many) formulas to which we will apply the reflection principle in the remaining part of this proof. Let W be a transitive set such that t, d, $(b_n\colon n\in\omega)$ and $\mathfrak{B}\in W$, and for which $(5)_\theta$ holds.

We now proceed with the proof of 3.2. If $d=0$ there is nothing to prove, so suppose $d\neq 0$. Then by 3.1, let U be an open set such that $d=U/I$ and U has no non-empty open meager subsets.

CLAIM.

$$\forall U_0 \subseteq U\, \exists V_0 \subseteq U_0\, \forall U_1 \subseteq V_0 \ldots \{n\colon \exists m\ V_m \leqslant A_n\}\in S. \tag{6}$$

Proof of claim. Apply Lemma 2.1 to our present $\mathfrak{B}$, W and d with $T = \{\mathfrak{B}, t, d, (b_n\colon n\in\omega)\}\cup\{b_n\colon n\in\omega\}$, to obtain a winning strategy F for player II in (4). We now describe player II's winning strategy in (6). As II plays the game in (6), he will simulate a play of the game in (4) in which he follows the winning strategy F.

Suppose I plays $U_0 \subseteq U$ in (6). Since U has no non-empty open meager subsets, $U_0/I\in\mathfrak{B}-\{0\}$ and so U_0/I is a legal play of the game in (4). Say F tells II to play (v_0, a_0) in (4). Then there exists $V_0\subseteq U_0$ such that $v_0 = V_0/I$. II plays V_0 (in (6)).

Now say I plays $U_1\subseteq V_0$. II simulates that I has played U_1/I in (4), and if F tells II to play (v_1, a_1) where $V_1\subseteq U_1$ and $v_1 = V_1/I$, II plays V_1.

Play continues in this manner, producing a play $(U_0, V_0, U_1, V_1, \ldots)$ of the game in (6), and a simulated play

$$(U_0/I, V_0/I, a_0, U_1/I, V_1/I, a_1, \ldots)$$

of the game in (4) in which II has followed the winning strategy F. Thus, if we put

$$M = \{\mathfrak{B}, t, d, (b_n\colon n\in\omega)\}\cup\{b_n\colon n\in\omega\}\cup\{U_n/I, V_n/I, a_n\colon n\in\omega\}$$

and

$$D = \{b\in B\colon V_n/I \leqslant b \text{ for some } n\in\omega\},$$

(1), (2) and (3) of Section 2 hold.

Since the axiom of extensionality is true, by (5), W is extensional, and so by (1), M is extensional. Thus we may let $'\colon M\to N$ be the transitive collapse of M, i.e. $'$ is an $\in$-isomorphism of M onto the transitive set N.

Since $\mathfrak{B}$ is a complete Boolean algebra (and recall that $\mathfrak{B}\in M$), by (5) and (1) $M\models$"$\mathfrak{B}$ is a complete Boolean algebra" and so, since $'$ is an isomorphism, $N\models$"$\mathfrak{B}'$ is a complete Boolean algebra."

Let $D^\backslash = \{b'\in B'\colon b\in D\cap M\}$. (Note that we cannot speak of D' since in general $D\notin M$.)

SUBCLAIM.

$D^\backslash$ is a N-generic ultrafilter over $\mathfrak{B}'$.

Proof of subclaim. That $D^\backslash$ is an ultrafilter over $\mathfrak{B}'$ follows easily from (2). To prove genericity, suppose $C\in N$ and $C\subseteq D^\backslash$. We show that $\Pi^{\mathfrak{B}'} C\in D^\backslash$.

Indeed, since $C\in N$, $C = A'$ for some $A\in M$. Since $A'\subseteq D^\backslash$, $A'\subseteq B'$ and so by isomorphism, (1) and (5), $A\subseteq B$. Since $\mathfrak{B}$ is complete, $\Pi^{\mathfrak{B}} A$ exists. Thus by (3)

$$\Pi^{\mathfrak{B}} A\in D\cap M \quad \text{or} \quad (\exists b\in A\cap M)\, \bar{b}\in D.$$

Suppose, for contradiction, that the latter holds. Then $b'\in A'$, so $b'\in D^\backslash$, so $b\in D$. This is impossible, since $\bar{b}\in D$ and D is a filter.

Therefore $\Pi^{\mathfrak{B}} A\in D\cap M$, so $(\Pi^{\mathfrak{B}} A)'\in D^\backslash$. A short argument shows that $(\Pi^{\mathfrak{B}} A)' = \Pi^{\mathfrak{B}'} C$. Thus $\Pi^{\mathfrak{B}'} C\in D^\backslash$, and the subclaim is established.

Hence $N[D^\backslash]$ is a generic extension of N.

Recall that we put $\|\Phi(r, \check{t})\|_{\mathfrak{B}} = d$. As it is a basic fact about forcing that "limited forcing is definable", and it is easily checked that r, $\check{t}$, $\mathfrak{B}$ and $d\in M$ by (5), (1) and the fact that $'$ is an isomorphism, we have

$$\|\Phi(r', \check{t}')\|_{\mathfrak{B}'} = d'.$$

Now $d'\in D^\backslash$ (indeed $d\geqslant V_0/I$ and so $d\in D$), so by the "truth lemma" for forcing

$$N[D^\backslash]\models\Phi(s, t')$$

where $s = \{n\in\omega\colon b'\in D^\backslash\}$. Recall that Φ was assumed to be a Σ^1_2 formula, and so by Mostowski's absoluteness theorem, $\Phi(x, t')$ is true, i.e.

$$s = \{n\in\omega\colon b'\in D^\backslash\}\in S. \tag{7}$$

Now, unscrambling definitions,

$$\begin{aligned} b'_n\in D^\backslash \quad &\text{if and only if } b_n\in D,\\ &\text{if and only if } \exists m\ V_m/I\leqslant b_n,\\ &\text{if and only if } \exists m\ U_m\leqslant A_n. \end{aligned}$$

Hence, by (7), $\{n\in\omega\colon \exists m\, U_m\leqslant A_n\}\in S$. This shows that our strategy for the game in (6) is indeed a winning strategy, and completes the proof of claim (6).

To complete the proof, we borrow a device (due to Vaught) from our [10]. According to Proposition 4.3 (the resolving lemma) of [10], from (6) it follows that there is a *winning* strategy G for player II in the game in (6) such that, for any play $(U_0, V_0, U_1, V_1, \ldots)$ in which player II plays according to G, for all $n\in\omega$, $V_n\subseteq A_n$ or $V_n\cap A_n = \emptyset$.

CLAIM. *G is a winning strategy for the Banach–Mazur game*

$$\forall U_0\subseteq U\,\exists V_0\subseteq U_0\,\forall U_1\subseteq V_0 \ldots \bigcap_{n\in\omega} V_n\subseteq \mathcal{O}^X_S(A). \tag{8}$$

Proof of claim. Let $(U_0, V_0, \ldots)$ be a play of the game in (8) in which player II has followed the strategy G. Suppose $x \in \bigcap_{n\in\omega} V_n$.

Fix $n \in \omega$. If $x \in A_n$, then $V_n \subseteq A_n$, so certainly $\exists m V_m \leqslant A_n$.

Conversely, suppose $\exists m V_m \leqslant A_n$. If $V_n \cap A_n = \emptyset$, one sees easily that one of the sets V_m, V_n is meager, contradicting the fact that U has no non-empty open meager subsets. Thus $V_n \subseteq A_n$, and so $x \in A_n$.

Hence for all $n \in \omega$, $x \in A_n$ if and only if $\exists m V_m \leqslant A_n$. Now since G was a winning strategy for (6), we have $\{n\colon x \in A_n\} \in S$, that is, $x \in \mathcal{O}_S^X(A)$. This completes the proof of the claim.

Thus by (8) and the Banach–Mazur theorem, $\mathcal{O}_S^X(A) \geqslant U$. Recalling that $U/I = \|\Phi(r, \check{t})\|_{\mathfrak{B}}$, we have

$$\mathcal{O}_S^X(A)/I \geqslant \|\Phi(r, \check{t})\|_{\mathfrak{B}},$$

and the proof of 3.2 is complete.

As promised, from 3.2 we obtain a short proof of 3.3

THEOREM 3.3. *Absolutely Δ_2^1 Boolean operations preserve the Baire property.*

In detail, let $A = (A_n\colon n \in \omega)$ be a sequence of subsets of X with the Baire property. Suppose $S \subseteq \mathcal{P}(\omega)$ is an absolutely Δ_2^1 set; say Φ and Ψ are Σ_2^1 formulas and $t \subseteq \omega$ such that

for all $x \subseteq \omega$, $x \in S$ if and only if $\Phi(x, t)$ holds if and only if $\sim\Psi(x, t)$ holds, and

(9) *for all complete Boolean algebras $\mathfrak{B}$, $\|\forall x \subseteq \check{\omega}(\Phi(x, \check{t}) \leftrightarrow \sim\Psi(x, \check{t}))\|_{\mathfrak{B}} = 1$.*

Then

1) *$\mathcal{O}_S^X(A)$ has the Baire property.*

2) *If r is the Boolean BP/I-valued subset of $\check{\omega}$ such that, for all $n \in \omega$, $r(\check{n}) = A_n/I$, then $\mathcal{O}_S^X(A)/I = \|\Phi(r, \check{t})\|_{BP/I}$.*

Proof. First, note that 2) implies 1). Next, observe that, using (1.1), 2) reduces to the case where all of the sets A_n are open. With this reduction, all of the hypotheses of 3.2 are satisfied, and we have

$$\mathcal{O}_S^X(A)/I \geqslant \|\Phi(r, \check{t})\|_{BP/I}. \tag{10}$$

Likewise, the hypotheses of 3.2 are satisfied with Ψ for Φ and $\sim S$ for S. Thus

$$\mathcal{O}_{\sim S}^X(A)/I \geqslant \|\Psi(r, \check{t})\|_{BP/I}. \tag{11}$$

It is routine to check that $\mathcal{O}_{\sim S}^X(A) = \sim\mathcal{O}_S^X(A)$. Also, by (9), $\overline{\|\Psi(r, \check{t})\|_{BP/I}} = \|\Phi(r, \check{t})\|_{BP}$. Thus, taking complements in (11) we have

$$\|\Phi(r, \check{t})\|_{BP/I} \geqslant \mathcal{O}_S^X(A)/I. \tag{12}$$

(10) and (12) yield the conclusion of 2), and we are done.

Remarks. 1) It is clear from the proof of 3.3 that in (9) the phrase "for all complete Boolean algebras $\mathfrak{B}$" may be weakened to "for $\mathfrak{B} = BP/I$".

2) Let $(X, \mathcal{M}, \mu)$ be an arbitrary σ-finite complete measure space. By a theorem in [8], there exists a topology $\mathfrak{J}$ for X with the property that

"$(X, \mathcal{M}, \mu)$-measurable $= (X, \mathfrak{J})$-Baire property"

and

"$(X, \mathcal{M}, \mu)$-measure zero $= (X, \mathfrak{J})$-meager".

Using the topological space $(X, \mathfrak{J})$, 3.3 yields at once 3.3′, which is 3.3 with "σ-finite measure space" replacing "topological space", "measurability" replacing "the Baire property", $\mathcal{M}$ replacing BP, and letting I be the ideal of sets of measure zero. In particular, all absolutely Δ_2^1 Boolean operations preserve measurability.

In the next section we shall, by another method, establish a generalization of 3.3′ (but not of 3.3) originally due to Vaught.

3) Let X be the space of infinite subsets of ω with the Ellentuck topology (see [2]). 3.3, along with the results of [2], yield at once the result that absolutely Δ_2^1 sets are Ramsey. This extends the result of Silver and Mathias that analytic sets are Ramsey.

Section 4. Other fields of sets. Let X be an arbitrary non-empty set, $\mathfrak{J}$ a σ-field of subsets of X, and I a σ-ideal on X such that $I \subsetneqq \mathfrak{J}$. Our main object in this section is to prove the following theorem, due to R. Vaught:

THEOREM 4.1. *Suppose the Boolean algebra $\mathfrak{J}/I$ satisfies the countable chain condition. Then $\mathfrak{J}$ is invariant under all absolutely Δ_2^1 Boolean operations.*

4.1 overlaps with but does not include 3.3. However, 4.1 does include 3.3′, as the Boolean algebra "measurable/measure zero" satisfies the countable chain condition for any σ-finite measure space X.

Our proof of 4.1 is accomplished by assigning new (non-topological) meanings to the topological terminology used in Theorems 3.2 and 3.3 (in particular "open", "meager", and "Baire property") in such a way that our proofs of 3.2 and 3.3 are valid for these new meanings. The reinterpreted 3.3 will immediately imply 4.1.

For the remainder of this section, fix arbitrary X, $\mathfrak{J}$ and I as above. As usual, we form the Boolean algebra $\mathcal{P}(X)/I$, and subalgebra $\mathfrak{B} = \mathfrak{J}/I$. We also add the hypothesis

(13) $\mathfrak{B}$ satisfies the countable chain condition.

Let $A, B \subseteq X$. We say A is *open* if $A \in \mathfrak{J} \sim I$ or if $A = \emptyset$. We say A is *meager* if $A \in I$.

(Aside: The space $\underline{X}$ consisting of X along with its open sets as just defined, is not in general a topological space, though as we shall see, X

satisfies a few key theorems of topology, suitably interpreted. It is true, that if one applies one of the usual definitions of 'meager' (in terms of 'open') to our $\underline{X}$, our notion of 'meager' is obtained (even without assuming (13); see [9]), but we will not need this fact).

Say $A \leqslant B$ if $A \sim B$ is meager. We say $A \equiv B$ if $A \leqslant B$ and $B \leqslant A$. A has the *Baire property* if $A \equiv \mathcal{O}$ for some open set $\mathcal{O}$.

Let BP be the class of sets with the Baire property. Then one easily see that

4.2. $BP = \mathfrak{I}$, *so* $BP/I = \mathfrak{I}/I (= \mathfrak{B})$.

We shall need a few simple facts.

4.3. *Every collection of disjoint open sets is countable.*

Indeed, 4.3 is equivalent (13), as an easy argument shows. As a consequence of 4.3, we obtain 4.4.

4.4. *The class of open sets is closed under disjoint unions.*

4.5. $\mathfrak{B}$ *is a complete Boolean algebra.*

4.5 follows from a general theorem about Boolean algebras: A σ-complete Boolean algebra satisfying the countable chain condition is complete (see [11]). Alternatively, 4.5 can be obtained directly from 4.4, without using (13).

Finally, a version of the Banach–Mazur theorem holds. Let U, V usually subscripted, range over non-empty open sets.

THEOREM 4.6. *Let* $A \subseteq X$. *For all non-empty open sets* U, $A \geqslant U$ *if and only if*

(14) $$\forall U_0 \subseteq U \exists V_0 \subseteq U_0 \forall U_1 \subseteq V_0 \ldots \bigcap_{n\in\omega} V_n \subseteq A.$$

Proof. Suppose $A \geqslant U$, i.e., $U \sim A \in I$. Then, in particular,

$$U \cap A = U \sim (U \sim A) \in \mathfrak{I}.$$

Say player I plays $U_0 \subseteq U$. We claim that $U_0 \cap A$ is nonempty and open. Indeed, $U_0 \cap A = U_0 \cap (U \cap A) \in \mathfrak{I}$. Also $U_0 \cap A \notin I$, for otherwise $U_0 = (U_0 \cap A) \cup (U_0 \sim A) \subseteq (U_0 \cap A) \cup (U \sim A) \in I$, contradicting the fact that U_0 is open. Thus $U_0 \cap A \in \mathfrak{I} \sim I$; player II plays $V_0 = U_0 \cap A$. Then, no matter how play continues, $\bigcap_{n\in\omega} V_n \subseteq A$. This proves (14).

Our proof of the converse copies the proof in [7] for topological spaces. Suppose F is a winning strategy for player II in (14). By a partial F-play, we mean a sequence $z = (U_0, V_0, \ldots, U_n, V_n)$ such that $U_0 \supseteq V_0 \supset \ldots \supseteq V_n$, and II has played according to F. Another partial F-play $z' = (U_0', V_0', \ldots, U_m', V_m')$ is said to be disjoint from z if $V_n \cap V_m' = \emptyset$.

Let S_0 be a maximal disjoint set of partial F-plays (U_0, V_0), and let $T_0 = \bigcup\{V_0\colon$ for some $U_0, (U_0, V_0) \in S_0\}$. Then by 4.4, T_0 is open. Thus $U \sim T_0 \in \mathfrak{I}$. Actually, $U \sim T_0 \in I$, for otherwise $U \sim T_0$ is non-empty and open, so for some V we could add $(U \sim T_0, V)$ to S_0, contradicting the maximality of S_0.

Now let S_1 be a maximal disjoint set of partial F-plays (U_0, V_0, U_1, V_1) such that $(U_0, V_0) \in S_0$, and let

$$T_1 = \bigcup\{V_1\colon \text{for some } U_0, V_0, U_1, (U_0, V_0, U_1, V_1) \in S_1\}.$$

Again by 4.4, T_1 is open. We claim that $U \sim T_1 \in I$. It suffices to show that $T_0 \sim T_1 \in I$, for then $U \sim T_1 \subseteq (U \sim T_0) \cup (T_0 \sim T_1) \in I$.

Indeed, suppose for contradiction that $T_0 \sim T_1$ is non-empty and open. Now $T_0 \sim T_1 = \bigcup\{V_0 \sim T_1\colon$ for some $U_0 (U_0, V_0) \in S_0\}$; by 4.3, this is a countable union, so for some $(U_0, V_0) \in S_0$, $V_0 \sim T_1$ is non-empty and open. But then we could add $(U_0, V_0, V_0 \sim T_1, V)$ to S_1 for some V, contradicting the maximality of S_1. This proves the claim.

Continuing in this manner, obtain a sequence of partial F-plays, $S_0, S_1, \ldots$, and a sequence of open sets, $T_0, T_1, \ldots$, such that, for all i, $U \sim T_i \in I$. Thus $U \sim \bigcap_{i\in\omega} T_i \in I$. We complete the proof by showing that $\bigcap_{i\in\omega} T_i \subseteq A$.

Let $x \in \bigcap_{i\in\omega} T_i$. Since $x \in T_0$, there is a unique $(U_0, V_0) \in S_0$ such that $x \in V_0$. Since $x \in T_1$, this is a unique $(U_0', V_0', U_1, V_1) \in S_1$ such that $x \in V_1$. But then $(U_0', V_0') \in S_0$ by construction, and so $x \in V_0'$; thus by uniqueness, $(U_0', V_0') = (U_0, V_0)$. Continuing in this way, obtain a play $(U_0, V_0, U_1, V_1, \ldots)$ of (14) according to the winning strategy F. Therefore $\bigcap_{i\in\omega} V_i \subseteq A$, so $x \in A$.

Thus $\bigcap_{i\in\omega} T_i \subseteq A$, so since $U \sim \bigcap_{i\in\omega} T_i \in I$, $U \sim A \in I$, that is, $A \geqslant U$, as desired.

The reader may now read the statements and proofs of Theorems 3.2 and 3.3 using the notions of open, Baire property, meager, BP, I, and $\leqslant$ of this section. The only needed change is the reference to 3.1 just before (6). In our current context, we may take U to be any open set such that $d = U/I$, as there are no non-empty open meager sets. Indeed, if U is non-empty and open then, by definition, $U \notin I$, so U is not meager.

Recalling 4.2, then, the first line of 3.3 as now interpreted, implies that $\mathfrak{I}$ is invariant under all absolutely Δ_2^1 Boolean operations. Thus we have 4.1.

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On Borel-measurable collections of countable-dimensional sets

by

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Abstract. Let B be a Borel set in the product $S\times T$ of compact metrizable spaces, whose vertical sections $B(s)$ are countable-dimensional (i.e. unions of countably many zero-dimensional sets) G_δ-sets in T. It is an open question whether the small transfinite dimension ind of the vertical sections of B is bounded, i.e. if $\sup\{\operatorname{ind} B(s)\colon s\in S\}<\omega_1$. We show that a certain additional assumption about B (an existence of a Borel-measurable, point-finite, sectionwise separation for B, see Definition 3.2) guarantees that this is true.

§ 1. Preliminaries. In this paper we consider only separable metrizable spaces and "compactum" means "compact space". Our terminology concerning analytic sets follows [K] and the terminology related to dimension theory follows [A–P], [E1] and [Na].

1.1. Terminology and notation. A closed set L in a space X *separates* two disjoint sets A and B in X if $X\setminus L = U\cup V$, U and V being disjoint open sets with $A\subset U$ and $B\subset V$. We denote by ω the set of natural numbers, I is the real unit interval and Fin ω is the set of all non-empty finite subsets of ω. We identify the power set $2^{\mathrm{Fin}\,\omega}$ with the Cantor cube $\{0,1\}^{\mathrm{Fin}\,\omega}$, i.e. we identify each subset of Fin ω with its characteristic function and we consider the characteristic functions with pointwise topology. The symbol $|A|$ stands for the cardinality of the set A. A sequence $\langle A_i\colon i\in\omega\rangle$ of subsets of X is *point-finite* if for each $x\in X$ the set $\{i\in\omega\colon x\in A_i\}$ is finite (thus we exclude the possibility that one set occurs in the sequence infinitely many times). Given a set E in the product $S\times T$ we denote by $E(s)$ the vertical section $\{t\in T\colon (s,t)\in E\}$ of the set E at the point $s\in S$.

1.2. Countable-dimensional sets and the small transfinite dimension. A space X is *countable-dimensional* if $X=\bigcup_{i=1}^{\infty} X_i$, X_i being zero-dimensional.

The small transfinite dimension ind is the ordinal-valued function obtained through the extension of the classical Menger–Urysohn inductive dimension by transfinite induction. If the transfinite dimension is not defined for X, we write $\operatorname{ind} X=\infty$; since our spaces have always a countable base, if $\operatorname{ind} X\neq\infty$, then $\operatorname{ind} X<\omega_1$.