Corollary 3.6. *Let* (X, T) *be a point-transitive flow where* X *is a sphere, real or complex projective space, or lens space (of dimension greater than one), and* T *is a connected abelian Lie group. Then* (X, T) *satisfies all conclusions of Theorem* 2.15 *and Proposition* 3.5.

References

[1] J. Auslander, *On the proximal relation in topological dynamics*, Proc. Amer. Math. Soc. 11 (1960), pp. 890–895.
[2] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, 1969.
[3] H. Furstenberg, *The structure of distal flows*, Amer. J. Math. 85 (1963), pp. 477–515.
[4] W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ. Vol. 36, Amer. Math. Soc., Providence, 1955.
[5] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Academic Press, 1963.
[6] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, 1961.
[7] P. J. Kahn and A. W. Knapp, *Equivariant maps onto minimal flows*, Math. Sys. Theory 2 (1968), pp. 319–324.
[8] H. B. Keynes, *The structure of weakly mixing minimal transformation groups*, Illinois J. Math. 15 (1971), pp. 475–489.
[9] – and J. B. Robertson, *Eigenvalue theorems in topological transformation groups*, Trans. Amer. Math. Soc. 139 (1969), pp. 359–369.
[10] K. Kuratowski, *Topology*, Vol. II, Academic Press, 1968.
[11] L. S. Pontriagin, *Topological Groups*, 2nd ed., Gordon and Breach, 1966.
[12] P. A. Schweitzer, *Counterexamples to the Seifert conjecture and opening closed leaves of foliations*, Ann. of Math. 100 (1974), pp. 386–400.
[13] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, 1966.
[14] M. D. Wochele, *Thesis*, University of Massachusetts, 1972.

Received 19 February 1979;
in revised form 17 August 1981

Connectivity properties in hyperspaces and product spaces

by

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Abstract. In this paper connectedness, local connectedness, and point-wise local connectedness and connectedness im kleinen in hyperspaces of product spaces and product spaces of hyperspaces are investigated and the relationships between these connectivity properties in hyperspaces of product spaces and product spaces of hyperspaces are determined. In order to include as many spaces as possible, the results in this paper are stated and proved for R_0 and R_1 topological spaces.

1. Introduction. One of the earliest results about connectivity properties in hyperspaces, due to Wojdyslawski [7] in 1939, is that for a metric continuum (X, T), $(2^X, E(X))$ is locally connected (l.c.) iff (X, T) is l.c. Since 1939 mathematicians have continued the investigation of connectivity properties in hyperspaces. In this paper connectivity properties in hyperspaces of product spaces and product spaces of hyperspaces are investigated. In order to include as many spaces as possible, the results in this paper are stated and proved for weak topological spaces. Listed below are definitions and theorems that will be utilized in this paper.

Definition 1.1. A space (X, T) is R_0 iff for each $0 \in T$ and $x \in 0$, $\overline{\{x\}} \subset 0$ [1].

Definition 1.2. A space (X, T) is R_1 iff for each pair $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [1].

Definition 1.3. Let (X, T) be a space, let $A \subset X$, and define 2^X, $C(X)$, $K(X)$, $S(A)$, and $I(A)$ as follows: $2^X = \{F \subset X|\ F \text{ is nonempty and closed}\}$, $C(X) = \{F \in 2^X|\ F \text{ is connected}\}$, $K(X) = \{F \in 2^X|\ F \text{ is compact}\}$, $S(A) = \{F \in 2^X|\ F \subset A\}$, and $I(A) = \{F \in 2^X|\ F \cap A \neq \emptyset\}$. Then the Vietoris topology on 2^X, denoted by $E(X)$, is the smallest topology on 2^X which satisfies the conditions that if $G \in T$, then $S(G) \in E(X)$ and $I(G) \in E(X)$ [6].

Theorem 1.1. *The product of an arbitrary family of nonempty topological spaces is* R_0 *iff each factor space is* R_0 [4].

Theorem 1.2. *If* (X,T) *is* R_1, *then* (X, T) *is* R_0 [5].

Theorem 1.3. *If* (X, T) *is* R_0, *then the following are equivalent*: (a) X *is connected*, (b) 2^X *is connected, and* (c) $K(X)$ *is connected* [2].

THEOREM 1.4. *If* (X, T) *is* R_0 *and* $B \in C(X)$, *then* 2^X *is l.c. (connected im kleinen (c.i.k.)) at* B *iff for each* $U \in T$ *such that* $B \subset U$, *there exists an open connected set* V *such that* $B \subset V \subset U$ *(for each* $U \in T$ *such that* $B \subset U$, *the component of* U *containing* B *contains* B *in its interior)* [2].

THEOREM 1.5. *If* (X, T) *is* R_0 *and* $M \in 2^X$, *then the following are equivalent*: (a) X *is l.c. (c.i.k.) at each element of* M, (b) 2^X *is l.c. (c.i.k.) at each element of* $C(M)$, (c) 2^X *is l.c. (c.i.k.) at each element of* $K(M)$, (d) $K(X)$ *is l.c. (c.i.k.) at each element of* $K(M)$, (e) $K(X)$ *is l.c. (c.i.k.) at each element of* $\{\overline{\{x\}} \mid x \in M\}$, *and* (f) 2^X *is l.c. (c.i.k.) at each element of* $\{\overline{\{x\}} \mid x \in M\}$ [2].

THEOREM 1.6. *If* (X, T) *is* R_0 *and* C *is a component of* X, *then the following are equivalent*: (a) 2^X *is l.c. at* C, (b) C *is closed open in* X, *and* (c) 2^X *is c.i.k. at* C [2].

THEOREM 1.7. *If* (X,T) *is* R_0 *and* $C \in K(X) \cap C(X)$, *then* 2^X *is l.c. (c.i.k.) at* C *iff* $K(X)$ *is l.c. (c.i.k.) at* C [2].

THEOREM 1.8. *If* (X, T) *is* R_0, $B \in K(X)$, *and* 2^X *or* $K(X)$ *is l.c. (c.i.k.) at each component of* B, *then* 2^X *and* $K(X)$ *are l.c. (c.i.k.) at* B [2].

THEOREM 1.9. *The product of an arbitrary family of nonempty topological spaces is* R_1 *iff each factor space is* R_1 [5].

THEOREM 1.10. *If* (X, T) *is* R_1 *and* $A \in K(X)$, *then the following are equivalent*: (a) 2^X *is l.c. at* A, (b) 2^X *is l.c. at each component of* A, (c) *if* C *is a component of* A *and* $U \in T$ *such that* $C \subset U$, *then there exists an open connected set* V *such that* $C \subset V \subset U$, (d) $K(X)$ *is l.c. at each component of* A, *and* (e) $K(X)$ *is l.c. at* A [3].

THEOREM 1.11. *If* (X, T) *is locally compact* R_1 *and* $A \in K(X)$, *then the following are equivalent*: (a) $K(X)$ *is c.i.k. at* A, (b) $K(X)$ *is c.i.k. at each component of* A, (c) *for each component* C *of* A, *if* $U \in T$ *such that* $C \subset U$, *then the component of* U *containing* C *contains* C *in its interior*, (d) 2^X *is c.i.k. at each component of* A, *and* (e) 2^X *is c.i.k. at* A [3].

2. Connectivity properties and product spaces. The first result follows from Theorem 1.1 and Theorem 1.3.

COROLLARY 2.1. *If* (X_α, T_α) *is nonempty and* R_0 *for all* $\alpha \in A$, *then the following are equivalent*: (a) (X_α, T_α) *is connected for all* $\alpha \in A$, (b) $2^{\prod_{\alpha \in A} X_\alpha}$ *is connected*, (c) $K(\prod_{\alpha \in A} X_\alpha)$ *is connected*, (d) $\prod_{\alpha \in A} 2^{X_\alpha}$ *is connected, and* (e) $\prod_{\alpha \in A} K(X_\alpha)$ *is connected.*

THEOREM 2.1. *For each* $\alpha \in A$ *let* (X_α, T_α) *be a nonempty* R_0 *space and let* $C_\alpha \in C(X_\alpha)$. *If* $2^{\prod_{\alpha \in A} X_\alpha}$ *is l.c. (c.i.k.) at* $\prod_{\alpha \in A} C_\alpha$, *then* $\prod_{\alpha \in A} 2^{X_\alpha}$ *is l.c. (c.i.k.) at* $\{W_\alpha\}_{\alpha \in A}$, *where*

$$W_\alpha = \begin{cases} C_\alpha & \text{if } \alpha \in F, \\ X_\alpha & \text{if } \alpha \notin F \end{cases}$$

and $F \in \mathscr{A} = \{B \subset A \mid B \supset \{\alpha \in A \mid X_\alpha \text{ is not connected}\}\}$, *and* $\{\alpha \in A \mid X_\alpha$ *is not connected*$\}$ *is finite.*

Proof. For each $\beta \in A$ let $P_\beta \colon \prod_{\alpha \in A} X_\alpha \to X_\beta$ be the projection function. Consider the case that $2^{\prod_{\alpha \in A} X_\alpha}$ is l.c. at $\prod_{\alpha \in A} C_\alpha$. Let $\beta \in A$ and let $0 \in T_\beta$ such that $C_\beta \subset 0$. For each $\alpha \in A$ let

$$B_\alpha = \begin{cases} 0 & \text{if } \alpha = \beta, \\ X_\alpha & \text{if } \alpha \neq \beta. \end{cases}$$

Then $\prod_{\alpha \in A} C_\alpha \subset \prod_{\alpha \in A} B_\alpha$, which is open in $\prod_{\alpha \in A} X_\alpha$, and by Theorem 1.4, there exists an open connected set $\mathscr{V}$ such that $\prod_{\alpha \in A} C_\alpha \subset \mathscr{V} \subset \prod_{\alpha \in A} B_\alpha$. Then $C_\beta \subset P_\beta(\mathscr{V}) \subset 0$, where $P_\beta(\mathscr{V})$ is open connected. Hence, by Theorem 1.4, 2^{X_β} is l.c. at C_β. Also, since $P_\alpha(\mathscr{V}) = X_\alpha$ except for finitely many $\alpha \in A$, then $F_1 = \{\alpha \in A \mid X_\alpha$ is not connected$\}$ is finite. Let $F \in \mathscr{A}$. For each $\alpha \in A - F$, X_α is a closed open component and by Theorem 1.6, 2^{X_α} is l.c. at $X_\alpha = W_\alpha$, for each $\alpha \in F$, 2^{X_α} is l.c. at $X_\alpha = W_\alpha$, and for each $\alpha \in A - F_1$, X_α is connected and by Theorem 1.3, 2^{X_α} is connected. Therefore 2^{X_α} is l.c. at W_α for all $\alpha \in A$ and 2^{X_α} is connected except for finitely many $\alpha \in A$, which implies $\prod_{\alpha \in A} 2^{X_\alpha}$ is l.c. at $\{W_\alpha\}_{\alpha \in A}$.

By a similar argument, the theorem follows for c.i.k.

THEOREM 2.2. *For each* $\alpha \in A$ *let* (X_α, T_α) *be nonempty* R_0 *and let* $M_\alpha \in 2^{X_\alpha}$. *Then the following are equivalent*: (a) X_α *is l.c. (c.i.k.) at each element of* M_α *for all* $\alpha \in A$ *and* X_α *is connected except for finitely many* $\alpha \in A$, (b) $\prod_{\alpha \in A} X_\alpha$ *is l.c. (c.i.k.) at each element of* $\prod_{\alpha \in A} M_\alpha$, (c) $2^{\prod_{\alpha \in A} X_\alpha}$ *is l.c. (c.i.k.) at each element of* $\{\prod_{\alpha \in A} \overline{\{x_\alpha\}} \mid x_\alpha \in M_\alpha$ *for all* $\alpha \in A\}$, (d) $K(\prod_{\alpha \in A} X_\alpha)$ *is l.c. (c.i.k.) at each element of* $\{\prod_{\alpha \in A} \overline{\{x_\alpha\}} \mid x_\alpha \in M_\alpha$ *for all* $\alpha \in A\}$, (e) $K(\prod_{\alpha \in A} X_\alpha)$ *is l.c. (c.i.k.) at each element of* $K(\prod_{\alpha \in A} M_\alpha)$, (f) $2^{\prod_{\alpha \in A} X_\alpha}$ *is l.c. (c.i.k.) at each element of* $K(\prod_{\alpha \in A} M_\alpha)$, (g) $2^{\prod_{\alpha \in A} X_\alpha}$ *is l.c. (c.i.k.) at each element of* $C(\prod_{\alpha \in A} M_\alpha)$, (h) $\prod_{\alpha \in A} 2^{X_\alpha}$ *is l.c. (c.i.k.) at each element of* $\{\{W_\alpha\}_{\alpha \in A} \mid W_\alpha \in C(M_\alpha)$ *for all* $\alpha \in A\}$, (i) $\prod_{\alpha \in A} 2^{X_\alpha}$ *is l.c. (c.i.k.) at each element of* $\{\{K_\alpha\}_{\alpha \in A} \mid K_\alpha \in K(M_\alpha)$ *for all* $\alpha \in A\}$, (j) $\prod_{\alpha \in A} 2^{X_\alpha}$ *is l.c. (c.i.k.) at each element of* $\{\{\overline{\{x_\alpha\}}\}_{\alpha \in A} \mid x_\alpha \in M_\alpha$ *for all* $\alpha \in A\}$, (k) $\prod_{\alpha \in A} K(X_\alpha)$

is l.c. (c.i.k.) at each element of $\{\{K_\alpha\}_{\alpha\in A} |\ K_\alpha \in K(M_\alpha)$ *for all* $\alpha \in A\}$, *and* (l) $\prod_{\alpha\in A} K(X_\alpha)$ *is l.c. (c.i.k.) at each element of* $\{\{\overline{\{x_\alpha\}}\}_{\alpha\in A} |\ x_\alpha \in M_\alpha$ *for all* $\alpha\in A\}$.

Proof. Consider the statement of the theorem for l.c. The straightforward proof that (a) and (b) are equivalent is omitted. By Theorem 1.5 (b) through (g) are equivalent and by Theorem 2.1 (g) implies (h).

(h) implies (i): For each $\alpha\in A$ let $K_\alpha \in K(M_\alpha)$. Let $\beta\in A$ and let $C_\beta \in C(M_\beta)$. For each $\alpha\neq\beta$ let $x_\alpha\in M_\alpha$. For each $\alpha\in A$ let

$$W_\alpha = \begin{cases} C_\beta & \text{if } \alpha=\beta, \\ \overline{\{x_\alpha\}} & \text{if } \alpha\neq\beta. \end{cases}$$

Then for each $\alpha\in A$, $W_\alpha \in C(M_\alpha)$ and $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{W_\alpha\}_{\alpha\in A}$, which implies 2^{X_α} is l.c. at W_α for all $\alpha\in A$ and 2^{X_α} is connected except for finitely many $\alpha\in A$. Thus 2^{X_α} is l.c. at each element of $C(M_\beta)$ and by Theorem 1.5 2^{X_β} is l.c. at each element of $K(M_\beta)$, which implies 2^{X_β} is l.c. at K_β. Therefore 2^{X_α} is l.c. at K_α for all $\alpha\in A$ and 2^{X_α} is connected except for finitely many $\alpha\in A$, which implies $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$.

(i) implies (j): Since $\{\{\overline{\{x_\alpha\}}\}_{\alpha\in A} |\ x_\alpha\in M_\alpha$ for all $\alpha\in A\} \subset \{\{K_\alpha\}_{\alpha\in A} |\ K_\alpha\in K(M_\alpha)$ for all $\alpha\in A\}$, then $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at each element of $\{\{\overline{\{x_\alpha\}}\}_{\alpha\in A} |\ x_\alpha\in M_\alpha$ for all $\alpha\in A\}$.

(j) implies (k): For each $\alpha\in A$ let $K_\alpha\in K(M_\alpha)$. Let $\beta\in A$ and let $x_\beta\in M_\beta$. For each $\alpha\neq\beta$ let $x_\alpha\in M_\alpha$. Then $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{\overline{\{x_\alpha\}}\}_{\alpha\in A}$, which implies 2^{X_α} is l.c. at $\overline{\{x_\alpha\}}$ for all $\alpha\in A$ and 2^{X_α} is connected except for finitely many $\alpha\in A$. Hence 2^{X_β} is l.c. at each element of $\{\overline{\{x\}} |\ x\in M_\beta\}$ and by Theorem 1.5 $K(X_\beta)$ is l.c. at each element of $K(M_\beta)$, which implies $K(X_\beta)$ is l.c. at K_β. Also, since 2^{X_α} is connected except for finitely many $\alpha\in A$, then by Theorem 1.3 $K(X_\alpha)$ is connected except for finitely many $\alpha\in A$. Therefore $K(X_\alpha)$ is l.c. at K_α for all $\alpha\in A$ and $K(X_\alpha)$ is connected except for finitely many $\alpha\in A$, which implies $\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$.

(k) implies (l): Since $\{\{\overline{\{x_\alpha\}}\}_{\alpha\in A} |\ x_\alpha\in M_\alpha$ for all $\alpha\in A\} \subset \{\{K_\alpha\}_{\alpha\in A} |\ K_\alpha\in K(M_\alpha)$ for all $\alpha\in A\}$, then $\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at each element of $\{\{\overline{\{x_\alpha\}}\}_{\alpha\in A} |\ x_\alpha\in M_\alpha$ for all $\alpha\in A\}$.

(l) implies (a): Let $\beta\in A$ and let $x_\beta\in M_\beta$. For each $\alpha\neq\beta$ let $x_\alpha\in M_\alpha$. Then $\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at $\{\overline{\{x_\alpha\}}\}_{\alpha\in A}$, which implies $K(X_\alpha)$ is l.c. at $\overline{\{x_\alpha\}}$ for all $\alpha\in A$ and $K(X_\alpha)$ is connected except for finitely many $\alpha\in A$. Then by Theorem 1.5 X_β is l.c. at x_β. Therefore X_β is l.c. at each element of M_β and since

$K(X_\alpha)$ is connected except for finitely many $\alpha\in A$, then by Theorem 1.3 X_α is connected except for finitely many $\alpha\in A$.

By a similar argument the theorem follows when l.c. is replaced by c.i.k.

If M_α is a component of X_α for all $\alpha\in A$, then the 24 statements in Theorem 2.2 are equivalent and each of the statements imply M_α is closed open for all $\alpha\in A$ and $\prod_{\alpha\in A} M_\alpha$ is closed open in $\prod_{\alpha\in A} X_\alpha$. Also, if $M_\alpha = X_\alpha$ for all $\alpha\in A$, then the 24 statements are equivalent and each of the statements imply components of X_α are closed open for all $\alpha\in A$ and components of $\prod_{\alpha\in A} X_\alpha$ are closed open.

Corollary 2.2. *For each* $\alpha\in A$ *let* (X_α, T_α) *be nonempty compact* R_0. *Then the following are equivalent*: (a) X_α *is l.c. for all* $\alpha\in A$ *and* X_α *is connected except for finitely many* $\alpha\in A$, (b) $2^{\prod_{\alpha\in A} X_\alpha}$ *is l.c.*, (c) $\prod_{\alpha\in A} 2^{X_\alpha}$ *is l.c., and* (d) 2^{X_α} *is l.c. for all* $\alpha\in A$ *and* 2^{X_α} *is connected except for finitely many* $\alpha\in A$.

Theorem 2.3. *For each* $\alpha\in A$ *let* (X_α, T_α) *be nonempty* R_0 *and let* $M_\alpha\in C(X_\alpha)$ *such that* X_α *is l.c. (c.i.k.) at each element of* M_α *for all* $\alpha\in A$. *Then the following are equivalent*: (a) $2^{\prod_{\alpha\in A} X_\alpha}$ *is l.c. (c.i.k.) at* $\prod_{\alpha\in A} M_\alpha$, (b) $\prod_{\alpha\in A} 2^{X_\alpha}$ *is l.c. (c.i.k.) at* $\{M_\alpha\}_{\alpha\in A}$, *and* (c) X_α *is connected except for finitely many* $\alpha\in A$.

The proof is straightforward using the previous results and is omitted.

Lemma 2.1. *For each* $\alpha\in A$ *let* (X_α, T_α) *be a nonempty topological space, let* K_α *be a nonempty compact subset of* X_α, *and let* $\mathcal{O}$ *be open in* $\prod_{\alpha\in A} X_\alpha$ *such that* $\prod_{\alpha\in A} K_\alpha \subset \mathcal{O}$. *Then for each* $\alpha\in A$ *there exists* $M_\alpha\in T_\alpha$ *such that* $M_\alpha = X_\alpha$ *except for finitely many* $\alpha\in A$ *and* $\prod_{\alpha\in A} K_\alpha \subset \prod_{\alpha\in A} M_\alpha \subset \mathcal{O}$.

Proof. Since a base for the weak topology on $\prod_{\alpha\in A} X_\alpha$ is $\mathcal{B} = \{\prod_{\alpha\in A} O_\alpha |\ O_\alpha\in T_\alpha$ for all $\alpha\in A$ and $O_\alpha = X_\alpha$ except for finitely many $\alpha\in A\}$, then for each $\{x_\alpha\}_{\alpha\in A}\in\prod_{\alpha\in A} K_\alpha$, let $\prod_{\alpha\in A} O_{x_\alpha}\in\mathcal{B}$ such that $\{x_\alpha\}_{\alpha\in A}\in\prod_{\alpha\in A} O_{x_\alpha}\in\mathcal{O}$. Then $\{\prod_{\alpha\in A} O_{x_\alpha}\}\ \{x_\alpha\}_{\alpha\in A}\in\prod_{\alpha\in A} K_\alpha$ is an open cover of $\prod_{\alpha\in A} K_\alpha$ and there exists a finite subcover $\{\prod_{\alpha\in A} O_{x(i)_\alpha}\}_{i=1}^n$. Let $F = \{\alpha\in A |\ O_{x(i)_\alpha}\neq X_\alpha$ for some $i\in\{1, \ldots, n\}\}$, which is finite. For each $\alpha\in A$ and $y_\alpha\in K_\alpha$ let

$$N_{y_\alpha} = \begin{cases} X_\alpha & \text{if } \alpha\in A-F, \\ \bigcap O_{x(i)_\alpha} & \text{if } \alpha\in F, \end{cases} \qquad y_\alpha\in O_{x(i)_\alpha}$$

and let

$$M_\alpha = \begin{cases} X_\alpha & \text{if } \alpha\in A-F, \\ \bigcup N_{y_\alpha} & \text{if } \alpha\in F, \end{cases} \qquad y_\alpha\in K_\alpha.$$

Then $M_\alpha \in T_\alpha$ for all $\alpha \in A$, $M_\alpha = X_\alpha$ except for finitely many $\alpha \in A$, and $\prod_{\alpha\in A} K_\alpha \subset \prod_{\alpha\in A} M_\alpha \subset \mathcal{O}$.

THEOREM 2.4. *For each $\alpha \in A$ let (X_α, T_α) be nonempty R_0 and let $K_\alpha \in K(X_\alpha) \cap C(X_\alpha)$. Then the following are equivalent:* (a) *X_α is connected except for finitely many $\alpha \in A$ and if O_α is open in X_α such that $K_\alpha \subset O_\alpha$, then there exists an open connected set C_α such that $K_\alpha \subset C_\alpha \subset O_\alpha$ for all $\alpha \in A$,* (b) *$\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$,* (c) *$\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$,* (d) *$2^{\prod_{\alpha\in A} X_\alpha}$ is l.c. at $\prod_{\alpha\in A} K_\alpha$, and* (e) *$K(\prod_{\alpha\in A} X_\alpha)$ is l.c. at $\prod_{\alpha\in A} K_\alpha$.*

Proof. (a) implies (b): By Theorem 1.3 $K(X_\alpha)$ is connected except for finitely many $\alpha \in A$, by Theorem 1.4 2^{X_α} is l.c. at K_α for all $\alpha \in A$, and by Theorem 1.7 $K(X_\alpha)$ is l.c. at K_α for all $\alpha \in A$, which implies $\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$.

(b) implies (c): Since $\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$, then $K(X_\alpha)$ is connected except for finitely many $\alpha \in A$ and $K(X_\alpha)$ is l.c. at K_α for all $\alpha \in A$, which implies 2^{X_α} is connected except for finitely many $\alpha \in A$ and 2^{X_α} is l.c. at K_α for all $\alpha \in A$. Thus $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$.

(c) implies (d): Since $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$, then 2^{X_α} is connected except for finitely many $\alpha \in A$, which implies X_α is connected except for finitely many $\alpha \in A$, and 2^{X_α} is l.c. at K_α for all $\alpha \in A$. Let $\mathcal{O}$ be open in $\prod_{\alpha\in A} X_\alpha$ such that $\prod_{\alpha\in A} K_\alpha \subset \mathcal{O}$. By Lemma 2.1 for each $\alpha \in A$ there exists $M_\alpha \in T_\alpha$ such that $M_\alpha = X_\alpha$ except for finitely many $\alpha \in A$ and $\prod_{\alpha\in A} K_\alpha \subset \prod_{\alpha\in A} M_\alpha \subset \mathcal{O}$. For each $\alpha \in F = \{\alpha \in A \mid M_\alpha \neq X_\alpha$ or X_α is not connected$\}$ let C_α be open connected in X_α such that $K_\alpha \subset C_\alpha \subset M_\alpha$. For each $\alpha \in A$ let

$$B_\alpha = \begin{cases} C_\alpha & \text{if } \alpha \in F, \\ X_\alpha & \text{if } \alpha \notin F. \end{cases}$$

Then $\prod_{\alpha\in A} B_\alpha$ is open connected in $\prod_{\alpha\in A} X_\alpha$ and $\prod_{\alpha\in A} K_\alpha \subset \prod_{\alpha\in A} B_\alpha \subset \mathcal{O}$. Thus by Theorem 1.4 $2^{\prod_{\alpha\in A} X_\alpha}$ is l.c. at $\prod_{\alpha\in A} K_\alpha$.

(d) implies (e): By Theorem 1.7 $K(\prod_{\alpha\in A} X_\alpha)$ is l.c. at $\prod_{\alpha\in A} K_\alpha$.

(e) implies (a): By Theorem 1.7 $2^{\prod_{\alpha\in A} X_\alpha}$ is l.c. at $\prod_{\alpha\in A} K_\alpha$. Then by Theorem 2.1 $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$ and X_α is connected except for finitely many $\alpha \in A$. Since $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{K_\alpha\}_{\alpha\in A}$, then 2^{X_α} is l.c. at K_α for all $\alpha \in A$, which implies if $O_\alpha \in T_\alpha$ such that $K_\alpha \subset O_\alpha$, then there exists an open connected set C_α such that $K_\alpha \subset C_\alpha \subset O_\alpha$ for all $\alpha \in A$.

THEOREM 2.5. *For each $\alpha \in A$ let (X_α, T_α) be nonempty R_0 and let $K_\alpha \in K(X_\alpha) \cap C(X_\alpha)$. Then the following are equivalent:* (a) *X_α is connected except for finitely many $\alpha \in A$ and if $O_\alpha \in T_\alpha$ such that $K_\alpha \subset O_\alpha$, then the component of O_α containing K_α contains K_α in its interior for all $\alpha \in A$,* (b) *$\prod_{\alpha\in A} K(X_\alpha)$ is c.i.k. at $\{K_\alpha\}_{\alpha\in A}$,* (c) *$\prod_{\alpha\in A} 2^{X_\alpha}$ is c.i.k. at $\{K_\alpha\}_{\alpha\in A}$,* (d) *$2^{\prod_{\alpha\in A} X_\alpha}$ is c.i.k. at $\prod_{\alpha\in A} K_\alpha$, and* (e) *$K(\prod_{\alpha\in A} X_\alpha)$ is c.i.k. at $\prod_{\alpha\in A} K_\alpha$*

The theorem follows by an argument similar to that for Theorem 2.4 and is omitted.

THEOREM 2.6. *For each $\alpha \in A$ let (X_α, T_α) be nonempty R_0 and let $C_\alpha \in K(X_\alpha)$ such that X_α is connected except for finitely many $\alpha \in A$ and if $\alpha \in A$, K_α is component of C_α, and $O_\alpha \in T_\alpha$ such that $K_\alpha \subset O_\alpha$, then there exists an open connected set B_α such that $K_\alpha \subset B_\alpha \in O_\alpha$. Then $\prod_{\alpha\in A} K(X_\alpha)$ and $\prod_{\alpha\in A} 2^{X_\alpha}$ are l.c. at $\{C_\alpha\}_{\alpha\in A}$ and $K(\prod_{\alpha\in A} X_\alpha)$ and $2^{\prod_{\alpha\in A} X_\alpha}$ are l.c. at $\prod_{\alpha\in A} C_\alpha$.*

Proof. Since X_α is connected except for finitely many $\alpha \in A$, then 2^{X_α} and $K(X_\alpha)$ are connected except for finitely many $\alpha \in A$. By Theorem 1.4 2^{X_α} is l.c. at each component of C_α and by Theorem 1.8 2^{X_α} and $K(X_\alpha)$ are l.c. at C_α. Hence $\prod_{\alpha\in A} K(X_\alpha)$ and $\prod_{\alpha\in A} 2^{X_\alpha}$ are l.c. at $\{C_\alpha\}_{\alpha\in A}$. Let $\mathcal{C}$ be a component of $\prod_{\alpha\in A} C_\alpha$. Then $\mathcal{C} = \prod_{\alpha\in A} K_\alpha$, where K_α is a component of C_α for all $\alpha \in A$, and by Theorem 2.4, $2^{\prod_{\alpha\in A} X_\alpha}$ and $K(\prod_{\alpha\in A} X_\alpha)$ are l.c. at $\mathcal{C}$. Then by Theorem 1.8 $2^{\prod_{\alpha\in A} X_\alpha}$ and $K(\prod_{\alpha\in A} X_\alpha)$ are l.c. at $\prod_{\alpha\in A} C_\alpha$.

THEOREM 2.7. *For each $\alpha \in A$ let (X_α, T_α) be nonempty R_0 and let $C_\alpha \in K(X_\alpha)$ such that X_α is connected except for finitely many $\alpha \in A$ and if $\alpha \in A$, K_α is a component of C_α, and $O_\alpha \in T_\alpha$ such that $K_\alpha \subset O_\alpha$, then the component of O_α containing K_α contains K_α in its interior. Then $\prod_{\alpha\in A} K(X_\alpha)$ and $\prod_{\alpha\in A} 2^{X_\alpha}$ are c.i.k. at $\{C_\alpha\}_{\alpha\in A}$ and $K(\prod_{\alpha\in A} X_\alpha)$ and $2^{\prod_{\alpha\in A} X_\alpha}$ are c.i.k. at $\prod_{\alpha\in A} C_\alpha$.*

The theorem follows by an argument similar to that for Theorem 2.6 and is omitted.

THEOREM 2.8. *For each $\alpha \in A$ let (X_α, T_α) be nonempty R_1 and let $C_\alpha \in K(X_\alpha)$ for all $\alpha \in A$. Then the following are equivalent:* (a) *X_α is connected except for finitely many $\alpha \in A$ and if $\alpha \in A$, K_α is a component of C_α, and $O_\alpha \in T_\alpha$*

such that $K_\alpha \subset O_\alpha$, *then there exists an open connected set* B_α *such that* $K_\alpha \subset B_\alpha \subset O_\alpha$, (b) $\prod_{\alpha\in A} K(X_\alpha)$ *is l.c. at* $\{C_\alpha\}_{\alpha\in A}$, (c) $\prod_{\alpha\in A} 2^{X_\alpha}$ *is l.c. at* $\{C_\alpha\}_{\alpha\in A}$, (d) $K(\prod_{\alpha\in A} X_\alpha)$ *is l.c. at* $\prod_{\alpha\in A} C_\alpha$, *and* (e) $2^{\prod_{\alpha\in A} X_\alpha}$ *is l.c. at* $\prod_{\alpha\in A} C_\alpha$.

Proof. By Theorem 2.6 (a) implies (b).

(b) implies (c): Since $\prod_{\alpha\in A} K(X_\alpha)$ is l.c. at $\{C_\alpha\}_{\alpha\in A}$, then $K(X_\alpha)$ is connected except for finitely many $\alpha\in A$ and $K(X_\alpha)$ is l.c. at C_α for all $\alpha\in A$. Then by Theorem 1.3 2^{X_α} is connected except for finitely many $\alpha\in A$ and by Theorem 1.10 2^{X_α} is l.c. at C_α for all $\alpha\in A$. Thus $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{C_\alpha\}_{\alpha\in A}$.

(c) implies (d): Since $\prod_{\alpha\in A} 2^{X_\alpha}$ is l.c. at $\{C_\alpha\}_{\alpha\in A}$, then 2^{X_α} is connected except for finitely many $\alpha\in A$ and 2^{X_α} is l.c. at C_α for all $\alpha\in A$. Then by Theorem 1.3 X_α is connected except for finitely many $\alpha\in A$, by Theorem 1.10 if $\alpha\in A$, K_α is a component of C_α, and $O_\alpha\in T_\alpha$ such that $K_\alpha\subset O_\alpha$, then there exists an open connected set B_α such that $K_\alpha\subset B_\alpha\subset O_\alpha$, and by Theorem 2.6 $2^{\prod_{\alpha\in A} X_\alpha}$ is l.c. at $\prod_{\alpha\in A} C_\alpha$.

(d) implies (e): Since $\prod_{\alpha\in A} X_\alpha$ is R_1, $\prod_{\alpha\in A} C_\alpha \in K(\prod_{\alpha\in A} X_\alpha)$, and $2^{\prod_{\alpha\in A} X_\alpha}$ is l.c. at $\prod_{\alpha\in A} C_\alpha$, then by Theorem 1.10 $K(\prod_{\alpha\in A} X_\alpha)$ is l.c. at $\prod_{\alpha\in A} C_\alpha$.

(e) implies (a): Let $\beta\in A$, let K_β be a component of C_β, and let $O_\beta\in T_\beta$ such that $K_\beta\subset O_\beta$. For each $\alpha\neq\beta$ let K_α be a component of C_α. Then $\prod_{\alpha\in A} K_\alpha$ is a component of $\prod_{\alpha\in A} C_\alpha$ and by Theorem 1.10 $2^{\prod_{\alpha\in A} X_\alpha}$ is l.c. at $\prod_{\alpha\in A} K_\alpha$. Then by Theorem 2.4 X_α is connected except for finitely many $\alpha\in A$ and there exists an open connected set B_β such that $K_\beta\subset B_\beta\subset O_\beta$.

THEOREM 2.9. *For each* $\alpha\in A$ *let* (X_α, T_α) *be nonempty locally compact* R_1 *and let* $C_\alpha\in K(X_\alpha)$ *for all* $\alpha\in A$, *where* X_α *is compact except for finitely many* $\alpha\in A$. *Then the following are equivalent*: (a) X_α *is connected except for finitely many* $\alpha\in A$ *and if* $\alpha\in A$, K_α *is a component of* C_α, *and* $O_\alpha\in T_\alpha$ *such that* $K_\alpha\subset O_\alpha$, *then the component of* O_α *containing* K_α *contains* K_α *in its interior*, (b) $\prod_{\alpha\in A} K(X_\alpha)$ *is c.i.k. at* $\{C_\alpha\}_{\alpha\in A}$, (c) $\prod_{\alpha\in A} 2^{X_\alpha}$ *is c.i.k. at* $\{C_\alpha\}_{\alpha\in A}$, (d) $K(\prod_{\alpha\in A} X_\alpha)$ *is c.i.k at* $\prod_{\alpha\in A} C_\alpha$, *and* (e) $2^{\prod_{\alpha\in A} X_\alpha}$ *is c.i.k. at* $\prod_{\alpha\in A} C_\alpha$.

The theorem follows by an argument similar to that for Theorem 2.8 and is omitted.

References

[1] A. Davis, *Indexed systems of neighborhoods for general topological spaces*, Amer. Math. Monthly 68 (1961), pp. 886–893.

[2] C. Dorsett, *Local Connectedness, Connectedness im kleinen, and other properties of hyperspaces of R_0 Spaces*, accepted by Mat. Vesnik.

[3] — *R_0 spaces, R_1 spaces, and hyperspaces*, Dissertation, North Texas State University, Dec. 1976.

[4] K. Dube, *A note on R_0 topological spaces*, Mat. Vesnik 11 (1974), pp. 203–208.

[5] W. Dunham, *Weakly Hausdorff spaces*, Kyungpook Math. J. 15 (1975), pp. 41–50.

[6] O. Frink, *Topologies in lattices*, Trans. Amer. Math. Soc. 51 (1942), pp. 569–582.

[7] M. Wojdyslawski, *Retractes absolus et hyperspaces des continus*, Fund. Math. 32 (1939), pp. 184–192.

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Received 1 December 1980