Mutually generic classes and incompatible expansions

by

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Abstract. A theorem of Mostowski states that every countable model of ZFC is the set-theoretic universe of 2^{ω_1} mutually incompatible models of Gödel–Bernays class theory, and conjectures that 2^{ω_1} may be improved to 2^{2^ω}, assuming Martin's axiom. We settle this conjecture in the affirmative, without set-theoretic hypotheses.

Let M be a countable model of ZFC set theory, or of its predicative second-order version Gödel–Bernays plus choice for sets (GBC). By a *C-extension* of M, we mean an extension N of M, $N \models$ GBC, with no new sets: $V^N = V^M$ (for $M \models$ GBC; $V^N = M$ in the case $M \models$ ZFC). Let us restrict ourselves, without loss of generality, to models N of GB in which $\in^N \restriction (V^N \times \{\text{proper classes of } N\})$ is just the membership relation.

The following theorem is proved in Mostowski [3]:

THEOREM (Mostowski). *Let M be a countable model of* ZFC *or of* GBC. *Then there is a family $\langle M_\alpha \colon \alpha < 2^{\omega_1} \rangle$ of C-extensions of M such that for $\alpha \neq \beta$, $M_\alpha \cup M_\beta$ is not contained in a C-extension of M. In fact, we may take each M_α to be a maximal C-extension of M.*

On page 337 of [3], Mostowski conjectured that 2^{ω_1} may be replaced by 2^{2^ω}, assuming Martin's axiom. In fact, Martin's axiom is not needed. Our purpose in this paper is to prove the following:

THEOREM. *Mostowski's Theorem above holds with 2^{2^ω} replacing 2^{ω_1}.*

Remarks. 1. Results of Keisler and Kunen in [2] and Shelah [6] show that if ZF is consistent, then there exists a model of ZFC of power $\aleph_1$ whose only expansion to a model of GB is <u>Def</u> (M) (defined below). Hence the hypothesis of countability is to a certain extent essential. In fact, it suffices that the class of ordinals of M has cofinality ω; see Remark 4 at the end of this paper.

2. Mostowski noticed [3, § 5] that this theorem also applies to non-standard models of Peano Arithmetic. That applies to our improvement as well, for the same reasons.

3. Various notions of mutual genericity have proven useful, perhaps originating with Solovay's work on iterated forcing. Our use of mutually

generic classes is very similar to work of Schmerl [4]. More precisely, our use differs from Mostowski's (and is similar to Schmerl's) in the way it uses infinitely many mutually generic classes at once.

DEFINITION 1. Let K be a language containing $\in$ and any set of unary relation symbols. We define ZF^K to be the theory ZF formulated in K, so that the axiom schemas apply to K-formulas (not just to $\{\in\}$-formulas). For $M \models ZF^K$, set

$$\mathrm{Def}(M) = \{X \subseteq M\colon X \text{ is definable in } M \text{ by a (first-order) } K\text{-formula with parameters in } M\};$$

$$\underline{\mathrm{Def}}(M) = (\mathrm{Def}(M), |M|, E), \text{ where}$$

$$xEy \text{ iff } \begin{cases} x \in^M y & \text{for } x, y \in |M|, \\ x \in y & \text{for } x \in M,\ y \in \mathrm{Def}(M). \end{cases}$$

It is well-known that if $M \models ZF^K$, then $\underline{\mathrm{Def}}(M) \models GB$. Of course, if $R \in K$ then $\{x \in |M|\colon M \models R(x)\} \in \mathrm{Def}(M)$. Every model of GB is naturally a K-structure, where each class is the interpretation of a relation symbol in K. So we may write $\underline{\mathrm{Def}}(M)$ for $M \models GB$ and even $\underline{\mathrm{Def}}(M, \mathscr{F})$ for $\mathscr{F} \subseteq \mathrm{Power}(M)$, where $M \models GB$.

The idea of Mostowski's proof is to build a complete binary tree $\langle M_s\colon s \in {}^{(\omega_1)}2\rangle$ of height ω_1 of expansions of M to models of GB (M a given countable model of ZFC). If $s, t \in {}^{(<\omega_1)}2$ and $s \subseteq t$, then $M_s \subseteq M_t$, and distinct functions $f, g \in {}^{(\omega_1)}2$ give incompatible expansions M_f and M_g, where $M_f = \bigcup_{\alpha<\omega_1} M_{f\restriction\alpha}$, $M_g = \bigcup_{\alpha<\omega_1} M_{g\restriction\alpha}$. So, the key idea is to see how to branch to get incompatible expansions. This is accomplished by taking M_s and adjoining generic classes X and Y to get $M_{s\frown 0} = \underline{\mathrm{Def}}(M_s, \{X\})$ and $M_{s\frown 1} = \underline{\mathrm{Def}}(M_s, \{Y\})$. The classes X and Y are constructed carefully so that $X \cap Y$ is contained and unbounded in the class of ordinals of M, and has order type ω. That guarantees the incompatibility of $M_{s\frown 0}$ and $M_{s\frown 1}$. The genericity of X guarantees that $\underline{\mathrm{Def}}(M_s, X) \models GB$; similarly for $\underline{\mathrm{Def}}(M_s, Y)$.

In the present paper, we construct "mutually generic" classes $\langle X_f\colon f \in {}^{\omega}2\rangle$ of M. Actually, that would be fine if we only wanted one expansion of M. (In fact, $\underline{\mathrm{Def}}(M)$ would suffice, and we would not need generic classes!) So instead we actually construct classes $\langle G_f^i\colon f \in {}^{\omega}2,\ i \in 2\rangle$ such that for each $h\colon {}^{\omega}2 \to 2$, the family $\langle G_f^{h(f)}\colon f \in {}^{\omega}2\rangle$ is "mutually generic". Set $M^h = \underline{\mathrm{Def}}(M, \{G_f^{h(f)}\colon f \in {}^{\omega}2\})$; then $M^h \models GB$. To ensure that these expansions are pairwise incompatible, we construct the classes G_f^i so that for each $f \in {}^{\omega}2$, $G_f^0 \cap G_f^1$ is contained and unbounded in ORD^M and has order type ω. We give further details in the proofs below.

DEFINITION 2. Work in ZFC. We define a partial order $\boldsymbol{P} = (P, \leqslant)$ (actually a proper class) as follows. P contains all functions $f\colon \alpha \to 2$, where α may be any ordinal. We define $f \leqslant g$ iff $f \supseteq g$.

For any $n \in \omega$, set $\boldsymbol{P}^n = (P^n, \leqslant_n)$, where $\langle f_0, \ldots, f_{n-1}\rangle \leqslant_n \langle g_0, \ldots, g_{n-1}\rangle$ iff $f_i \supseteq g_i$ for all $i < n$. For $p \in P^n$, we may write $p = \langle p_0, \ldots, p_{n-1}\rangle$. Now fix $M \models ZFC^K$ (cf. Definition 1). Our forcing language contains all sentences of $K \cup \{P_0, \ldots, P_{n-1}\} \cup \{c_m\colon m \in M\}$, where the P_i are new unary relation symbols and the c_m are new constant symbols. We define the relation $p \Vdash_n \varphi$, for $p \in (P^n)^M$ (P^n in the sense of M) and φ in the forcing language, by induction on complexity as follows:

$p \Vdash_n \varphi$ iff φ, if φ is atomic, no P_i occurring in φ,
$p \Vdash_n P_i(c_m)$ iff $p_i(m) = 1$,
$p \Vdash_n \varphi \wedge \psi$ iff $(p \Vdash_n \varphi) \wedge (p \Vdash_n \psi)$,
$p \Vdash_n \neg\varphi$ iff $(\forall q \leqslant p)\neg(q \Vdash_n \varphi)$,
$p \Vdash_n \forall x \varphi$ iff $xp \Vdash_n \varphi(c_x)$,

"Class forcing" such as this was used by Felgner [1], and independently by P. Cohen, R. Jensen, S. Kripke and R. Solovay, to show that GB+ +[global choice] is a conservative extension of ZFC. We omit details of the proofs of the following standard lemmas, as they may be observed in [1] or [3].

LEMMA 1 (Definability Lemma). *Let M be a model of ZFC^K. For fixed $n \in \omega$ and first-order $\varphi(x_0, \ldots, x_{n-1}) \in K \cup \{P_0, \ldots, P_{n-1}\}$, the relation*

$$\{\langle p, m_0, \ldots, m_{n-1}\rangle\colon p \Vdash \varphi(c_{m_0}, \ldots, c_{m_{n-1}})\}$$

is definable in M.

Proof. Immediate by induction on complexity of φ. ■

DEFINITION 3. Fix $n \in \omega$, and let M be a countable model of ZF^K. A subset $D \subseteq (P^n)^M$ is *dense* if $(\forall p \in P^n)(\exists q \in P^n)(q \leqslant_n p \wedge q \in D)$. A subset $G \subseteq (P^n)^M$ is *M-generic* if:

(i) $(\forall p \in G)(\forall q \in G)(\exists r \in G)(r \leqslant_n p \wedge r \leqslant_n q)$;
(ii) $(\forall p \in G)(\forall q \geqslant_n p)(q \in G)$;
(iii) $G \cap D \neq \emptyset$ for all dense $D \subseteq (P^n)^M$ which are definable in M.

We write $M[G]$ for the structure $(M, G_0, \ldots, G_{n-1})$ for the language $K \cup \{P_0, \ldots, P_{n-1}\}$, where $G_i = \{m \in |M|\colon (\exists p \in G)\ p_i(m) = 1\}$. As usual, one can prove that generic sets exist. But we shall prove something stronger in Lemma 5. First, we have the usual Truth Lemma:

LEMMA 2 (Truth Lemma). *Fix $n \in \omega$, and let M be a countable model of ZFC^K. Then for all M-generic $G \subseteq P^n$, all formulas φ of $K \cup \{P_0, \ldots, P_{n-1}\}$, and all $m_1, \ldots, m_i \in M$,*

$$M[G] \models \varphi[\vec{m}] \quad \textit{iff} \quad (\exists p \in G)(p \Vdash \varphi(c_{m_1}, \ldots, c_{m_i})).$$

Proof. Again, an easy induction on complexity of φ. The genericity of G gets us past the negation step. ■

LEMMA 3. *Fix $n \in \omega$, let M be a model of ZFC^K, and let $G \subseteq (P^n)^M$ be M-generic. Then $M[G] \models ZFC^{K \cup \{P_0, \ldots, P_{n-1}\}}$.*

Proof. This is well-known for $n=1$, and the general case is the same. ■

Definition 4. Let $\mathscr{G}=\{G_i\colon i\in I\}$ be a family of subsets of P^M, M a model of ZFC^K. $\mathscr{G}$ is *mutually M-generic* if for all $n\in\omega$ and distinct $i_0,\ldots,i_{n-1}\in I$, $G_{i_0}\times\ldots\times G_{i_{n-1}}$ is an M-generic subset of $(P^n)^M$. Set

$$M\ulcorner\mathscr{G}\urcorner=\bigcup\{M[G_{i_0}\times\ldots\times G_{i_{n-1}}]\colon G_{i_0},\ldots,G_{i_{n-1}}\in\mathscr{G}\text{ and } n\in\omega\},$$

which is a structure for the language $K\cup\{P_i\colon i\in I\}$.

Lemma 4. *If $M\models\mathrm{ZFC}^K$, and $\mathscr{G}$ is mutually M-generic, then $M[\mathscr{G}]\models\mathrm{ZFC}^{K\cup\{P_i\colon i\in I\}}$. Hence $\underline{\mathrm{Def}}(M[\mathscr{G}])\models\mathrm{GBC}$.*

Proof. The first part is immediate by Lemma 3, since sentences are finite. It is well-known that the second part follows from the first. ■

Our main lemma now appears.

Lemma 5. *Let M be a countable model of ZFC^K, K countable. Then there exists a collection $\{G_f^i\colon f\in{}^\omega 2,\ i\in 2\}$ with the following properties. For $h\colon{}^\omega 2\to 2$, set $\mathscr{G}^h=\{G_f^{h(f)}\colon f\in{}^\omega 2\}$.*

(i) *For all $h\colon{}^\omega 2\to 2$, $\mathscr{G}^h$ is mutually M-generic.*

(ii) *If $g,h\colon{}^\omega 2\to 2$ and $g\neq h$, then there is no C-extension M^+ of M such that $\mathscr{G}^g\cup\mathscr{G}^h\subseteq M^+$.*

Before proving Lemma 5, we use it to prove the result we have been working toward. Here is a precise restatement.

Theorem. *Let M be a countable model of GBC. Then there exists a family $\{M^h\colon [h\colon{}^\omega 2\to 2]\}$ of models of GBC with the following properties.*

(a) *M^h is a C-extension of M.*

(b) *There is no C-extension M^+ of M such that $M^g\cup M^h\subseteq M^+$ for distinct $g,h\colon{}^\omega 2\to 2$.*

(c) *M^h has no proper C-extension, and $g\neq h\Rightarrow M^g\neq M^h$.*

Proof. Obviously (c) implies (b); however, we instead prove (b). Then we may obtain (c) if we replace each M^h by a maximal expansion of M^h, using Zorn's Lemma. (This is Marek's observation, Lemma 2.1 in [3].) So let us see how to define M^h to make (a) and (b) hold. Of course, Lemma 5 gives us the answer. Choose $\{G_f^i\colon f\in{}^\omega 2,\ i\in 2\}$ satisfying properties (i) and (ii) of Lemma 5. For $h\colon{}^\omega 2\to 2$, set $M^h=\underline{\mathrm{Def}}(M,\mathscr{G}^h)$; then (a) is clearly true. Also $M^h\models\mathrm{GB}$ by property (i) together with Lemma 4. Finally, (b) follows from property (ii). ■

Proof of Lemma 5. Let $\langle D_n\colon n\in\omega\rangle$ be an enumeration of $\bigcup\limits_{m\in\omega}\{D\subseteq P^m\colon D\text{ is dense in } P^m\}$, with infinite repetition. We will construct elements $p_s^i\in P$ for $s\in{}^{<\omega}2$ and $i\in 2$. This is done by induction on $|s|$, according to the following inductive hypotheses for $n\geqslant 1$. (Set $p_\emptyset^0=p_\emptyset^1=\emptyset$.)

(0) $\mathrm{dom}(p_s^i)=\mathrm{dom}(p_t^j)$ and $p_s^i\neq p_t^j$ for all distinct $\langle s,i\rangle,\langle t,j\rangle\in({}^n 2)\times 2$.

(1) Meeting dense sets: If $D_{n-1}\subseteq P^m$, $s_0,\ldots,s_{m-1}$ are members of ${}^n 2$ such that $s_i\upharpoonright(n-1)\neq s_j\upharpoonright(n-1)$ for $i\neq j$, and $f\in{}^m 2$, then $\langle p_{s_i}^{f(i)}\colon i<m\rangle\geqslant_m\vec{q}$ for some $\vec{q}\in D_{n-1}$.

(2) For all $s\in{}^n 2$, $|\{\beta\colon p_s^0(\beta)=1=p_s^1(\beta)\}|=n$.

Suppose for a moment that such a construction can be carried out. Then for $f\in{}^\omega 2$ and $i\in 2$, set $G_f^i=\{p\in P\colon(\exists n)(p\geqslant p_{f\upharpoonright n}^i)\}$. To check property (i) (mutual genericity), suppose D is dense in P^m, $h\in{}^m 2$, and $f_0,\ldots,f_{m-1}$ are distinct members of ${}^\omega 2$. Choose n such that $D=D_n$ and $f_i\upharpoonright n\neq f_j\upharpoonright n$ for all distinct $i,j<n$. Then inductive hypothesis (1) at $n+1$ gives us what we need. As for property (ii), if $g,h\colon{}^\omega 2\to 2$ and $g\neq h$, we may choose $f\in{}^\omega 2$ such that $g(f)\neq h(f)$. Then $G_f^{g(f)}\in\mathscr{G}^g$ and $G_f^{h(f)}\in\mathscr{G}^h$. Hence no C-extension M^+ of M can have $\mathscr{G}^g\cup\mathscr{G}^h\subseteq M^+$. For otherwise $G_f^{g(f)},G_f^{h(f)}\in M^+$ and hence $G_f^{g(f)}\cap G_f^{h(f)}$ is a cofinal class of ordinals having order type ω, which violates replacement.

It remains to construct the conditions p_s^i by induction on $|s|$. For fixed n the construction of p_s^i for $|s|=n$ is internal within M. The idea is this: given the conditions p_s^i for $|s|=n$, then to get $p_{s^\wedge 0}^i$ and $p_{s^\wedge 1}^i$, we first extend the conditions in order to meet D_n, while adding no new elements to $(p_s^0)^{-1}(1)\cap(p_s^1)^{-1}(1)$. Next, we add such an element. The last step is to split each p_s^i into $p_{s^\wedge 0}^i$ and $p_{s^\wedge 1}^i$

Suppose p_s^i is defined for all $i\in 2$, $s\in{}^n 2$. Fix the unique m such that $D_n\subseteq P^m$. Enumerate all tuples $\vec{a}=\langle s_0,\ldots,s_{m-1},f\rangle$ such that $s_0,\ldots,s_{m-1}$ are distinct members of ${}^n 2$ and $f\in{}^m 2$; let $\langle\vec{a}_i\colon i<k\rangle$ be such an enumeration. We successively extend the conditions p_s^i ($i\in 2$, $s\in{}^n 2$) in k stages $0,1,\ldots,l,\ldots,k$ to $(p_s^i)_l$, as follows. First, set $(p_s^i)_0=p_s^i$. In general, if $l<k$ then to define $(p_s^i)_{l+1}$ from $(p_s^i)_l$, first write a_l as $\langle s_0,\ldots,s_{m-1},f\rangle$. Extend the m-tuple $\langle(p_{s_0}^{f(0)})_l,\ldots,(p_{s_{m-1}}^{f(m-1)})_l\rangle$ to a tuple $\langle q_0,\ldots,q_{m-1}\rangle\in D_n$; we may do this because D_n is dense in P^m. Further extend these q_j to q_j' so that $\mathrm{dom}(q_j')=\alpha_{l+1}$ for some fixed α_{l+1} (all $j<m$). Now set $(p_{s_j}^{f(i)})_{l+1}=q_j'$ for all $j<m$. Finally, for $\langle i,s\rangle\notin\{\langle f(j),s_j\rangle\colon i,j<m\}$, extend $(p_s^i)_l$ to a function $(p_s^i)_{l+1}$ with domain α_{l+1}, by the rule: $(p_s^i)_{l+1}(\beta)=0$ for all $\beta\in\alpha_{l+1}\setminus\mathrm{dom}((p_s^i)_l)$.

Notice that throughout this procedure, we never add any new elements to $(p_s^0)_{l+1}^{-1}(1)\cap(p_s^1)_{l+1}^{-1}(1)$. Hence, we have (by the inductive hypothesis 2): $|\{\beta\colon(p_s^0)_k(\beta)=1=(p_s^1)_k(\beta)\}|=n$. We now extend the conditions $(p_s^i)_k$ ($s\in{}^n 2$, $i\in 2$) to conditions $p_{s^\wedge j}^i\colon(\alpha+2)\to 2$ (for $s\in{}^n 2$, $i\in 2$, $j\in 2$) by the rules:

(a) $p_{s^\wedge j}^i(\alpha_k)=1$ for all $s\in{}^n 2$, $i\in 2$, $j\in 2$.

(b) $p_{s^\wedge j}^i(\alpha_{k+1})=\begin{cases}0 & \text{if } i=j,\\ 1 & \text{if } i\neq j.\end{cases}$

Now (a) guarantees that $|\{\beta\colon p_s^0(\beta)=1=p_s^1(\beta)\}|$ is one greater than before, hence $=n+1$ which preserves inductive hypothesis 2. Part (b) preserves this property, and also guarantees the splitting required by inductive hypothesis 0. ■

Remark 4. As indicated in Remark 1, countability is not quite as essential as it appears: if ORD^M has cofinality ω, then our theorem remains true. (Similarly, for $M \models$[Peano Arithmetic], if M has cofinality ω then our theorem goes through.) For suppose $(\alpha_n: n<\omega)$ is cofinal. Basically, at stage n we construct the p_s^i $(s \in {}^n 2,\ i \in 2)$ so that every dense set is intersected, which is Σ_n definable with parameters in $R(\alpha_n)$ (the sets of rank $<\alpha_n$). This argument was previously carried out for arithmetic in Schmerl [4]. It is also shown there (Theorem 1.6) that if $\langle N_\nu: \nu \leqslant \alpha \rangle$ is a MacDowell–Specker chain, where $\mathrm{cf}(\alpha) > \omega$, then N_α has only one expansion to a model of predicative second-order extension $\Sigma_\infty^0 - \mathrm{CA}$ of PA. In a more recent paper Schmerl [5] has shown that in fact, if $S \subseteq |N_\alpha|$ and $\{x \in S: x <^{N_\alpha} a\}$ is definable in N_α for all $a \in |N_\alpha|$, then S is definable in N_α. (A similar result appears in Theorem 1.5 of [4], but only for regular cardinals α.)

References

[1] U. Felgner, *Comparison of the axioms of local and universal choice*, Fund. Math. 71 (1971), pp. 43–62.

[2] H. J. Keisler, *Models with tree structures*, Proc. Tarski Sympos., Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, R. I. (1974), pp. 331–348.

[3] A. Mostowski, *A remark on models of the Gödel–Bernays axioms for set theory*, Sets and Classes, North-Holland (1976), pp. 325–340.

[4] J. Schmerl, *Peano models with many generic classes*, Pacific J. Math. 46 (1973), pp. 523–536. (Errata: Pacific J. Math., to appear.)

[5] — *Recursively saturated, rather classless models of Peano Arithmetic*, in: *Logic Year 1979–80* (Univ. of Connecticut), Lec. Notes Math. 859 (1981), pp. 268–282.

[6] S. Shelah, *Models with second-order properties II: Trees with no undefined branches*, Ann. Math. Logic 14 (1978), pp. 73–87.

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Orderability from selections: Another solution to the orderability problem

by

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Abstract. We prove that a Tychonov space X is a GO-space iff X admits a certain type of (weak) selection.

0. Introduction. All spaces under discussion are Tychonov.

A space is called *orderable* iff its topology is generated by a linear ordering. In addition, a space is called a *generalized ordered space* (abbreviated GO-space) iff there exists a linear order $\leqslant$ on X such that every point in X has arbitrary small $\leqslant$-convex neighborhoods. It is well known that the class of GO-spaces coincides with the class of subspaces of orderable spaces. As far as we know, the most general characterization of GO-spaces was given by van Dalen & Wattel [1]:

A space X is GO-space iff X possesses an open subbase consisting of two nests.

In this paper we will give quite a different characterization of GO-spaces, namely, we give a characterization in terms of selections. This generalizes results from our paper [3] where the compact case was treated.

1. Preliminaries. Let X be a space and let 2^X denote the hyperspace of nonempty closed subsets of X. A *selection* for X is a map $F: 2^X \to X$ such that $F(A) \in A$ for all $A \in 2^X$. Let $X(2)$ denote the 2-fold symmetric product of X, i.e. the subspace of 2^X consisting of all non-empty closed subspaces of X consisting of at most two points. A *weak selection* for X is a map $s: X(2) \to X$ such that $s(A) \in A$ for all $A \in X(2)$. It is easy to see that X has a weak selection if and only if there is a map $s: X^2 \to X$ such that for all $x, y \in X$,

$$s(x, y) = s(y, x), \tag{1}$$

and

$$s(x, y) \in \{x, y\}. \tag{2}$$

Such a map $s: X^2 \to X$ will also be called a *weak selection*.

Michael [2] showed that for a continuum X the following statements