Corollary 3.6. Let \((X, T)\) be a point-transitive flow where \(X\) is a sphere, real or complex projective space, or lens space (of dimension greater than one), and \(T\) is a connected abelian Lie group. Then \((X, T)\) satisfies all conclusions of Theorem 2.15 and Proposition 3.5.

References


Received 19 February 1979;
in revised form 17 August 1981

Connectivity properties in hyperspaces and product spaces

by

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Abstract. In this paper connectedness, local connectedness, and point-wise local connectedness in the class of product spaces and product spaces of hyperspaces are investigated and the relationships between these connectivity properties in hyperspaces of product spaces and product spaces of hyperspaces are determined. In order to include as many spaces as possible, the results in this paper are stated and proved for \(R_0\) and \(R_1\) topological spaces.

1. Introduction. One of the earliest results about connectivity properties in hyperspaces, due to Wojdyslawski [7] in 1939, is that for a metric continuum \((X, T)\), \((2^X, E(X))\) is locally connected (l.c.) iff \((X, T)\) is l.c. Since 1939 mathematicians have continued the investigation of connectivity properties in hyperspaces. In this paper connectivity properties in hyperspaces of product spaces and product spaces of hyperspaces are investigated. In order to include as many spaces as possible, the results in this paper are stated and proved for weak topological spaces. Listed below are definitions and theorems that will be utilized in this paper.

Definition 1.1. A space \((X, T)\) is \(R_0\) iff for each \(e \in T\) and \(x \in e\), \(|x| = 0\) [1].

Definition 1.2. A space \((X, T)\) is \(R_1\) iff for each pair \(x, y \in X\) such that \(x \neq y\), there exist disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\) [1].

Definition 1.3. Let \((X, T)\) be a space, let \(A \subseteq X\), and define \(2^X\), \(C(X), K(X), S(A), I(A)\) as follows: \(2^X = \{F \subseteq X\} F\) is nonempty and closed, \(C(X) = \{F \subseteq 2^X\} F\) is connected, \(K(X) = \{F \subseteq 2^X\} F\) is compact, \(S(A) = \{F \subseteq 2^X\} F \subseteq A\), and \(I(A) = \{F \subseteq 2^X\} F \cap A \neq \phi\). Then the Vietoris topology on \(2^X\), denoted by \(E(X)\), is the smallest topology on \(2^X\) which satisfies the conditions that if \(G \subseteq T\), then \(S(G) \subseteq E(X)\) and \(I(G) \subseteq E(X)\) [6].

Theorem 1.1. The product of an arbitrary family of nonempty topological spaces is \(R_0\) iff each factor space is \(R_0\) [4].

Theorem 1.2. If \((X, T)\) is \(R_1\), then \((X, T)\) is \(R_0\) [5].

Theorem 1.3. If \((X, T)\) is \(R_0\), then the following are equivalent: (a) \(X\) is connected, (b) \(2^X\) is connected, and (c) \(K(X)\) is connected [2].
Theorem 1.4. If \((X, T)\) is \(R_0\) and \(B \in C(X)\), then \(2^X\) is lc. (connected im kleinen \(c.i.k\)) at \(B\) iff for each \(U \in T\) such that \(B \subseteq U\), there exists an open connected set \(V\) such that \(B \subseteq V \subseteq U\) (for each \(U \in T\) such that \(B \subseteq U\), the component of \(U\) containing \(B\) is in \(U\)) \([2]\).

Theorem 1.5. If \((X, T)\) is \(R_0\) and \(M \subseteq 2^X\), then the following are equivalent: (a) \(X\) is lc. \(c.i.k\) at each element of \(M\), (b) \(2^X\) is lc. \(c.i.k\) at each element of \(K(M)\), and (c) \(X\) is lc. \(c.i.k\) at each element of \(K(M)\), (d) \(X\) is lc. \(c.i.k\) at each element of \(\{x\in M\}|\{x\} \in M\}|\{x\} \in M\} \([2]\).

Theorem 1.6. If \((X, T)\) is \(R_0\) and \(C\) is a component of \(X\), then the following are equivalent: (a) \(C\) is lc. \(c.i.k\) at \(C\), (b) \(C\) is closed open in \(X\), and (c) \(2^X\) is lc. \(c.i.k\) at \(C \([2]\).

Theorem 1.7. If \((X, T)\) is \(R_0\) and \(C \subseteq X(X) \cap C(X)\), then \(2^X\) is lc. \(c.i.k\) at \(C\) if \(K(X)\) is lc. \(c.i.k\) at \(C \([2]\).

Theorem 1.8. If \((X, T)\) is \(R_0\), \(B \subseteq K(X)\), and \(2^B\) or \(K(X)\) is lc. \(c.i.k\) at each component of \(B\), then \(2^B\) and \(K(X)\) are lc. \(c.i.k\) at \(B \([2]\).

Theorem 1.9. The product of an arbitrary family of nonempty topological spaces \(R_0\) iff each factor space is \(R_0\) \([5]\).

Theorem 1.10. If \((X, T)\) is \(R_1\) and \(A \subseteq K(X)\), then the following are equivalent: (a) \(2^A\) is lc. \(c.i.k\) at \(A\), (b) \(2^A\) is lc. \(c.i.k\) at each component of \(A\), and (c) \(A\) is a component of \(A\) if \(U \in T\) such that \(A \subseteq U\), then there exists an open connected set \(V\) such that \(A \subseteq V \subseteq U\), \((U)\) is lc. \(c.i.k\) at each component of \(A\), and \(A\) is lc. \(c.i.k\) at \(A \([3]\).

Theorem 1.11. If \((X, T)\) is locally compact \(R_0\) and \(A \subseteq K(X)\), then the following are equivalent: (a) \(K(X)\) is lc. \(c.i.k\) at \(A\), (b) \(K(X)\) is lc. \(c.i.k\) at each component of \(A\), (c) \(K(X)\) is lc. \(c.i.k\) at each component of \(A\), if \(U \in T\) such that \(A \subseteq U\), then the component of \(U\) containing \(A\) contains \(A\) in its interior, (d) \(2^A\) is lc. \(c.i.k\) at each component of \(A\), and (e) \(2^A\) is lc. \(c.i.k\) at \(A \([3]\).

2. Connectivity properties and product spaces. The first result follows from Theorem 1.1 and Theorem 1.3.

Corollary 2.1. If \((X_\Lambda, T_\Lambda)\) is nonempty and \(R_0\) for all \(\Lambda \subseteq A\), then the following are equivalent: (a) \((X_\Lambda, T_\Lambda)\) is connected for all \(\Lambda \subseteq A\), (b) \(2^{\Lambda X_\Lambda}\) is connected, (c) \(K(\Lambda X_\Lambda)\) is connected, (d) \(\Lambda X_\Lambda\) is connected, and (e) \(\Lambda X_\Lambda\) is connected.

Theorem 2.1. For each \(\Lambda \subseteq A\) let \((X_\Lambda, T_\Lambda)\) be a nonempty \(R_0\) space and let \(C_\Lambda \subseteq C(X_\Lambda)\). If \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at \(\Lambda X_\Lambda\), then \(\Lambda X_\Lambda\) is lc. \(c.i.k\) at \(\Lambda X_\Lambda\) where

\[
W_\Lambda = \begin{cases} 
C_\Lambda & \text{if } \Lambda \notin F, \\
X_\Lambda & \text{if } \Lambda \notin F 
\end{cases}
\]

and \(F \subseteq A\) = \{\Lambda \subseteq A\} | \(B \subseteq \Lambda X_\Lambda\) \(X_\Lambda\) is not connected\}, \(\Lambda X_\Lambda\) is not connected\} is finite.

Proof. For each \(\beta \subseteq A\) let \(P_\beta: \Pi_{\Lambda \subseteq A} X_\Lambda \to X_\beta\) be the projection function.

Consider the case that \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at \(\Lambda X_\Lambda\). Let \(B \in A\) and let \(0 \subseteq \Lambda \subseteq T_\beta\) such that \(C_\beta \subseteq 0\). For each \(\Lambda \subseteq A\) let

\[
B_\Lambda = \begin{cases} 
0 & \text{if } \Lambda = \beta, \\
X_\Lambda & \text{if } \Lambda \neq \beta 
\end{cases}
\]

Then \(\Pi_{\Lambda \subseteq A} C_\Lambda = \Pi_{\Lambda \subseteq A} R_\Lambda\), which is open in \(\Pi_{\Lambda \subseteq A} X_\Lambda\), and by Theorem 1.4, there exists an open connected set \(\gamma\) such that \(\Pi_{\Lambda \subseteq A} C_\Lambda = \gamma \subseteq \Pi_{\Lambda \subseteq A} R_\Lambda\). Then \(C_\beta = P_\beta(\gamma) \subseteq 0\), where \(P_\beta(\gamma)\) is open connected. Hence, by Theorem 1.4, \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at \(\beta\). Also, since \(P_\beta(\gamma) = X_\beta\) except for finitely many \(\Lambda \subseteq A\), then \(F_\beta = \{\Lambda \subseteq A\} | X_\beta\) is not connected\} is finite. Let \(F \subseteq A\). For each \(\Lambda \subseteq A\), \(X_\Lambda\) is a closed open component and by Theorem 1.6, \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at \(\Lambda X_\Lambda\). Therefore \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at \(\Lambda X_\Lambda\) for all \(\Lambda \subseteq A\) and \(2^{\Lambda X_\Lambda}\) is connected except for finitely many \(\Lambda \subseteq A\), which implies \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at \(\Pi_{\Lambda \subseteq A} W_\Lambda\).

By a similar argument, the theorem follows for \(c.i.k\).

Theorem 2.2. For each \(\Lambda \subseteq A\) let \((X_\Lambda, T_\Lambda)\) be nonempty \(R_0\) and let \(M_\Lambda \subseteq 2^{X_\Lambda}\). Then the following are equivalent: (a) \(X_\Lambda\) is lc. \(c.i.k\) at each element of \(M_\Lambda\) for all \(\Lambda \subseteq A\) and \(X_\Lambda\) is connected except for finitely many \(\Lambda \subseteq A\), (b) \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (c) \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (d) \(K(\Pi_{\Lambda \subseteq A} X_\Lambda)=\Pi_{\Lambda \subseteq A} (K(X_\Lambda))\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (e) \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (f) \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (g) \(2^{\Lambda X_\Lambda}\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (h) \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (i) \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (j) \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\), (k) \(\Pi_{\Lambda \subseteq A} X_\Lambda\) is lc. \(c.i.k\) at each element of \(\Pi_{\Lambda \subseteq A} M_\Lambda\).
is lc. (c.i.k.) at each element of $\{K_\alpha\}_{\alpha \in A}$, and (i)
$\prod K(X_\alpha)$ is lc. (c.i.k.) at each element of $\{\{x_\alpha\}_{\alpha \in A} | x_\alpha \in M_\alpha \text{ for all } \alpha \in A\}$.

Proof. Consider the theorem for lc. The straightforward proof that (a) and (b) are equivalent is omitted. By Theorem 1.5 (b) through (g) are equivalent and by Theorem 2.1 (g) implies (h).

(h) implies (i): For each $\alpha \in A$ let $K_\alpha \in K(M_\alpha)$. Let $\beta \in A$ and let $C_\beta \in C(M_\beta)$. For each $\alpha \neq \beta$ let $x_\alpha \in M_\alpha$. For each $\alpha \in A$ let

$$W_\alpha = \begin{cases} C_\beta & \text{if } \alpha = \beta, \\ \{x_\alpha\} & \text{if } \alpha \neq \beta. \end{cases}$$

Then for each $\alpha \in A$, $W_\alpha \in C(M_\alpha)$ and $\prod W_\alpha$ is lc. at $\{W_\beta\}_{\beta \neq \alpha}$, which implies $2^{\times_\alpha}$ is lc. at $W_\alpha$ for all $\alpha \in A$ and $2^{\times_\alpha}$ is connected except for finitely many $\alpha \in A$. Thus $2^{\times_\alpha}$ is lc. at each element of $C(M_\alpha)$ and by Theorem 1.5 $2^{\times_\alpha}$ is lc. at each element of $K(M_\alpha)$, which implies $2^{\times_\alpha}$ is lc. at $K_\alpha$. Therefore $2^{\times_\alpha}$ is lc. at $K_\alpha$ for all $\alpha \in A$ and $2^{\times_\alpha}$ is connected except for finitely many $\alpha \in A$, which implies $\prod 2^{\times_\alpha}$ is lc. at $\{K_\alpha\}_{\alpha \in A}$.

(i) implies (j): Since $\{\{x_\alpha\}_{\alpha \in A} | x_\alpha \in M_\alpha \text{ for all } \alpha \in A\} \subset \{\{x_\beta\}_{\beta \in A} | K_\beta \in K(M_\beta) \text{ for all } \beta \in A\}$, then $\prod 2^{\times_\alpha}$ is lc. at each element of $\{\{x_\beta\}_{\beta \in A} | K_\beta \in K(M_\beta) \text{ for all } \beta \in A\}$.

(j) implies (k): For each $\alpha \in A$ let $K_\alpha \in K(M_\alpha)$. Let $\beta \in A$ and let $x_\alpha \in M_\alpha$. For each $\alpha \neq \beta$ let $x_\alpha \in M_\alpha$. Then $\prod 2^{\times_\alpha}$ is lc. at $\{x_\alpha\}_{\alpha \in A}$, which implies $2^{\times_\alpha}$ is lc. at $\{x_\alpha\}$ for all $\alpha \in A$. Hence $2^{\times_\alpha}$ is lc. at each element of $\{x_\alpha\}_{\alpha \in A}$ and by Theorem 1.5 $K(X_\alpha)$ is lc. at each element of $K(M_\alpha)$, which implies $K(X_\alpha)$ is lc. at $K_\alpha$. Also, since $2^{\times_\alpha}$ is connected except for finitely many $\alpha \in A$, then by Theorem 1.3 $K(X_\alpha)$ is connected except for finitely many $\alpha \in A$. Therefore $K(X_\alpha)$ is lc. at $K_\alpha$ for all $\alpha \in A$ and $K(X_\alpha)$ is connected except for finitely many $\alpha \in A$, which implies $\prod K(X_\alpha)$ is lc. at $\{K_\alpha\}_{\alpha \in A}$.

(k) implies (l): Since $\{\{x_\alpha\}_{\alpha \in A} | x_\alpha \in M_\alpha \text{ for all } \alpha \in A\} \subset \{\{x_\beta\}_{\beta \in A} | K_\beta \in K(M_\beta) \text{ for all } \beta \in A\}$, then $\prod K(X_\alpha)$ is lc. at each element of $\{\{x_\beta\}_{\beta \in A} | x_\beta \in M_\beta \text{ for all } \beta \in A\}$.

(l) implies (a): Let $\beta \in A$ and let $x_\beta \in M_\beta$. For each $\alpha \neq \beta$ let $x_\alpha \in M_\alpha$. Then $\prod K(X_\alpha)$ is lc. at $\{x_\alpha\}$, which implies $K(X_\beta)$ is lc. at $\{x_\beta\}$ for all $\beta \in A$ and $K(X_\beta)$ is connected except for finitely many $\beta \in A$. Then by Theorem 1.3 $X_\beta$ is lc. at $x_\beta$. Therefore $X_\beta$ is lc. at each element of $M_\beta$ and since $K(X_\beta)$ is connected except for finitely many $\beta \in A$, then by Theorem 1.3 $X_\beta$ is connected except for finitely many $\beta \in A$.

By a similar argument the theorem follows when lc. is replaced by c.i.k.

If $M_\alpha$ is a component of $X_\alpha$ for all $\alpha \in A$, Then the 24 statements in Theorem 2.2 are equivalent and each of the statements imply $M_\alpha$ is closed open for all $\alpha \in A$ and $\prod M_\alpha$ is closed open in $\prod X_\alpha$. Also, if $M_\alpha = X_\alpha$ for all $\alpha \in A$, then the 24 statements are equivalent and each of the statements imply components of $X_\alpha$ are closed open for all $\alpha \in A$ and components of $\prod X_\alpha$ are closed open.

**Corollary 2.2.** For each $\alpha \in A$ let $\{X_\alpha, V_\alpha\}$ be nonempty compact $R_0$. Then the following are equivalent: (a) $X_\alpha$ is lc. for all $\alpha \in A$ and $X_\alpha$ is connected except for finitely many $\alpha \in A$, (b) $\prod X_\alpha$ is lc. and $\prod 2^{\times_\alpha}$ is lc. except for finitely many $\alpha \in A$, (c) $\prod 2^{\times_\alpha}$ is lc. and $\prod X_\alpha$ is connected except for finitely many $\alpha \in A$.

**Theorem 2.3.** For each $\alpha \in A$ let $\{X_\alpha, V_\alpha\}$ be nonempty compact $R_0$ and let $M_\alpha \in C(X_\alpha)$ such that $X_\alpha$ is lc. (c.i.k.) at each element of $M_\alpha$ for all $\alpha \in A$. Then the following are equivalent: (a) $\prod \overline{\prod X_\alpha}$ is lc. (c.i.k.) at $\prod M_\alpha$, (b) $\prod 2^{\times_\alpha}$ is lc. (c.i.k.) at $\{M_\alpha\}_{\alpha \in A}$, and (c) $X_\alpha$ is connected except for finitely many $\alpha \in A$.

The proof is straightforward using the previous results and is omitted.

**Lemma 2.1.** For each $\alpha \in A$ let $\{x_\alpha, V_\alpha\}$ be a nonempty topological space, let $K_\alpha$ be a nonempty compact subset of $X_\alpha$, and let $\theta$ be open in $\prod X_\alpha$ such that $\prod K_\alpha \subset \theta$. Then for each $\alpha \in A$ there exists $M_\alpha \in V_\alpha$ such that $M_\alpha = X_\alpha$ except for finitely many $\alpha \in A$ and $\prod K_\alpha \subset \theta'$.

**Proof.** Since a base for the weak topology on $\prod X_\alpha$ is $\mathcal{B} = \{\prod O_\alpha | O_\alpha \in V_\alpha \text{ for all } \alpha \in A \text{ and } O_\alpha = X_\alpha \text{ except for finitely many } \alpha \in A\}$, then for each $\{x_\alpha\}_{\alpha \in A} \in \prod K_\alpha$, let $\prod O_\alpha \in \mathcal{B}$ such that $\{x_\alpha\}_{\alpha \in A} \in \prod O_\alpha \in \theta'$.

Then $\prod O_\alpha \in \{x_\alpha\}_{\alpha \in A} \subset \prod K_\alpha$ is an open cover of $\prod K_\alpha$ and there exists a finite subcover $\{\prod O_{\alpha_0}\}_{i=1}^n$. Let $F = \{\alpha \in A | O_{\alpha_0} \neq X_\alpha \text{ for some } i \in \{1, \ldots, n\}\}$, which is finite. For each $\alpha \in A$ and $y_\alpha \in K_\alpha$ let

$$N_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in F, \\ y_\alpha & \text{if } \alpha \notin F, \end{cases}$$

and let

$$M_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in F, \\ y_\alpha & \text{if } \alpha \notin F, \end{cases}$$

for $\alpha \in A$. Then $M_\alpha$ is lc. at each element of $M_\alpha$.
Then $M_x \in T_x$ for all $x \in A$, $M_x = X_x$ except for finitely many $x \in A$, and
\[ \prod_{x \in A} K_x = \prod_{x \in A} M_x = \emptyset. \]

**Theorem 2.4. For each $x \in A$ let $(X_x, T_x)$ be nonempty $R_0$ and let $K_x \in K(X_x) \cap C(X_x)$. Then the following are equivalent:**
(a) $X_x$ is connected except for finitely many $x \in A$ and if $O_x \in T_x$ then there exists an open connected set $C_x$ such that $K_x \subseteq C_x \subseteq O_x$ for all $x \in A$,
(b) \[ \prod_{x \in A} K_x \quad \text{is l.c. at} \quad \{K_x\}_{x \in A}, \]
(c) \[ \prod_{x \in A} K_x \quad \text{is l.c. at} \quad \{K_x\}_{x \in A}, \]
(d) \[ \prod_{x \in A} K_x \quad \text{is l.c. at} \quad \{K_x\}_{x \in A}, \]
(e) \[ \prod_{x \in A} K_x \quad \text{is l.c. at} \quad \{K_x\}_{x \in A}, \]
(f) $K(\prod_{x \in A} X_x)$ is l.c. at $\{K_x\}_{x \in A}$.

**Proof:** (a) implies (b): By Theorem 1.3 $K(X_x)$ is connected except for finitely many $x \in A$, by Theorem 1.4 $X_x$ is l.c. at $K_x$ for all $x \in A$, and by Theorem 2.6 $K(X_x)$ is l.c. at $K_x$ for all $x \in A$, which implies $\prod_{x \in A} K(X_x)$ is l.c. at $\{K_x\}_{x \in A}$.

(b) implies (c): Since $\prod_{x \in A} K(X_x)$ is l.c. at $\{K_x\}_{x \in A}$, then $K(X_x)$ is connected except for finitely many $x \in A$ and $K(X_x)$ is l.c. at $K_x$ for all $x \in A$, which implies $\prod_{x \in A} K(X_x)$ is l.c. at $\{K_x\}_{x \in A}$.

(c) implies (d): Since $\prod_{x \in A} K(X_x)$ is l.c. at $\{K_x\}_{x \in A}$, then $X_x$ is connected except for finitely many $x \in A$, which implies $X_x$ is connected except for finitely many $x \in A$, and $X_x$ is l.c. at $K_x$ for all $x \in A$. Let $\theta$ be open in $\prod_{x \in A} X_x$ such that $\prod_{x \in A} K_x \subseteq \theta$. By Lemma 2.1 for each $x \in A$ there exists $M_x \in T_x$ such that $M_x \subseteq X_x$ except for finitely many $x \in A$ and $\prod_{x \in A} K_x \subseteq \prod_{x \in A} M_x = \emptyset$. For each $x \in F = \{x \in A \mid M_x \notin X_x \}$ or $x \notin F$ let $C_x$ be open connected in $X_x$ such that $K_x \subseteq C_x \subseteq M_x$. For each $x \in A$ let $B_x = \begin{cases} C_x & \text{if } x \notin F, \\ X_x & \text{if } x \in F. \end{cases}$

Then $\prod_{x \in A} B_x$ is open connected in $\prod_{x \in A} X_x$ and $\prod_{x \in A} K_x \subseteq \prod_{x \in A} B_x \subseteq \theta$. Thus by Theorem 1.4 $\prod_{x \in A} B_x$ is l.c. at $\prod_{x \in A} K_x$.

(d) implies (e): By Theorem 1.7 $K(\prod_{x \in A} X_x)$ is l.c. at $\prod_{x \in A} K_x$.

(e) implies (a): By Theorem 1.7 $\prod_{x \in A} K_x$ is l.c. at $\prod_{x \in A} K_x$. Then by Theorem 2.1 $\prod_{x \in A} K_x$ is l.c. at $\{K_x\}_{x \in A}$ and $X_x$ is connected except for finitely many $x \in A$. Since $\prod_{x \in A} K_x$ is l.c. at $\{K_x\}_{x \in A}$, then $2^n$ is l.c. at $K_x$ for all $x \in A$, which implies if $O_x \in T_x$ such that $K_x \subseteq O_x$, then there exists an open connected set $C_x$ such that $K_x \subseteq C_x \subseteq O_x$ for all $x \in A$.

**Theorem 2.5. For each $x \in A$ let $(X_x, T_x)$ be nonempty $R_0$ and let $K_x \in K(X_x) \cap C(X_x)$. Then the following are equivalent:**
(a) $X_x$ is connected except for finitely many $x \in A$ and if $O_x \in T_x$ such that $K_x \subseteq O_x$, then the component of $O_x$ containing $K_x$ contains $K_x$ in its interior for all $x \in A$,
(b) $\prod_{x \in A} K(X_x)$ is c.i.k. at $\{K_x\}_{x \in A}$,
(c) $\prod_{x \in A} K(X_x)$ is c.i.k. at $\{K_x\}_{x \in A}$,
(d) $\prod_{x \in A} K(X_x)$ is c.i.k. at $\{K_x\}_{x \in A}$,
(e) $K(\prod_{x \in A} X_x)$ is c.i.k. at $\{K_x\}_{x \in A}$.

The theorem follows by an argument similar to that for Theorem 2.4 and is omitted.

**Theorem 2.6. For each $x \in A$ let $(X_x, T_x)$ be nonempty $R_0$ and let $C_x \in K(X_x)$ such that $X_x$ is connected except for finitely many $x \in A$ and if $x \in A$, $K_x$ is component of $C_x$, and $O_x \subseteq T_x$ such that $K_x \subseteq O_x$, then there exists an open connected set $B_x$ such that $K_x \subseteq B_x \subseteq O_x$. Then $\prod_{x \in A} K(X_x)$ and $\prod_{x \in A} K(X_x)$ are l.c. at $\{C_x\}_{x \in A}$ and $K(\prod_{x \in A} X_x)$ and $\prod_{x \in A} K(X_x)$ are l.c. at $\{C_x\}_{x \in A}$.

**Proof.** Since $X_x$ is connected except for finitely many $x \in A$, then $X_x$ and $K(X_x)$ are connected except for finitely many $x \in A$. By Theorem 1.4 $X_x$ is l.c. at each component of $C_x$ and $K(X_x)$ are l.c. at $C_x$. Hence $\prod_{x \in A} K(X_x)$ and $\prod_{x \in A} K(X_x)$ are l.c. at $\{C_x\}_{x \in A}$. Let $\theta$ be a component of $\prod_{x \in A} C_x$. Then $\theta = \prod_{x \in A} K_x$, where $K_x$ is a component of $C_x$ for all $x \in A$, and by Theorem 2.4 $\prod_{x \in A} K(X_x)$ and $K(\prod_{x \in A} X_x)$ are l.c. at $\theta$. Then by Theorem 1.8 $\prod_{x \in A} X_x$ and $K(\prod_{x \in A} X_x)$ are l.c. at $\prod_{x \in A} C_x$.

**Theorem 2.7. For each $x \in A$ let $(X_x, T_x)$ be nonempty $R_0$ and let $C_x \in K(X_x)$ such that $X_x$ is connected except for finitely many $x \in A$ and if $x \in A$, $K_x$ is component of $C_x$, and $O_x \subseteq T_x$ such that $K_x \subseteq O_x$, then the component of $O_x$ containing $K_x$ contains $K_x$ in its interior. Then $\prod_{x \in A} K(X_x)$ and $\prod_{x \in A} K(X_x)$ are c.i.k. at $\{C_x\}_{x \in A}$ and $K(\prod_{x \in A} X_x)$ and $\prod_{x \in A} K(X_x)$ are c.i.k. at $\{C_x\}_{x \in A}$.

The theorem follows by an argument similar to that for Theorem 2.6 and is omitted.

**Theorem 2.8. For each $x \in A$ let $(X_x, T_x)$ be nonempty $R_0$ and let $C_x \in K(X_x)$ for all $x \in A$. Then the following are equivalent:**
(a) $X_x$ is connected except for finitely many $x \in A$ and if $x \in A$, $K_x$ is a component of $C_x$, and $O_x \subseteq T_x$.
such that \( K_s \subseteq O_s \), then there exists an open connected set \( B_s \) such that 
\( K_s \subseteq B_s \subseteq O_s \). (b) \( \prod X_s \) is l.c. at \( \{C_s\}_{s \in A^*} \) and \( \prod 2^{s} \) is l.c. at \( \{C_s\}_{s \in A^*} \). (d) 
\( K(\prod X_s) \) is l.c. at \( \prod C_s \) and (e) \( \prod 2^{s} \) is l.c. at \( \prod C_s \).

Proof. By Theorem 2.6 (a) implies (b).

(b) implies (c): Since \( \prod K(X_s) \) is l.c. at \( \{C_s\}_{s \in A^*} \), then \( K(X_s) \) is connected except for finitely many \( s \in A \) and \( K(X_s) \) is l.c. at \( C_s \) for all \( s \in A \). Then by Theorem 1.3 \( 2^{s} \) is connected except for finitely many \( s \in A \) and by Theorem 1.10 \( 2^{s} \) is l.c. at \( C_s \) for all \( s \in A \). Thus \( \prod 2^{s} \) is l.c. at \( \{C_s\}_{s \in A^*} \).

(c) implies (d): Since \( \prod 2^{s} \) is l.c. at \( \{C_s\}_{s \in A^*} \), then \( 2^{s} \) is connected except for finitely many \( s \in A \) and \( 2^{s} \) is l.c. at \( C_s \) for all \( s \in A \). Then by Theorem 1.3 \( X_s \) is connected except for finitely many \( s \in A \), by Theorem 1.10 if \( s \in A \), \( K_s \) is a component of \( C_s \), and \( O_s \in T_s \) such that \( K_s \subseteq O_s \), then there exists an open connected set \( B_s \) such that \( K_s \subseteq B_s \subseteq O_s \), and by Theorem 2.6 \( \prod 2^{s} \) is l.c. at \( \{C_s\}_{s \in A^*} \).

(d) implies (e): Since \( \prod X_s \) is \( R \), \( \prod C_s \subseteq K(\prod X_s) \), and \( \prod 2^{s} \) is l.c. at \( \prod C_s \), then by Theorem 1.10 \( K(\prod X_s) \) is l.c. at \( \prod C_s \).

(e) implies (a): Let \( \beta \in A \), let \( K_s \) be a component of \( C_s \), and let \( O_s \in T_s \) such that \( K_s \subseteq O_s \). For each \( s \neq \beta \) let \( K_s \) be a component of \( C_s \). Then \( \prod K_s \) is a component of \( \prod C_s \) and by Theorem 1.10 \( \prod X_s \) is l.c. at \( \prod K_s \). Then by Theorem 2.4 \( X_s \) is connected except for finitely many \( s \in A \) and there exists an open connected set \( B_s \) such that \( K_s \subseteq B_s \subseteq O_s \).

Theorem 2.9. For each \( s \in A \) let \( (X_s, T_s) \) be nonempty locally compact \( R \), and let \( C_s \subseteq K(X_s) \) for all \( s \in A \), where \( X_s \) is compact except for finitely many \( s \in A \). Then the following are equivalent: (a) \( X_s \) is connected except for finitely many \( s \in A \) and if \( s \in A \), \( K_s \) is a component of \( C_s \), and \( O_s \in T_s \) such that \( K_s \subseteq O_s \), then the component of \( O_s \) containing \( K_s \) contains \( K_s \) in its interior, (b) \( \prod K(X_s) \) is c.i.k. at \( \{C_s\}_{s \in A^*} \), (c) \( \prod 2^{s} \) is c.i.k. at \( \{C_s\}_{s \in A^*} \), (d) \( K(\prod X_s) \) is c.i.k. at \( \prod C_s \), and (e) \( \prod 2^{s} \) is c.i.k. at \( \prod C_s \).

The theorem follows by an argument similar to that for Theorem 2.8 and is omitted.