Binary consistent choice on pairs and a generalization of Konig's infinity lemma

by

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Abstract. In this paper we answer in the affirmative the question of Cowan: Does $T_2 \to \mathrm{BPI}$? where T_2 is Cowan's generalization of Konig's infinity lemma restricted to trees of order 2. We also give a negative answer to the question: Does $F_2 \to \mathrm{BPI}$? where F_2 is a principle involving binary consistent choice on pairs and BPI is the Boolean prime ideal theorem.

1. In [2] R. Cowan generalizes Konig's infinity lemma. We begin by describing that generalization. For convenience we give the following definitions from [2].

A *tree* is a connected undirected graph without circuts one of whose vertices is designated as the origin. The number of vertices on the unique path connecting a vertex v with the origin is the level of v, $l(v)$. (Thus the set of vertices of a tree can be decomposed into an at most denumerable set of levels.) A vertex v' is a *successor of a vertex* v if v and v' are connected by an edge and $l(v') = l(v)+1$.

A tree is *finite* if its set of vertices is finite and locally finite if each vertex has only finitely many successors. A *branch in a tree* is a maximal path beginning at the origin. If v and v' are on the same branch, then v' *dominates* v if $l(v') \geqslant l(v)$. Konig's lemma states that any infinite locally finite tree has an infinite branch.

Let T be a collection of locally finite trees (not necessarily pairwise disjoint). By a *vertex* or a *level* of T, we mean a vertex or a level of some tree in T. Also if v and v' are vertices of T, then v' *dominates* v *in* T if v' dominates v in some tree in T. Let S be a set of vertices of T. S *pierces a level* l *of* T if $|S \cap l| = 1$. (For any set A, $|A|$ denotes the cardinal number of A.) S is *consistent* if for every v, v' in S there is a v'' which dominates them both in T. We can now state R. Cowan's generalization of Konig's lemma.

THEOREM 1.1. *Let T be a collection of locally finite trees such that for any finite set of levels of T, there is a consistent set of vertices piercing those levels. Then there is a consistent set of vertices piercing the entire set of levels of T.*

Let $\{A_i\}_{i\in I}$ be a collection of sets and R a symmetric binary relation on $\bigcup_{i\in I} A_i$. A choice function f for $\{A_i\}_{i\in I}$ is R-consistent if $f(i)Rf(j)$ for all i, j in I

with $i \neq j$. We will also consider the following theorem of Łoś and Ryll-Nardzewski:

THEOREM 1.2. *Let* $\{A_i\}_{i\in I}$ *be a collection of finite sets and* R *a symmetric binary relation on* $\bigcup_{i\in I} A_i$. *Suppose that for every finite* $W \subseteq I$, *there is an* R-*consistent choice function for* $\{A_i\}_{i\in W}$. *Then there is an* R-*consistent choice function for* $\{A_i\}_{i\in I}$.

It is known [1] and [6] that both of the above theorems are equivalent to the Boolean prime ideal theorem (BPI).

We now define the order of a collection of locally finite trees $T(o(T))$ to be the least cardinal $\mathfrak{k}$ such that no tree in T contains a vertex with more than $\mathfrak{k}$ successors and let T_n be the statement of Theorem 1.1 only for T with $o(T) = n$, n a positive integer. Also let F_n denote the statement of Theorem 1.2 where it is required that $|A_i| \leqslant n$ for all $i \in I$.

In [2] it is shown that $T_n \to F_n \to$ BPI for any integer $n \geqslant 3$ and the question is posed: Does $T_2 \to$ BPI or $F_2 \to$ BPI? The purpose of this paper is to answer both questions. In Section 2 we give a proof of BPI from T_2 and in Section 3 we construct a Fraenkel-Mostowski model of ZFU (Zermelo–Fraenkel set theory weakened to permit the existence of urelements) in which F_2 is true and BPI is false. Actually, in the model constructed, the axiom of choice for sets of three element sets fails, so it appears that F_2 is considerably weaker than BPI.

2. In this section we prove:

THEOREM 2.1. T_2 *implies the compactness theorem for propositional logic.*

Proof. Let K be an infinite set of propositional formulas such that every finite subset of K is satisfiable. Let P be the set of propositional variables occurring in K.

LEMMA. *If* P_0 *is any finite subset of* P *and* $K(P_0)$ *is* $\{x \in K$: *all the propositional variables in* x *are in* $P_0\}$ *then* $K(P_0)$ *is satisfiable.*

Proof. If $K(P_0)$ is not satisfiable, then for each truth assignment σ for the variables in P_0 there is an x_σ in $K(P_0)$ such that $\sigma(x_\sigma) = F$. Therefore $\{x_\sigma$: σ is a truth assignment for $P_0\}$ is a finite, nonsatisfiable set. This proves the lemma.

Now to complete the proof of Theorem 2.1 we follow the proof of Theorem 7 in [2]. Suppose that W is a finite subset of P. A sequence of subsets of W, $W_1, W_2, \ldots, W_k$ is a W-tower if W_1 is a singleton, $W_k = W$ and $W_{i+1} = W_i \cup \{x\}$ for some x, $i = 1, 2, \ldots, k-1$. For each W-tower we form a tree as follows: The origin is $\emptyset$, level $i+1$ is

$$F_{W_i} = \{\sigma\colon \sigma \text{ is a truth assignment for } W_i \text{ such that } \sigma(x) = T \text{ for all } x \in K_{W_i}\}.$$

$\sigma \in F_{W_i}$ is connected to $\sigma | W_{i-1}$ which belongs to $F_{W_{i-1}}$. Each vertex has at most two successors, therefore if T is the set of all such trees, $o(T) = 2$. If $F_{W_1}, F_{W_2}, \ldots, F_{W_m}$ is any finite set of levels of T, then $V = \bigcup_{i=1}^{m} W_i$ is a finite subset of P. By the lemma, therefore, $F_V \neq \emptyset$. Suppose $\sigma \in F_V$, then $\{\sigma|W_1, \sigma|W_2, \ldots, \sigma|W_m\}$ is a consistent set of vertices since σ dominates $\sigma|W_i$ in all the trees formed from V-towers containing W_i.

Further $\{\sigma|W_1, \sigma|W_2, \ldots, \sigma|W_m\}$ pierces each F_{W_i}, $i = 1, 2, \ldots, m$. Therefore by T_2 there is a consistent set F such that

$$|F \cap F_W| = 1 \quad \text{for all finite } W \subseteq P.$$

Since any two truth assignments in F are restrictions of the same truth assignment, F uniquely determines a truth assignment for P which satisfies K. This completes the proof of the theorem.

Since the compactness theorem for propositional logic implies BPI [1], we have

THEOREM 2.2. T_2 *implies* BPI.

3. In this section we prove that the implication $F_2 \to$ BPI does not hold by constructing a Fraenkel–Mostowski model in which F_2 is true and BPI fails.

Given a model M' of ZFU+AC which has U as its set of urelements, a permutation model M of ZFU is determined by a group G of permutations of U and a filter of subgroups Γ of G which satisfies

$$(\forall a \in U)(\exists H \in \Gamma)(\forall \varphi \in H)(\varphi(a) = a)$$

and

$$(\forall \varphi \in G)(\forall H \in \Gamma)(\varphi H \varphi^{-1} \in \Gamma).$$

Each permutation of U extends uniquely to a permutation of M' by $\in$-induction and for any $\varphi \in G$, we identify φ with its extension.

If H is a subgroup of G and $x \in M'$ and $(\forall \varphi \in H)(\varphi(x) = x)$ we say H fixes x. If it is also the case that $(\forall \varphi \in H)(\forall y \in x)(\varphi(y) = y)$ we say that H fixes x pointwise. The permutation model M determined by U, G and Γ consists of all those $x \in M'$ such that for every y in the transitive closure of x, there is some H in Γ such that H fixes y. We refer the reader to [4, p.46] for the proof that M is a model of ZFU.

For our proof we assume that M' is a model of ZFU+AC with a countable set of urelements U. We also assume for convenience that $U = \bigcup_{i\in\omega} U_i$ where $U_i \cap U_j = \emptyset$ if $i \neq j$ and $U_i = \{a_i, b_i, c_i\}$, $i = 0, 1, 2, \ldots$ For each $i \in \omega$, define $\eta_i\colon U_i \to U_i$ by $\eta_i(a_i) = b_i$, $\eta_i(b_i) = c_i$ and $\eta_i(c_i) = a_i$. G is then defined to be the group of permutations

$$G = \{\varphi\colon \varphi\colon U \xrightarrow[\text{onto}]{1-1} U \text{ and } (\forall i \in \omega)(\varphi|U_i = \eta_i \text{ or } \varphi|U_i = \eta_i^2 \text{ or } \varphi|U_i = 1_{U_i}\}$$

where 1_{U_i} is the identity permutation on U_i.

If S is any finite subset of ω we define the subgroup G_S of G by

$$G_S = \{\varphi \in G\colon (\forall i \in S)(\varphi \text{ fixes } U_i \text{ pointwise})\}$$

and the filter Γ of subgroups of G is

$$\Gamma = \{G_S\colon S \text{ is a finite subset of } \omega\}.$$

LEMMA 1. *G is commutative.*

This follows from the definition of G.

LEMMA 2. *For any $x \in M$, there is a smallest finite subset S of ω such that $(\forall \varphi \in G_S)(\varphi(x) = x)$.*

Proof. It suffices to show that the intersection of two subsets of ω satisfying the condition of the lemma is also such a subset. Suppose that G_{S_1} and G_{S_2} both fix x and suppose that $\psi \in G_{S_1 \cap S_2}$. To complete the proof we show that $\psi(x) = x$. Define $\varphi_1 \in G$ and $\varphi_2 \in G$ as follows:

$$\varphi_1(t) = \begin{cases} \psi(t) & \text{if } t \in U_i \text{ where } i \in S_2 - S_1, \\ t & \text{otherwise,} \end{cases}$$

$$\varphi_2(t) = \begin{cases} \psi(t) & \text{if } t \in U_i, \text{ where } i \notin S_2, \\ t & \text{otherwise.} \end{cases}$$

Then we have $\varphi_1 \in G_{S_1}$, $\varphi_2 \in G_{S_2}$ and $\psi = \varphi_1\varphi_2$. Therefore $\psi(x) = \varphi_1\varphi_2(x) = x$.

DEFINITION. If $x \in M$ and S is the smallest finite subset of ω such that G_S fixes x, then S is called the *support of x*.

The following lemma also follows from the definition of G:

LEMMA 3. *For any $\varphi \in G$, $\varphi^3 = 1_U$.*

For each $i \in \omega$ we define $\eta_i' = \eta_i \cup 1_{U-U_i}$ and we note that $\eta_i' \in G$.

THEOREM 3.1. BPI *is false in M.*

Proof. The set $X = \{U_i\colon i \in \omega\}$ has support $\emptyset$ and is therefore in M. No choice function for X is in M, for if f is such a choice function with support S, we choose an integer $i \notin S$ and a $\varphi \in G_S$ such that $\varphi | U_i = \eta_i$. Without loss of generality assume that $f(U_i) = a_i$. Then $\varphi(U_i) = U_i$ but $\varphi(f(U_i)) = \varphi(a_i) = b_i \neq f(U_i)$, hence φ does not fix f, a contradiction. Therefore the axiom of choice for sets of 3-element sets is false in the model and the theorem follows since BPI implies this form of AC.

THEOREM 3.2. *F_2 is true in M.*

Proof. Let A be any set of pairs in the model and R a binary relation ($R \in M$) such that the hypotheses of F_2 are satisfied. (I.e., for any finite subset B of A, B has an R-consistent choice function.) Let $S_0 \subseteq \omega$ be the support of $\langle A, R\rangle$. For each $t \in A$, $\mathrm{OB}_t = \{\varphi(t)\colon \varphi \in G_{S_0}\}$ is the orbit of t under the group G_{S_0} and we let OB be the set of orbits of elements of A under the group G_{S_0}. (Then each $t \in A$ is in exactly one orbit and OB is well-ordered in M since it is fixed pointwise by G_{S_0}.)

We apply F_2 in the model M' to get an R-consistent choice function g for A. g need not be in M, but we plan to modify g to get an r-consistent choice function f for A which is in M.

For each $t = \{a, b\} \in A$, define $\sup(t) = S - S_0$, where S is the support of t. For each finite $S \subseteq \omega$ such that $S \cap S_0 = \emptyset$ define

$$\mathrm{perm}(S) = \{\prod_{i \in S} (\eta_i')^{\Delta_i}\colon \Delta_i \in \{0, 1, 2\} \text{ for all } i \in S\}.$$

If $t \in A$, we will write $\mathrm{perm}(t)$ for $\mathrm{perm}(\sup(t))$. Note that $|\mathrm{perm}(t)| = 3^{|\sup(t)|}$ and is therefore an odd natural number. We also note that $\mathrm{perm}(t)$ is a subgroup of G_{S_0}. In addition we have:

LEMMA 4. *If $t \in A$ and $t' \in \mathrm{OB}_t$, then $t' = \psi(t)$ for some $\psi \in \mathrm{perm}(t)$.*

Proof. Suppose $t' \in \mathrm{OB}_t$, then $t' = \psi'(t)$ for some $\psi' \in G_{S_0}$. Define ψ by

$$\psi(x) \in \begin{cases} \psi'(x) & \text{if } x \in U_i \text{ for some } i \in \sup(t), \\ x & \text{otherwise.} \end{cases}$$

Then $\psi \in \mathrm{perm}(t)$ and further since for all x such that $x \in U_i$ for some $i \in \sup(t)$ $\psi^{-1}\psi'(x) = x$, we have $\psi^{-1}\psi'(t) = t$ so that $\psi(t) = \psi'(t) = t'$. This completes the proof of the lemma.

Now suppose $t = \{a, b\} \in A$. We define

$$\mathrm{perm}(t, a) = \{\psi \in \mathrm{perm}(t)\colon g(\psi(t)) = \psi(a)\}$$

and

$$\mathrm{perm}(t, b) = \{\psi \in \mathrm{perm}(t)\colon g(\psi(t)) = \psi(b)\}.$$

Then $|\mathrm{perm}(t)| = |\mathrm{perm}(t, a)| + |\mathrm{perm}(t, b)|$ therefore since $|\mathrm{perm}(t)|$ is odd, $|\mathrm{perm}(t, a)| \neq |\mathrm{perm}(t, b)|$. We can therefore define

$$f(t) = \begin{cases} a & \text{if } |\mathrm{perm}(t, a)| > |\mathrm{perm}(t, b)|, \\ b & \text{if } |\mathrm{perm}(t, b)| > |\mathrm{perm}(t, a)| \end{cases}$$

and f is defined for every $t \in A$.

LEMMA 5. *f is in M and G_{S_0} fixes f.*

Proof. Let $t = \{a, b\} \in A$ and let ψ' be any element of G_{S_0}. Define ψ as in the proof of Lemma 4. Then as in the proof of Lemma 4, $\psi(t) = \psi'(t)$ and further $\psi(a) = \psi'(a)$. (If not then $\psi^{-1}\psi'(a) = b$ while $\psi^{-1}\psi'(\{a, b\}) = \{a, b\}$ contradicting the fact from Lemma 3 that $(\psi^{-1}\psi')^3 = 1_U$.) So that

$$\begin{aligned} |\mathrm{perm}(t, a)| &= |\{\eta \in \mathrm{perm}(t)\colon g(\eta(t)) = \eta(a)\}| \\ &= |\{\eta\psi \in \mathrm{perm}(t)\colon g(\eta\psi(t)) = \eta\psi(a)\}| \\ &= |\{\eta \in \mathrm{perm}(t)\colon g(\eta(\psi(t))) = \eta\psi(a)\}| \\ &= |\mathrm{perm}(\psi'(t), \psi'(a))|. \end{aligned}$$

Similarly $|\operatorname{perm}(t, b)| = |\operatorname{perm}(\psi'(t), \psi'(b))|$. Therefore by the definition of f,

$$f(t) = a \leftrightarrow f(\psi'(t)) = \psi'(a)$$

proving Lemma 5.

Now the proof of Theorem 3.2 is completed by proving the following:

CLAIM. *f is R consistent.*

The proof is by contradiction. Suppose $t_1 = \{a, b\}$ and $t_2 = \{c, d\}$ are in A, that $f(t_1) = a, f(t_2) = c$ and further that aRc is false.

LEMMA 6. *Suppose that $\eta \in \operatorname{perm}(\sup(t_1) \cap \sup(t_2))$ and suppose that for some $\psi \in \operatorname{perm}(\sup(t_1) - \sup(t_2))$, $\psi\eta \in \operatorname{perm}(t_1, a)$, i.e., $g(\psi\eta(t_1)) = \psi\eta(a)$. Then for every $\varphi \in \operatorname{perm}(\sup(t_2) - \sup(t_1))$, $\varphi\eta \in \operatorname{perm}(t_2, d)$, i.e., $g(\varphi\eta(t_2)) = \varphi\eta(d)$.*

Proof. If not then for some φ perm$(\sup(t_2) - \sup(t_1))$, $\varphi\eta \in \operatorname{perm}(t_2, c)$ which means $g(\varphi\eta(t_2)) = \varphi\eta(c)$. Since g is R-consistent we have $\psi\eta(a) R \varphi\eta(c)$. Since R is fixed by G_{S_0}, we get

$$\eta^{-1}\psi^{-1}\varphi^{-1}\psi\eta(a) R \eta^{-1}\psi^{-1}\varphi^{-1}\varphi\eta(c).$$

Since G is commutative we have $\varphi^{-1}(a) R \psi^{-1}(c)$. But $\psi \in \operatorname{perm}(\sup(t_1) - \sup(t_2))$ and therefore, $\psi(c) = \psi^{-1}(c) = c$. Similarly $\varphi^{-1}(a) = a$ so we conclude that aRc, a contradiction. This completes the proof of Lemma 6.

We therefore conclude that for every $\eta \in \operatorname{perm}(\sup(t_1) \cap \sup(t_2))$ either:

(1) $$(\forall \psi \in \operatorname{perm}(\sup(t_1) - \sup(t_2)))(\psi\eta \in \operatorname{perm}(t_1, b))$$

or

(2) $$(\forall \varphi \in \operatorname{perm}(\sup(t_2) - \sup(t_1)))(\varphi\eta \in \operatorname{perm}(t_2, d)).$$

Therefore if

$$D_1 = |\{\eta \in \operatorname{perm}(\sup(t_1) \cap \sup(t_2))\text{: (1) holds}\}|$$

and

$$D_2 = |\{\eta \in \operatorname{perm}(\sup(t_1) \cap \sup(t_2))\text{: (2) holds}\}|,$$

then $D_1 + D_2 \geqslant |\operatorname{perm}(\sup(t_1) \cap \sup(t_2))|$. So either

$$2 \cdot D_1 \geqslant |\operatorname{perm}(\sup(t_1) \cap \sup(t_2))| \quad \text{or} \quad 2 \cdot D_2 \geqslant |\operatorname{perm}(\sup(t_1) \cap \sup(t_2))|.$$

In the first case we would have $|\operatorname{perm}(t_1, b)| > |\operatorname{perm}(t_1, a)|$ (since $\operatorname{perm}(t_1) = \operatorname{perm}(\sup(t_1))$ can be written as $\operatorname{perm}(\sup(t_1)) = \{\psi\eta\colon \psi \in \operatorname{perm}(\sup(t_1) - \sup(t_2))$ and $\eta \in \operatorname{perm}(\sup(t_1) \cap \sup(t_2))\}$) and this contradicts $f(t_1) = a$. Similarly in the second case we would have $|\operatorname{perm}(t_2, d)| > |\operatorname{perm}(t_2, c)|$ contradicting $f(t_2) = c$. This proves the claim and therefore Theorem 3.2.

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