# Monotone decompositions of hereditarily unicoherent continua via set functions and quasi-orders

by

G. R. Gordh, Jr. (Sacramento, Ca.) and Eldon J. Vought (Chico, Ca.)

**Abstract.** The following theorem is obtained by applying the FitzGerald–Swingle theory of core decompositions to a pointed version of the aposyndetic set function $T$. THEOREM. *If $X$ is a hereditarily unicoherent continuum, then for each point $p$ of $X$ there exists a unique minimal monotone upper semi-continuous decomposition $\mathscr{D}_p = \{D(x)\}$ of $X$ such that $X/\mathscr{D}_p$ is a dendroid which is smooth at $D(p)$*. It is also shown that the decomposition $\mathscr{D}_p$ may be viewed as the level set decomposition of a quasi-order $\leqslant_p^*$ termed the generalized weak cutpoint order. An explicit description of $\leqslant_p^*$ is provided for continua which satisfy a strong aposyndetic property.

**1. Introduction.** Let $X$ denote a hereditarily unicoherent metric continuum. The theory of core decompositions due to FitzGerald and Swingle using the aposyndetic set function $T$ ([2], Theorem 2.7) together with the observation that every semi-locally connected hereditarily unicoherent continuum is a dendrite (e.g., [5], Theorem 1) yields

THEOREM A. *There exists a unique minimal monotone upper semi-continuous decomposition $\mathscr{A}$ of $X$ such that $X/\mathscr{A}$ is a dendrite.*

If $X$ is smooth at $p$ in the sense of [3], then the weak cutpoint order $\leqslant_p$ is closed, and the level set decomposition $\mathscr{D}_p$ is upper semi-continuous and monotone. According to Corollary 4.1 of [4], we have.

THEOREM B. *If $X$ is smooth at $p$, then there exists a unique minimal monotone upper semi-continuous decomposition $\mathscr{D}_p = \{D(x)\}$ of $X$ such that $X/\mathscr{D}_p$ is a dendroid which is smooth at $D(p)$.*

The main purpose of this paper is to establish the following result, which is closely related to Theorems A and B.

THEOREM 1. *If $X$ is a hereditarily unicoherent metric continuum, then for each point $p$ of $X$ there exists a unique minimal monotone upper semi-continuous decomposition $\mathscr{D}_p = \{D(x)\}$ of $X$ such that $X/\mathscr{D}_p$ is a dendroid which is smooth at $D(p)$.*

Observe that in Theorem A the quotient space $X/\mathscr{A}$ is a dendroid which is smooth at each point (i.e., a dendrite). Thus Theorem 1 may be viewed as a pointed version of Theorem A. In this spirit it is shown that Theorem 1 can be obtained by applying the theory of core decompositions to a pointed version of the aposyndetic set function $T$ denoted by $T_p$.

On the other hand, Theorem 1 may be viewed as a generalization of Theorem B. In this spirit it is shown that the decomposition $\mathscr{D}_p$ in Theorem 1 can be obtained as the level set decomposition of a closed quasi-order $\leqslant_p^*$ termed the *generalized weak cutpoint order*.

Theorem 1 fills a gap between Theorem A and the following result ([8], Theorem 5).

THEOREM C. *There exists a unique minimal monotone upper semi-continuous decomposition $\mathscr{G}$ of $X$ such that $X/\mathscr{G}$ is a semi-aposyndetic dendroid.*

Since dendrites are smooth at each point ([1], Theorem 6), and smooth dendroids are semi-aposyndetic ([1], Corollary 4), it follows that $\mathscr{G} \leqslant \mathscr{D}_p \leqslant \mathscr{A}$ for each $p$ in $X$ where $\leqslant$ denotes refinement.

Finally it is shown that the generalized weak cutpoint order $\leqslant_p^*$ can be simply and explicitly described for continua on which the aposyndetic set function $K$ is sufficiently well-behaved.

For simplicity the results have been stated above for metric spaces; however, all results are actually established in the setting of Hausdorff spaces.

**2. Preliminaries.** By a *continuum* we mean a compact connected Hausdorff space. The continuum $X$ is said to be *hereditarily unicoherent* in case each subcontinuum of $X$ is unicoherent. If $A$ and $B$ are subsets (or points) of such a continuum, then $AB$ will denote the unique subcontinuum which is irreducible with respect to containing $A \cup B$. An *arc* (not necessarily metrizable) is a continuum with exactly two non-separating points. An *arboroid* (or *dendroid* in the metric setting) is an arcwise connected hereditarily unicoherent continuum. A *tree* (or *dendrite* in the metric setting) is a locally connected hereditarily unicoherent continuum.

A pointed hereditarily unicoherent continuum $(X, p)$ is said to be *smooth* [3] if for each net of points $x_n$ convering to $x$ in $X$, the net of subcontinua $px_n$ converges to $px$. A smooth arboroid $(X, p)$ is called a *generalized tree* [9]. Metrizable generalized trees are termed *smooth dendroids* [1].

We shall make use of the notion of *aposyndesis* due to F. B. Jones (see [7] for survey articles and an extensive bibliography). If $A$ and $B$ are subsets (or points) of a continuum $X$, then $X$ is said to be *aposyndetic* at $A$ with respect to $B$ if there is a continuum-neighborhood $Q$ of $A$ which misses $B$. The continuum $X$ is called *semi-aposyndetic* if for each pair of distinct points $x$ and $y$ of $X$ either $X$ is aposyndetic at $x$ with respect to $y$ or at $y$ with respect to $x$.

PROPOSITION 2.1 ([4], Theorem 2.3). *If the continuum $X$ is hereditarily unicoherent and semi-aposyndetic, then $X$ is an arboroid.*

The continuum $X$ is said to be *aposyndetic toward the point* $p$ [6] provided that $X$ is aposyndetic at $q$ with respect to $r$ whenever $r$ does not cut $p$ from $q$ (i.e., whenever $p$ and $q$ can be joined by a subcontinuum missing $r$).



PROPOSITION 2.2 ([6], Theorem 6). *Let $(X, p)$ be a hereditarily unicoherent continuum. Then $(X, p)$ is smooth if and only if $X$ is aposyndetic toward $p$.*

Let $A$ be a subset of the continuum $X$. Then $T(A)$ denotes the set $\{x \in X\colon X$ is not aposyndetic at $x$ with respect to $A\}$. Let $T^0(A) = A$, $T^1(A) = T(A)$, and for each natural number $n$, let $T^n(A) = T(T^{n-1}(A))$. If $A$ is a subcontinuum of $X$, then $T^n(A)$ is also a subcontinuum of $X$ (e.g., [2], Lemma 1.3). Dually, $K(A) = \{x \in X\colon X$ is not aposyndetic at $A$ with respect to $x\}$. Let $K^0(A) = A$, $K^1(A) = K(A)$, and $K^n(A) = K(K^{n-1}(A))$. If $A$ is a subcontinuum of $X$, then $K^n(A)$ need not be connected; however, when $X$ is hereditarily unicoherent it is easy to see that $K^n(A)$ is a subcontinuum of $X$.

PROPOSITION 2.3. *Let $X$ be a hereditarily unicoherent continuum, and let $A$ be a subcontinuum of an open set $V$ such that $K(A) = A$. Then there is a nest of subcontinua $\{Q_i\}_{i=0}^{\infty}$ (i.e., $Q_0 \supseteq \mathrm{Int}(Q_0) \supseteq Q_1 \supseteq \mathrm{Int}(Q_1) \supseteq Q_2 \supseteq \ldots$) such that $A \subseteq \mathrm{Int}(Q_i) \subseteq Q_i \subseteq V$ for each $i$.*

Proof. For each point $z$ in $X \backslash V$ there is a continuum-neighborhood $Q(z)$ of $A$ which misses $z$. By compactness there are finitely many points $z_1, \ldots, z_k$ such that $Q_0 = Q(z_1) \cap \ldots \cap Q(z_k) \subseteq V$. The set $Q_1$ is obtained in the same way, using $\mathrm{Int}(Q_0)$ in place of $V$. Clearly the process can be continued for each natural number $i$.

By a *quasi-order* $\leqslant$ on a continuum $X$ we mean a reflexive and transitive relation. A *zero* is a point $p$ such that $p \leqslant x$ for each $x$ in $X$. The *level set* of the point $x$ is the set $E(x) = \{y \in X\colon x \leqslant y$ and $y \leqslant x\}$. The *lower set* of $x$ is the set $L(x) = \{y \in X\colon y \leqslant x\}$. The quasi-order $\leqslant$ is *closed* provided it is closed as a subset of $X \times X$. A subset $C$ of $X$ is termed a *chain* if whenever $x$ and $y$ belong to $C$, then either $x \leqslant y$ or $y \leqslant x$. If $x \leqslant y$ in $X$, then the *interval* $[x, y]$ is the set $\{z \in X\colon x \leqslant z \leqslant y\}$.

**3. Decompositions via the set function $T_p$.** Let $X$ denote a hereditarily unicoherent continuum containing a fixed point $p$.

We define the set function $T_p$ by the equation $T_p(A) = pA \cap T(A)$ for each $A \subseteq X$. If $T_p(A) = A$, then $A$ is said to be *$T_p$-closed.*

Clearly $X$ is a tree if and only if each point of $X$ is $T$-closed. Restating Corollary 3.6 of [4] we have

PROPOSITION 3.1. *$(X, p)$ is a generalized tree if and only if each point of $X$ is $T_p$-closed.*

If $A \subseteq B \subseteq X$, then $A \subseteq T_p(A) \subseteq T_p(B)$. Consequently, $T_p$ is *expansive* in the sense of [2]. According to Theorem 2.5 of [2] there exists a unique minimal upper semi-continuous decomposition $\mathscr{D}_p$ of $X$ such that the elements of $\mathscr{D}_p$ are $T_p$-closed. Furthermore, the decomposition $\mathscr{D}_p$ is monotone. To see this, let $D$ be an element of $\mathscr{D}_p$, and let $C$ be a component of $D$. Since $T_p(C) = pC \cap T(C)$ and $T(C)$ is a continuum (see Section 2), it follows that $T_p(C)$ is a continuum. But $C \subseteq T_p(C) \subseteq T_p(D) = D$, and hence $C = T_p(C)$. Consequently the decomposition $\mathscr{D}_p^*$ of $X$ into components of elements of $\mathscr{D}_p$ is upper semi-continuous with $T_p$-closed elements. So $\mathscr{D}_p = \mathscr{D}_p^*$ and $\mathscr{D}_p$ is monotone. We have established

PROPOSITION 3.2. *There exists a unique minimal upper semi-continuous decomposition $\mathscr{D}_p$ of X such that the elements of $\mathscr{D}_p$ are $T_p$-closed. Furthermore, $\mathscr{D}_p$ is monotone.*

Next it will be shown that the decomposition $\mathscr{D}_p$ of Proposition 3.2 is that required in Theorem 1. In the Hausdorff setting Theorem 1 can be restated as follows.

THEOREM 1. *If $(X, p)$ is a hereditarily unicoherent Hausdorff continuum, then there exists a unique minimal monotone upper semi-continuous decomposition $\mathscr{D}_p = \{D(x)\}$ of X such that $(X/\mathscr{D}_p, D(p))$ is a generalized tree.*

Proof. Let $\mathscr{D}_p$ denote the decomposition of Proposition 3.2, and let $f\colon X \to X/\mathscr{D}_p$ be the natural quotient map.

CLAIM 1. *$X/\mathscr{D}_p$ is an arboroid.*

Since monotone maps preserve hereditary unicoherence, it suffices to prove that $X/\mathscr{D}_p$ is semi-aposyndetic (see Proposition 2.1). Let $A$ and $B$ be distinct elements of $\mathscr{D}_p$. Clearly it is enough to show that either $T(A) \cap B = \emptyset$ or $T(B) \cap A \in \emptyset$. Suppose neither equality holds, and let $C = (T(A) \cup B) \cap \cap (A \cup T(B))$. By hereditary unicoherence, $C$ is a continuum containing $A$ and $B$. Similarly, $C \cap (pA \cup pB)$ is a continuum containing $A$ and $B$. By elementary set algebra $C \cap (pA \cup pB)$ becomes

$$(A \cup B) \cup (pA \cap T(A) \cap T(B)) \cup (pB \cap T(B) \cap T(A)).$$

Since $A$ and $B$ are $T_p$-closed, it follows that

$$C \cap (pA \cup pB) = (A \cup B) \cup (A \cap T(B)) \cup (B \cap T(A)) = A \cup B.$$

This contradiction establishes the claim.

CLAIM 2. *$(X/\mathscr{D}_p, D(p))$ is a generalized tree.*

According to Proposition 2.2 it suffices to show that $X/\mathscr{D}_p$ is aposyndetic toward $f(p)$. Suppose that $x \notin f(p)y$. Let $A = f^{-1}(x)$ and $B = f^{-1}(y)$. Note that $pB \subseteq f^{-1}(f(p)y) \subseteq X \backslash A$. To establish aposyndesis at $y$ with respect to $x$ in $X/\mathscr{D}_p$ it suffices to demonstrate that $B \cap T(A) = \emptyset$. Suppose $B \cap T(A) \neq \emptyset$, and observe that $pA$ and $pB \cup T(A)$ are each continua containing $p$ and $A$. By hereditary unicoherence $pA \cap (pB \cup T(A))$ must be a continuum containing $p$ and $A$. However, since $A$ is $T_p$-closed,

$$pA \cap (pB \cup T(A)) = (pA \cap pB) \cup (pA \cap T(A)) = (pA \cap pB) \cup A \subseteq pB \cup A$$

where $pB$ and $A$ are disjoint closed sets. This is a contradiction.

Now suppose that $\mathscr{F}_p = \{F(x)\}$ is any monotone upper semi-continuous decomposition of $X$ such that $(X/\mathscr{F}_p, F(p))$ is a generalized tree.

CLAIM 3. *$\mathscr{D}_p$ refines $\mathscr{F}_p$.*



Let $g\colon X \to X/\mathscr{F}_p$ be the natural quotient map. According to Proposition 3.2 it is enough to show that each element $F$ of $\mathscr{F}_p$ is $T_p$-closed. Suppose that $T_p(F) \neq F$. Choose $x \in T_p(F) \backslash F$ and note that $x \in pF \backslash F$. From the irreducibility of $pF$, the monotonicity of $g$ and the hereditary unicoherence of $X$, it follows that $g(x) \in g(p)g(F)$. Since $X/\mathscr{F}_p$ is arcwise connected and aposyndetic toward $g(p)$, there is a continuum-neighborhood $H$ of $g(x)$ which misses $g(F)$. So $g^{-1}(H)$ is a continuum-neighborhood of $x$ which misses $F$. Consequently $x \notin T(F)$, so $x \notin T_p(F)$. This contradiction establishes that $F$ is $T_p$-closed.

The conclusion of the theorem now follows immediately from Claims 1, 2 and 3.

**4. Decompositions via the quasi-order $\leqslant_p^*$.** Let $X$ be a hereditarily unicoherent continuum containing a fixed point $p$. We say that a quasi-order $\leqslant$ on $X$ is *p-admissible* in case (1) $\leqslant$ is closed, (2) $p$ is a zero of $\leqslant$, (3) each level set $E(x)$ is connected, and (4) each lower set $L(x)$ is a connected chain. It follows immediately that the level sets and the lower sets of $\leqslant$ are continua. Furthermore, if $r \leqslant s$, then the interval $[r, s]$ is a continuum.

Recall that the quasi-order $\leqslant_p$ on $X$ defined by setting $x \leqslant_p y$ whenever $x \in py$ is called the *weak cutpoint order with respect to p.*

PROPOSITION 4.1. *$(X, p)$ is smooth if and only if the weak cutpoint order $\leqslant_p$ with respect to $p$ is p-admissible.*

Proof. If $\leqslant_p$ is $p$-admissible, hence closed, then $(X, p)$ is smooth by Theorem 3.1 of [4]. That $\leqslant_p$ is $p$-admissible when $(X, p)$ is smooth can be seen from the proof of Corollary 4.1 in [4].

PROPOSITION 4.2. *If $\{\leqslant_\alpha\colon \alpha \in A\}$ is any collection of p-admissible quasi-orders on X, then $\leqslant = \bigcap \{\leqslant_\alpha\colon \alpha \in A\}$ is a p-admissible quasi-order on X.*

Proof. Part (1) and (2) of the definition are clearly valid. Part (3) follows from hereditary unicoherence and the observation that $E(x) = \bigcap \{E_\alpha(x)\colon \alpha \in A\}$. Similarly $L(x) = \bigcap \{L_\alpha(x)\colon \alpha \in A\}$ is connected. It remains only to show that $L(x)$ is a chain for each $x$ in $X$. Suppose that $L(x)$ is not a chain for some $x$. Then there are points $z$ and $w$ in $L(x)$ which are not related by $\leqslant$. Note that $p < z < x$ and $p < w < x$. Choose $\gamma$ and $\beta$ in $A$ such that $z \nleqslant_\gamma w$ and $w \nleqslant_\beta z$. Since $L_\gamma(x)$ and $L_\beta(x)$ are chains, it follows that $w <_\gamma z$ and $z <_\beta w$. Thus $L_\gamma(w) \cap [z, x]_\gamma = \emptyset$ and $L_\beta(z) \cap [w, x]_\beta = \emptyset$. Let $Z$ be the continuum $L_\beta(z) \cup [z, x]_\gamma$ which contains $p$ and $x$; and let $W$ be the continuum $L_\gamma(w) \cup [w, x]_\beta$ which contains $p$ and $x$. By hereditary unicoherence $Z \cap W$ is a continuum containing $p$ and $x$. But

$$\begin{aligned} Z \cap W &= (L_\beta(z) \cup [z, x]_\gamma) \cap (L_\gamma(w) \cup [w, x]_\beta) \\ &= (L_\beta(z) \cap L_\gamma(w)) \cup ([z, x]_\gamma \cap [w, x]_\beta) \subseteq L_\beta(z) \cup [w, x]_\beta \end{aligned}$$

where $L_\beta(z)$ and $[w, x]_\beta$ are disjoint closed sets containing $p$ and $x$ respectively. Thus $L(x)$ is a chain and the proof is complete.

DEFINITION. We say that $x \leqslant_p^* y$ in $X$ provided that $x \leqslant_\alpha y$ in $X$ for every $p$-admissible quasi-order $\leqslant_\alpha$ on $X$. According to Proposition 4.2, $\leqslant_p^*$ is a well-defined $p$-admissible quasi-order on $X$. We call it the *generalized weak cutpoint order with respect to p*.

The proof of the next result shows that Theorem 1 may be viewed as a corollary of Proposition 4.2.

THEOREM 2. *The decomposition $\mathscr{E}_p = \{E(x)\}$ of $X$ into level sets of the generalized weak cutpoint order $\leqslant_p^*$ coincides with the decomposition $\mathscr{D}_p$ of Theorem* 1.

Proof. We first show that $\mathscr{D}_p$ refines $\mathscr{E}_p$. Let $\leqslant(\mathscr{E}_p)$ denote the quotient order on $X/\mathscr{E}_p$. It is easy to verify that $\leqslant(\mathscr{E}_p)$ is a closed partial order with zero $E(p)$ and arcs for lower sets of elements distinct from $E(p)$. Thus $\leqslant(\mathscr{E}_p)$ is a $E(p)$-admissible partial order; in fact, $\leqslant(\mathscr{E}_p)$ is the weak cutpoint order $\leqslant_{E(p)}$ on $X/\mathscr{E}_p$ with respect to $E(p)$. Thus, by Proposition 4.1, $(X/\mathscr{E}_p, E(p))$ is a generalized tree. By Theorem 1, $\mathscr{D}_p$ refines $\mathscr{E}_p$.

Now let $\leqslant_{D(p)}$ be the weak cutpoint order on $X/\mathscr{D}_p$ with respect to $D(p)$. Define an order $\leqslant$ on $X$ by letting $x \leqslant y$ if $D(x) \leqslant_{D(p)} D(y)$ in $X/\mathscr{D}_p$. Using the fact that $\leqslant_{D(p)}$ is a $D(p)$-admissible partial order it is easy to verify that $\leqslant$ is a $p$-admissible quasi-order on $X$ whose level set decomposition in $\mathscr{D}_p$. By the definition of $\leqslant_p^*$ it follows that $\leqslant_p^*$ is contained in $\leqslant$. This means that $\mathscr{E}_p$ refines $\mathscr{D}_p$ as required.

**5. An explicit description of $\leqslant_p^*$ when $K$ is finitely stable.** As above, let $X$ denote a hereditarily unicoherent continuum containing a fixed point $p$. We say that the aposyndetic set function $K$ is *n-stable on* $(X, p)$ if $K^n(P) = K^{n-1}(P)$ for each subcontinuum $P$ of $X$ which contains $p$.

Theorem 3.1 of [3] implies that $K$ is 1-stable on $(X, p)$ whenever $(X, p)$ is smooth. Observe that in this case $x \leqslant_p^* y$ if and only if $x \in K^1(py)$ (i.e., $x \in py$). Our final result generalizes this fact.

THEOREM 3. *If $K$ is n-stable on $(X, p)$, then $x \leqslant_p^* y$ if and only if $x \in K^n(py)$.*

Proof. Define $x \leqslant y$ in case $x \in K^n(py)$. We first establish that $\leqslant$ is a $p$-admissible quasi-order on $X$. To see that $\leqslant$ is transitive, assume that $x \leqslant y$ and $y \leqslant z$. Then $y \in K^n(pz)$ and hence $py \subseteq K^n(pz)$. Thus $K^n(py) \subseteq K^{2n}(pz) = K^n(pz)$. Since $x \leqslant y$, it follows that $x \in K^n(pz)$ and hence that $x \leqslant z$.

To see that $\leqslant$ is closed, let $(x_j, y_j)$ be a net in $X \times X$ converging to $(x, y)$ such that $x_j \leqslant y_j$ but $x \nleqslant y$. Thus $x \notin K^n(py)$, and since $K^n(py)$ is $K$-closed there exists a nest of subcontinua $\{Q_i\}_{i=0}^\infty$ containing $K^n(py)$ and missing $x$ (see Proposition 2.3). Choose $j$ large enough so that $x_j \notin Q_0$ and $y_j \in \operatorname{Int}(Q_n)$. Thus $py_j \subseteq Q_n$ so $K(py_j) \subseteq K(Q_n) \subseteq Q_{n-1}$ and, by induction, $K^n(py_j) \subseteq Q_0$. Now $x_j \notin K^n(py_j)$ contrary to the assumption that $x_j \leqslant y_j$.

Clearly $p$ is a zero of $\leqslant$.

Now suppose that some level set $E(x)$ of $\leqslant$ is not connected, and let $C_1$ and $C_2$ denote distinct components of $E(x)$ with $x \in C_1$. Observe that $C_1C_2 \subseteq K^n(px)$ and choose $z \in C_1C_2 \setminus E(x)$. Since $z \leqslant x$ it follows that $K^n(pz) \cap E(x) = \emptyset$. Let $\{Q_i\}_{i=0}^\infty$ be a nest of subcontinua containing $K^n(pz)$ but missing $E(x)$ (see Proposition 2.3). Observe that in the relative topology on $C_1C_2$ the continuum $Q_n \cap C_1C_2$ has non-void interior and hence separates $C_1C_2$ into exactly two components $B_1$ and $B_2$ containing $C_1$ and $C_2$, respectively. Observe that $px \subseteq Q_n \cup B_1$ and hence $px \cap C_2 = \emptyset$. The existence of the nest $\{Q_i\}_{i=0}^\infty$ implies that $T^n(C_2) \cap Q_n = \emptyset$. Since $T^n(C_2)$ is a continuum (as noted in Section 2) and $X$ is hereditarily unicoherent, it follows that $T^n(C_2) \cap B_1 = \emptyset$. But $px \subseteq Q_n \cup B_1$, so $K^n(px) \cap C_2 = \emptyset$, which contradicts the fact that $E(x) \subseteq K^n(px)$. Thus each level set is connected.

Each lower set $L(x)$ is of the form $K^n(px)$ and hence connected. It remains to show that each lower set $L(x)$ is a chain. Suppose not, and let $y$ and $z$ be points of $L(x)$ such that $y \notin K^n(pz)$ and $z \notin K^n(py)$. Let $\{Q_i(y)\}_{i=0}^\infty$ be a nest of continua containing $K^n(py)$ and missing $\{x, z\}$, and let $\{Q_i(z)\}_{i=0}^\infty$ be a nest of continua containing $K^n(pz)$ and missing $\{x, y\}$. Let $H = px \setminus (Q_n(y) \cup Q_n(z))$ and observe that $H$ is connected since $px$ is irreducible. Without loss of generality we may assume that $\operatorname{Cl}(H) \cap Q_n(y) \neq \emptyset$ and thus that $px \subseteq Q_n(y) \cup H$. Since $z \in K^n(px)$ it follows that $T^n(z) \cap px \neq \emptyset$ and, since $T^n(z) \cap Q_n(y) = \emptyset$, it follows that $T^n(z) \cap px \subseteq H$. Note that $K^n(pz) \cap \operatorname{Cl}(H) = \emptyset$. Thus $K^n(pz) \cup T^n(z)$ meets $px$ in a disconnected set which contradicts hereditary unicoherence.

Since $\leqslant$ is $p$-admissible and the generalized weak cutpoint order $\leqslant_p^*$ is the smallest $p$-admissible quasi-order on $X$, it follows that $\leqslant_p^*$ is a subsets of $\leqslant$. Suppose $x \leqslant y$, but $x \nleqslant_p^* y$. Let $f\colon X \to X/\mathscr{D}_p$ be the natural quotient map where $\mathscr{D}_p$ is the decomposition of Theorems 1 and 2. Since $f(p)f(y)$ is $K$-closed in $X/\mathscr{D}_p$ by Theorem 3.1 of [3], there is a nest of continua $\{Q_i\}_{i=0}^\infty$ containing $f(p)f(y)$ and missing $f(x)$. Using the nest of continua $\{f^{-1}(Q_i)\}_{i=0}^\infty$ in $X$, one sees that $x \notin K^n(py)$ and hence $x \nleqslant y$. This contradiction shows that $\leqslant$ and $\leqslant_p^*$ agree as required.



### References


[1] J. J. Charatonik and C. Eberhart, *On smooth dendroids*, Fund. Math. 67 (1970), pp. 297–322.

[2] R. W. FitzGerald and P. M. Swingle, *Core decompositions of continua*, Fund. Math. 61 (1967), pp. 33–50.

[3] G. R. Gordh, Jr., *On decompositions of smooth continua*, Fund. Math. 75 (1972), pp. 51–60.

[4] — *Concerning closed quasi-orders on hereditarily unicoherent continua*, Fund. Math. 78 (1973), pp. 61–73.

[5] — *Aposyndesis in hereditarily unicoherent continua*, in: *General Topology and Modern Analysis* (edited by L. F. McAuley and M. M. Rao), Academic Press, New York 1981, pp. 53–63.

[6] — *Aposyndesis and the notion of smoothness in continua*, ibid., pp. 65–70.

[7] L. F. McAuley and M. M. Rao (editors), *General Topology and Modern Analysis*, Academic Press, New York 1981.
[8] E. J. Vought, *Monotone decompositions of Hausdorff continua*, Proc. Amer. Math. Soc. 56 (1976), pp. 371–376.
[9] L. E. Ward, Jr., *Mobs, trees, and fixed points*, Proc. Amer. Math. Soc. 8 (1957), pp. 798–804.

G. R. Gordh, Jr
DEPARTMENT OF MATHEMATICS AND STATISTICS
CALIFORNIA STATE UNIVERSITY
Sacramento, California 95819
and (current address)
DEPARTMENT OF MATHEMATICS
GUILFORD COLLEGE
Greensboro, North Carolina 27410

Eldon J. Vought
DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY
Chico, California 95929

*Received 26 October 1981*

---

# On absolutely $\Delta_2^1$ operations

by

**Kenneth Schilling** (Los Angeles, Ca.)

**Abstract.** Every absolutely $\Delta_2^1$ Boolean operation preserves the Baire property in all topological spaces, and, as a consequence, measurability in all $\sigma$-finite complete measure spaces.

It is a classical theorem that the operation $(A)$ preserves the Baire property in all topological spaces, and measurability in all $\sigma$-finite complete measure spaces.

R. Solovay (unpublished) introduced the class of *absolutely $\Delta_2^1$ sets* (to be defined in the next section) in Polish spaces, and proved that they have the Baire property, and are Lebesgue measurable. Solovay's results were rediscovered and extended by Fenstad and Normann [3], who showed that an absolutely $\Delta_2^1$ set in an analytic space is measurable with respect to any $\sigma$-finite, complete, regular Borel measure.

In order to extend these results further, R. Vaught (unpublished; announced at Wrocław, 1977) considered the *absolutely $\Delta_2^1$ Boolean operations*, and showed that these operations preserve the Baire property in any topological space satisfying the countable chain condition, and measurability with respect to any $\sigma$-finite complete measure.

The main result here, in analogy with and extending the classical theorem cited above, is

THEOREM 3.3. *All absolutely $\Delta_2^1$ Boolean operations preserve the Baire property in all topological spaces.*

From 3.3, using a theorem in [8], we directly infer the part of Vaught's result dealing with measure.

Now let $\mathfrak{I}$ be an arbitrary $\sigma$-field of sets on a set $X$, and let $I$ be a $\sigma$-ideal on $X$ such that $I \subset \mathfrak{I}$. Vaught proved

THEOREM 4.1. *If the Boolean algebra $\mathfrak{I}/I$ satisfies the countable chain condition, then $\mathfrak{I}$ is invariant under all absolutely $\Delta_2^1$ Boolean operations.*

We cannot infer 4.1 directly from 3.3. However, we do show that, by introducing a simple device, the pattern of our proof of 3.3 carries over into a new proof of 4.1.

Most of the material herein appears in the author's doctoral dissertation [9]. I am grateful to my thesis advisor, Robert Vaught, for his help in all aspects of its preparation.