The equicontinuous structure relation of a unicoherent point-transitive flow

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Abstract. We prove that if (X, T) is a point-transitive transformation group, where X is a compact unicoherent space and T is a connected abelian Lie group, then (X, T) has no nontrivial equicontinuous or distal homomorphic images. Hence unicoherent point-transitive flows are highly nonrecursive and possess dynamical properties similar to unicoherent minimal flows.

In connection to this result, we discuss the existence and properties of point-transitive continuous flows $(S^n, \boldsymbol{R})$, $n \geqslant 2$, on the n-sphere. Concrete examples of unicoherent point-transitive flows $(S^n, \boldsymbol{R}^k)$, $n \geqslant k \geqslant 2$, are given. We show how our theory applies to these examples. In addition, we study the question: For $n \geqslant 2$, what is the least positive integer k such that point-transitive actions $(S^n, \boldsymbol{R}^k)$ exist?

I. Introduction. An open question in topological transformation group theory is: Do point-transitive actions (actions such that some orbit is dense) of the additive group of real numbers $\boldsymbol{R}$ on higher-dimensional spheres S^n, $n \geqslant 2$, exist? The question of existence of such flows is closely related to a conjecture of Seifert, made in 1950, that each continuous flow $(S^3, \boldsymbol{R})$ has a closed orbit. Although Seifert's conjecture has been proved false [12], the question of existence of point-transitive continuous flows $(S^3, \boldsymbol{R})$ remains open.

In this paper we show that if (X, T) is a point-transitive transformation group, where X is a compact unicoherent space and T is a connected abelian Lie group, then (X, T) has no nontrivial equicontinuous or distal homomorphic images. Hence (X, T) is highly nonequicontinuous and highly nondistal. In particular, if point-transitive continuous flows $(S^n, \boldsymbol{R})$, $n \geqslant 2$, do exist, they are highly nonequicontinuous, highly nondistal, and could not be built up as products or extensions of simpler equicontinuous or distal flows.

DEFINITION 1.1. A *transformation group* (or *flow*) is a triple (X, T, π) such that:

1) X is a nonempty compact Hausdorff space,
2) T is a separated topological group,
3) $\pi\colon X \times T \to X$ is a continuous map satisfying
 i) $\pi(x, 1) = x$ $(x \in X,\ 1 = \text{identity element of } T)$;
 ii) $\pi(x, t_1 t_2) = \pi(\pi(x, t_1), t_2)$ $(x \in X;\ t_1, t_2 \in T)$.

X (T) is called the *phase space* (*phase group*) of (X, T, π).

We frequently suppress π and write (X, T) instead of (X, T, π); also if $x \in X$ and $t \in T$ we usually write the abbreviated notation xt for $\pi(x, t)$. In certain instances, the action of a transformation group is denoted by $*$; in this case we write $x*t$ for $*(x, t)$.

If $T = \boldsymbol{R}$, the additive group of real numbers with its usual topology, we call (X, T) a *continuous flow*. We sometimes write the shortened notation $X \cong Y$ if X is homeomorphic to Y, and $G \approx H$ if G and H are isomorphic topological groups.

DEFINITION 1.2. Let (X, T) be a transformation group.

1) Let $x \in X$. The *orbit* of x, denoted by xT, is the subset $\{xt \mid t \in T\}$ of X.

2) (X, T) is said to be *point-transitive* if some orbit of (X, T) is dense in X.

3) (X, T) is said to be *minimal* if each orbit of (X, T) is dense in X. Of course, if (X, T) is minimal, then (X, T) is point-transitive.

4) Let $A \subseteq X$ and $S \subseteq T$. A is said to be *S-invariant* if $AS = \{as \mid a \in A, s \in S\} \subseteq A$.

DEFINITION 1.3. Let (X, T) and (Y, T) be transformation groups, and let $\varphi\colon X \to Y$ be a continuous map.

1) φ is said to be a *homomorphism* of (X, T) to (Y, T) if $\varphi(xt) = \varphi(x)t$ $(x \in X, t \in T)$. If $\varphi\colon X \to Y$ is a surjective homomorphism, we write $\varphi\colon (X, T) \rightrightarrows (Y, T)$. If such a surjective homomorphism φ exists, we write $(X, T) \rightrightarrows (Y, T)$ and call (Y, T) a *homomorphic image* of (X, T).

2) If $\varphi\colon (X, T) \rightrightarrows (Y, T)$ is a homeomorphism, we write $\varphi\colon (X, T) \simeq (Y, T)$ and call φ an *isomorphism* of (X, T) to (Y, T). If such an isomorphism φ exists, we write $(X, T) \simeq (Y, T)$.

Remark 1.4. If $\varphi\colon (X, T) \rightrightarrows (Y, T)$ and (X, T) is point-transitive (minimal), then (Y, T) is point-transitive (minimal). If R is a closed T-invariant equivalence relation on (X, T) there is a canonical action of T on the quotient space X/R, and we denote this induced flow by $(X/R, T)$.

DEFINITION 1.5. A transformation group (X, T) is said to be *quasi-separable* if (X, T) is point-transitive and $(X, T) \simeq \varprojlim_{\alpha \in \Omega}(X_\alpha, T)$, the inverse limit transformation group of an inverse system $\big((X_\alpha, T);\ \varphi_\alpha^\beta, \beta \geqslant \alpha\big)_{\alpha \in \Omega}$ of point-transitive transformation groups having compact metric phase spaces [8, Theorem 2.2].

In particular, a point-transitive transformation group with compact metric phase space is quasi-separable.

PROPOSITION 1.6. *Let (X, T) be a point-transitive flow such that T is a connected abelian Lie group. Then (X, T) is quasi-separable.*

Proof. According to [14, Proposition 1.2], each point-transitive flow (X, T) with σ-compact phase group is quasi-separable. From [11, p. 415], each connected abelian Lie group is topologically isomorphic to a σ-compact group $\boldsymbol{R}^n \times \boldsymbol{T}^k$ for some $n \geqslant 0$ and $k \geqslant 0$. Therefore (X, T) is quasi-separable.

II. Main Theorem of unicoherent point-transitive flows. To lay further ground work for our results, we define several more specialized terms of topological dynamics and discuss the topological property, unicoherence.

DEFINITION 2.1. Let (X, T) be a transformation group. (X, T) is said to be (uniformly) *equicontinuous* if for each index $\alpha \in \boldsymbol{U}$, the unique uniform structure of X, there exists $\beta \in \boldsymbol{U}$ such that $\beta T \subseteq \alpha$. It follows from [2, Proposition 4.4] that (X, T) is equicontinuous if and only if (X, T) is almost periodic.

DEFINITION 2.2. The *proximal relation* P of (X, T) is defined to be $\bigcap \{\alpha T \mid \alpha \in \boldsymbol{U}\}$. Two points $x, y \in X$ are said to be *proximal* if for each $\alpha \in \boldsymbol{U}$, there exists $t \in T$ such that $(xt, yt) \in \alpha$. (X, T) is said to be *distal* if $P = \Delta$, the diagonal of $X \times X$, and $x \in X$ is a *distal point* if $xP = \{x\}$.

DEFINITION 2.3. The *equicontinuous structure relation* (*distal structure relation*) of (X, T) is defined to be the smallest closed T-invariant equivalence relation Σ (S) on X such that $(X/\Sigma, T)$ is equicontinuous ($(X/S, T)$ is distal). We call X/Σ (X/S) the *equicontinuous structure space* (*distal structure space*) of (X, T), and we note that (X, T) is equicontinuous (distal) if and only if $\Sigma = \Delta$ $(S = \Delta)$.

Remark 2.4. If (Y, T) is an equicontinuous (distal) homomorphic image of (X, T), then (Y, T) is also a homomorphic image of $(X/\Sigma, T)$ ($(X/S, T)$), respectively. If $(X, T) \rightrightarrows (Y, T)$ and (X, T) is equicontinuous (distal), then (Y, T) is equicontinuous (distal).

DEFINITION 2.5. A point $x \in X$ is said to be *almost periodic* under T if given $\alpha \in \boldsymbol{U}$, there exist a syndetic subset A of T such that $xA \subseteq x\alpha$. (X, T) is *pointwise almost periodic* if and only if each point of X is almost periodic.

Remark 2.6. From [4, Theorem 4.10], $x \in X$ is almost periodic if and only if $\overline{xT}$ is a minimal set. If (X, T) is distal, then (X, T) is pointwise almost periodic, and if $x \in X$ is a distal point, then x is almost periodic [1, Lemma 2].

DEFINITION 2.7. (X, T) is said to be *locally almost periodic* if given $x \in X$ and a neighborhood U of x, there exists a neighborhood V of x and a syndetic subset A of T such that $VA \subseteq U$. If (X, T) is locally almost periodic, then (X, T) is pointwise almost periodic.

DEFINITION 2.8. A *unicoherent space* is a compact, connected, locally path-connected Hausdorff space such that each continuous map $\varphi\colon X \to \boldsymbol{S}^1$ is homotopic to a constant map.

Remark 2.9. (See [10, p. 438], [13, p. 103]). Let X be a compact, connected, locally path-connected Hausdorff space. The following statements are equivalent.

1) X is unicoherent.

2) For each pair A, B of closed connected subsets of X such that $A \cup B = X$, the intersection $A \cap B$ is connected.

3) For each $n \geqslant 1$, each continuous map $\varphi\colon X \to T^n$ is homotopic to a constant map, where $T^n = (S^1)^n$ is the n-dimensional torus.

4) For each $n \geqslant 1$, each continuous map $\varphi\colon X \to T^n$ induces the zero homomorphism $\varphi_{\#}\colon \pi_1(X) \to \pi_1(T^n)$ between fundamental groups.

A flow (X, T) is called a *unicoherent flow* if X is unicoherent.

Several preparatory lemmas will be required before we can state the main theorem of this section.

LEMMA 2.10. *Let G be a compact connected abelian topological group such that $G \neq \{0\}$. Then there exists a closed subgroup H of G such that the quotient group G/H is topologically isomorphic to T^k for some positive integer k.*

Proof. By [5, Theorem 9.5], there exists a closed proper subgroup H of G such that G/H is topologically isomorphic to $T^k \times F$ for some nonnegative integer k and some finite abelian group F. Since G is connected, G/H and F are also connected, which implies $F = \{0\}$ and $k > 0$.

LEMMA 2.11 (Gottschalk–Hedlund–Ellis structure theorem for equicontinuous minimal flows [4, Theorem 4.48]). *Let (X, T) be a flow with abelian phase group T. The following two statements are equivalent.*

1) *(X, T) is equicontinuous and minimal.*

2) *There exists a group structure $(X, *)$ on X which makes X an abelian topological group, and there exists a continuous group homomorphism $\varphi\colon T \to X$ such that*

i) *$\varphi(T)$ is dense in X.*

ii) *(X, T) is isomorphic to the transformation group $(X, \varphi, *, T)$ whose action is defined by:*

$$(x, t) \to x*\varphi(t) \qquad (x \in X,\ t \in T).$$

The next lemma is essential to the proof of our main theorem.

LEMMA 2.12 ([7, Theorem 2.2]). *Let $\varphi\colon (X, T) \rightrightarrows (Y, T)$ be a homomorphism of a flow (X, T) onto a minimal flow (Y, T). Suppose that X is compact, connected, and locally arcwise connected, that Y is locally arcwise connected and semi-locally 1-connected ([13, p. 78]), and that T is a connected Lie group. Then $\varphi_{\#}(\pi_1(X))$ has finite index in $\pi_1(Y)$.*

The next lemma is an immediate consequence of the "Hahn–Mazurkiewicz theorem" [6, Theorem 3-30].

LEMMA 2.13. *Each compact locally path-connected Hausdorff space is locally arcwise connected. Therefore, each unicoherent space is locally arcwise connected.*

Our final preparatory lemma states the primary result of this section.

LEMMA 2.14. *Let (X, T) be a unicoherent point-transitive flow where T is a connected abelian Lie group. Then (X, T) has no nontrivial equicontinuous homomorphic images, or equivalently, $\Sigma = X \times X$.*

Proof. We show that X/Σ, the equicontinuous structure space of (X, T), is a one-point space.

Suppose on the contrary that X/Σ contains more than one point and consider the equicontinuous flow $(X/\Sigma, T)$. Since $(X, T) \rightrightarrows (X/\Sigma, T)$, Remark 1.4 implies that $(X/\Sigma, T)$ is point-transitive. Since $(X/\Sigma, T)$ is equicontinuous, $(X/\Sigma, T)$ is also pointwise almost periodic. Therefore $(X/\Sigma, T)$ is a minimal set by Remark 2.6.

By Lemma 2.11, there exists a group structure $*$ on $G = X/\Sigma$ which makes $G = X/\Sigma$ a compact abelian topological group, and there exists a continuous group homomorphism $\varphi\colon T \to G$ such that $(X/\Sigma, T)$ is isomorphic to the flow $(G, \varphi, *, T)$ whose action is defined by: $(x, t) \to x*\varphi(t)$ $(x \in G,\ t \in T)$. By Lemma 2.10, there exists a closed subgroup H of G such that G/H is topologically group isomorphic to T^k for some $k \geqslant 1$. There is a naturally induced transformation group $(G/H, T)$ whose action is defined by: $(xH, t) \to x*\varphi(t)H$ $(x \in G,\ t \in T)$, and the canonical projection $p\colon G \to G/H$ is a homomorphism of $(G, \varphi, *, T)$ onto $(G/H, T)$. Since G/H is homeomorphic to $T^k = (S^1)^k$, a standard result of homotopy theory implies that $\pi_1(G/H) \approx \pi_1((S^1)^k) \approx Z^k$ [14, Lemma 1.4].

Since $(X, T) \rightrightarrows (X/\Sigma, T)$ and $(X/\Sigma, T) \simeq (G, \varphi, *, T)$, there is a homomorphism $\psi\colon (X, T) \rightrightarrows (G, \varphi, *, T)$. Consequently, $p\psi\colon (X, T) \rightrightarrows (G/H, T)$. Now X is locally arcwise connected by Lemma 2.13, G/H is locally arcwise connected and semi-locally 1-connected and $(G/H, T)$ is minimal. Therefore Lemma 2.12 implies $(p\psi)_{\#}(\pi_1(X))$ has finite index in $\pi_1(G/H) \approx Z^k$. But X is unicoherent and $G/H \cong T^k$, so $(p\psi)_{\#}\colon \pi_1(X) \to \pi_1(G/H)$ equals zero by Remark 2.9.

We have reached a contradiction. Therefore, X/Σ is a one-point space.

We are now ready to state our "Main Theorem" which describes the structure relations of a unicoherent point-transitive flow.

THEOREM 2.15. *Let (X, T) be a nontrivial unicoherent point-transitive flow where T is a connected abelian Lie group. The following conclusions hold:*

(1) *$\Sigma = X \times X$.*

(2) *(X, T) has no nontrivial equicontinuous homomorphic images; in particular, (X, T) is not equicontinuous.*

(3) *$S = X \times X$.*

(4) *(X, T) has no nontrivial distal homomorphic images; in particular, (X, T) is not distal.*

(5) *(X, T) is not locally almost periodic.*

(6) *No distal point of (X, T) can have dense orbit in X.*

(7) *The set $\{\varphi \mid \varphi\colon (X, T) \simeq (X, T)\}$ of automorphisms of (X, T) is not universally transitive on X.*

Proof. Conclusions (1) and (2) follow directly from Lemma 2.14.

To prove (3), we show that the distal structure space X/S is a one-point space. Suppose on the contrary that X/S contains more than one point, and consider the induced distal flow $(X/S, T)$. Since $(X, T) \precsim (X/S, T)$, Remark 1.4 implies that $(X/S, T)$ is point-transitive. Therefore Proposition 1.6 implies $(X/S, T) \simeq \varprojlim_{\beta\in\Omega}(Y_\beta, T)$, the inverse limit of point-transitive flows $\{(Y_\beta, T)\}_{\beta\in\Omega}$ having compact metric phase spaces.

Since $(X/S, T)$ is nontrivial, there exists $\alpha\in\Omega$ such that (Y_α, T) is nontrivial. Now Definition 2.3 and Remark 2.6 together imply that $(X/S, T)$ is distal and minimal. Hence $\varprojlim_{\beta\in\Omega}(Y_\beta, T)$ is also distal and minimal. Now p_α: $\varprojlim_{\beta\in\Omega}(Y_\beta, T) \precsim (Y_\alpha, T)$ is a homomorphism, so (Y_α, T) is distal and minimal by Remarks 2.4 and 1.4, respectively. Since (Y_α, T) is compact, metrizable, and nontrivial, a result of H. Furstenberg implies that (Y_α, T) has a nontrivial equicontinuous homomorphic image (Z_α, T) [3, p. 499]. Therefore, we have $(X, T) \precsim (X/S, T) \simeq \varprojlim_{\beta\in\Omega}(Y_\beta, T) \precsim (Y_\alpha, T) \precsim (Z_\alpha, T)$, which implies (X, T) has a nontrivial equicontinuous homomorphic image. This contradicts conclusion (2).

Conclusion (4) follows immediately from (3).

To prove (5), suppose that (X, T) is locally almost periodic. Then Definition 2.7 implies that (X, T) is pointwise almost periodic. Let $x\in X$ have dense orbit in X. Then x is an almost periodic point, so Remark 2.6 implies that (X, T) is minimal. But this contradicts Theorem 1.10 of [14], which states that a unicoherent minimal abelian flow cannot be locally almost periodic.

To prove (6), suppose $x\in X$ is a distal point. Then Remark 2.6 implies x is almost periodic. If x has dense orbit, then (X, T) is a minimal set by Remark 2.6. But this again contradicts Theorem 1.10 of [14].

Conclusion (7) is a consequence of conclusion (4) and [4, Theorem 10.06].

An additional conclusion to Theorem 2.15 is the following: Each continuous complex-valued function f: $X\to C$ such that $f(xt) = \chi(t)f(x)$ for some character χ of T $(x\in X,\ t\in T)$ is a constant. This conclusion is implied by Lemma 2.14 and [9, Lemma 3.3] when X is metrizable, and the proof used for the metric case can be readily extended to a proof of the quasi-separable case. This result suggests an approach one might use to prove that a point-transitive continuous flow (S^3, R) could not exist.

Remark 2.16. Let (Y, T) be a nontrivial homomorphic image of a unicoherent point-transitive flow (X, T), where T is a connected abelian Lie group. Then (Y, T) satisfies each conclusion of Theorem 2.15.

Proof. Conclusions (1) and (2) follow immediately from Lemma 2.14. The remaining conclusions are proved using [8] and Theorem 1.10 of [14].

III. Examples and discussion. In this section we construct some examples of unicoherent point-transitive flows (S^n, R^k), $n\geqslant k\geqslant 2$, and use Theorem 2.15 to discuss their dynamical properties.

We conjecture that the proximal relation of a unicoherent point-transitive flow of Theorem 2.15 is necessarily dense. Certainly if such a flow in addition were minimal, Theorem 1.10 of [14], which proves several key properties of a unicoherent minimal flow, states that P is dense and $\Sigma = S = X\times X$. We think however that P might still be dense when (X,T) is point-transitive because the conclusion $\Sigma = S = X\times X$ still holds in that case, just as it does when (X, T) is minimal.

We begin our discussion with a basic example of a unicoherent point-transitive flow.

Example 3.1. Let $T = R^n$, $n\geqslant 2$. Then $(\hat{T}, T)$ the induced action of T on its one-point compactification $\hat{T} = T\cup\{\infty\}$, has the properties:

(1) $(\hat{T}, T)$ is a unicoherent point-transitive flow with two orbits, one of which, $\{\infty\}$, is a fixed point.

(2) The proximal relation P of $(\hat{T}, T)$ equals $\hat{T}\times\hat{T}$.

Proof. The one-point compactification $\hat{T} = R^n\cup\{\infty\}$ of R^n is homeomorphic to S^n, which is unicoherent. Now T and $\{\infty\}$ are the only orbits of T. Hence $(\hat{T}, T)$ is point-transitive and has two orbits, one of which, $\{\infty\}$, is a fixed point. Therefore (1) is proved.

To prove $P = \hat{T}\times\hat{T}$, let $x, y\in R^n\cup\{\infty\}$ and $t_m = (m, m, \dots, m)\in R^n$, $m\in Z$, $m\geqslant 0$. Then $xt_m\to\infty$ and $yt_m\to\infty$ as m becomes infinitely large. Hence $(x, y)\in P$, proving (2).

We note that the flow just constructed has closed proximal relation and is proximally equicontinuous. An open question related to constructing unicoherent point-transitive flows is: Given $n\geqslant 2$, what is the least positive integer k such that there exists a point-transitive action of R^k on S^n? In particular, for $n\geqslant 2$, does there exist a point-transitive flow (S^n, R)? The next examples show that, in constructing such flows, some reduction in dimension from $k = n$ is indeed possible. A preliminary result is required.

Lemma 3.2. *A minimal continuous flow* (T^k, R) *on the k-dimensional torus* $T^k = \{(z_1, z_2, \dots, z_k) \mid z_i\in C,\ |z_i| = 1,\ 1\leqslant i\leqslant k\}$ *is defined by*: $((z_1, z_2, \dots, z_k), t) \to (z_1 e^{i\alpha_1 t}, z_2 e^{i\alpha_2 t}, \dots, z_k e^{i\alpha_k t})$ $(t\in R)$, *where* $\{\pi, \alpha_1, \alpha_2, \dots, \alpha_n\}\subseteq R$ *is linearly independent over* Q.

Proof. That each orbit of (T^k, R) is dense is just a restatement of the Kronecker approximation theorem (see [5, p. 435]).

We use Lemma 3.2 to construct a unicoherent point-transitive flow having a dense but nonclosed proximal relation.

EXAMPLE 3.3. For $n \geq 3$, let $S^{2n-2} = C^{n-1} \cup \{\infty\}$, the one-point compactification of $C^{n-1} \cong R^{2n-2}$, and let $\{\pi, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \subseteq R$ be linearly independent over Q. Then a point-transitive flow (S^{2n-2}, R^n) is defined by: $((z_1, z_2, \ldots, z_{n-1}), (t_1, t_2, \ldots, t_n)) \to (2^{t_1} z_1 e^{i\alpha_1 t_n}, 2^{t_2} z_2 e^{i\alpha_2 t_n}, \ldots, 2^{t_{n-1}} z_{n-1} e^{i\alpha_{n-1} t_n})$ $(z_i \in C,\ 1 \leq i \leq n-1;\ t_j \in R,\ 1 \leq j \leq n)$, $(\infty, \vec{t}) \to \infty\ (\vec{t} \in R^n)$. Then:

(1) This flow has infinitely many orbits, two of which, $\{0\}$ and $\{\infty\}$, are fixed points.

(2) $P = S^{2n-2} \times S^{2n-2} - \{(0, \infty), (\infty, 0)\}$. Therefore P is dense in $S^{2n-2} \times S^{2n-2}$ but is not closed.

Proof. (1) The map of $C^{n-1} \times R^n$ to C^{n-1} we have defined satisfies all properties required of an action, and the extended action of R^n to $C^{n-1} \cup \{\infty\} = S^{2n-2}$ is easily seen to be continuous at ∞. A standard dimension-theoretic argument shows that (S^{2n-2}, R^n) has infinitely many orbits, and Lemma 3.2 implies that each point $(z_1, z_2, \ldots, z_{n-1}) \in C^{n-1}$, $z_i \neq 0$ $(1 \leq i \leq n-1)$, has dense orbit.

(2) To show $P = S^{2n-2} \times S^{2n-2} - \{(0, \infty), (\infty, 0)\}$, let $(x, y) \in C^{n-1} \times C^{n-1}$ and $t_n = (-n, -n, \ldots, -n)$, $n \in Z$, $n \geq 0$. Then $xt_n \to 0$ and $yt_n \to 0$ as n becomes infinitely large. Hence $(x, y) \in P$.

Let $(x, y) \in S^{2n-2} \times S^{2n-2} - (C^{n-1} \times C^{n-1} \cup \{(0, \infty), (\infty, 0)\})$, and $t_n = (n, n, \ldots, n) \in R^n$, $n \in Z$, $n \geq 0$. Then $xt_n \to \infty$ and $yt_n \to \infty$ as n becomes infinitely large. Therefore $(x, y) \in P$. Now 0 and ∞ are not proximal in (S^{2n-2}, R^n) because they are distinct fixed points. Hence $P = S^{2n-2} \times S^{2n-2} - \{(0, \infty), (\infty, 0)\}$. Consequently, P is dense in $S^{2n-2} \times S^{2n-2}$ but is not closed.

Our next result shows how the techniques of Examples 3.1 and 3.3 may be combined to produce flows having dense closed proximal relations.

EXAMPLE 3.4. For $n \geq 3$, let $S^{2n-3} = (C^{n-2} \times R) \cup \{\infty\}$, the one-point compactification of $C^{n-2} \times R \cong R^{2n-3}$, and let $\{\pi, \alpha_1, \alpha_2, \ldots, \alpha_{n-2}\} \subseteq R$ be linearly independent over Q. Then a point-transitive flow (S^{2n-3}, R^n) is defined by: $((z_1, z_2, \ldots, z_{n-2}, t), (t_1, t_2, \ldots, t_n)) \to (2^{t_1} z_1 e^{i\alpha_1 t_{n-1}}, 2^{t_2} z_2 e^{i\alpha_2 t_{n-1}}, \ldots, 2^{t_{n-2}} z_{n-2} e^{i\alpha_{n-2} t_{n-1}}, t + t_n)$ $(z_i \in C,\ 1 \leq i \leq n-2,\ t \in R;\ t_j \in R,\ 1 \leq j \leq n)$, $(\infty, \vec{t}) \to \infty$ $(\vec{t} \in R^n)$. Then:

(1) (S^{2n-3}, R^n) is point-transitive and has infinitely many orbits, one of which, $\{\infty\}$, is a fixed point.

(2) $P = S^{2n-3} \times S^{2n-3}$. Therefore P is closed.

Proof. A straightforward exercise shows that (S^{2n-3}, R^n) satisfies all properties required of a transformation group, and Lemma 3.2 implies that each point $(z_1, z_2, \ldots, z_{n-2}, t) \in C^{n-2} \times R$, $z_i \neq 0$ $(1 \leq i \leq n-2)$, has dense orbit. By duplicating the proof of Example 3.1, it follows that $P = S^{2n-3} \times S^{2n-3}$. Therefore P is closed.

Although similar in construction, the two previous examples differ in the following respect: In Example 3.4, the proximal relation is closed; in Example 3.3, it is not closed. Hence the flow of Example 3.4 is proximally equicontinuous, but that of Example 3.3 is not proximally equicontinuous. Our results show that for $n \geq 3$, a Euclidean group of dimension less than n can act point-transitively on S^n. For instance, Example 3.3 shows that R^8 can act point-transitively on S^{14}. Additional information concerning this question would be desirable.

Examples 3.1, 3.3, and 3.4 support the conjecture, raised earlier, that the proximal relation of a unicoherent point-transitive flow is necessarily dense. Our next result lends added support to this conjecture.

PROPOSITION 3.5. *Let (X, T) be a nontrivial unicoherent point-transitive flow where T is a connected abelian Lie group. If either (X, T) is weakly mixing or (X, T) has a fixed point, then P is dense in $X \times X$.*

Proof. If (X, T) is weakly mixing, Proposition 1.6 implies (X, T) is quasi-separable. Now [14, Lemma 2.20] states that the product flow $(X \times X, T)$, defined by $((x, y), t) \to (xt, yt)$ $(x, y \in X,\ t \in T)$, of a weakly mixing quasi-separable flow is also quasi-separable. Hence $(X \times X, T)$ is point-transitive. Let $(x, y) \in X \times X$ be a point having dense orbit. Then $\overline{(x, y) T} \cap \Delta \neq \emptyset$, implying that $(x, y) T \subseteq P$. Therefore P is dense in $X \times X$.

If (X, T) has a fixed point, it follows that $yT \times yT \subseteq P$, where y is some point of X having dense orbit. Hence P is dense in $X \times X$.

Thus, if the hypothesis of Theorem 2.15 is strengthened with the added assumption that (X, T) is weakly mixing or has a fixed point, the conclusion that P is dense readily follows. We note that the proof of Proposition 3.5 primarily depends upon this added assumption and not upon the unicoherence of X. However, our results do suggest that if (X, T) satisfies the hypothesis of Theorem 2.15, then (X, T) might have a fixed point or be weakly mixing. This conclusion certainly holds if X is a compact polyhedron of nonzero Euler characteristic and $T = R$ [13, p. 197] or if (X, T) is minimal [14, Theorem 1.8]. If this conjecture were true, so would be our original conjecture that the proximal relation of a unicoherent point-transitive flow is dense.

IV. Summary. The results of Theorem 2.15 give useful information about the properties a unicoherent point-transitive flow must possess. In particular, if a point-transitive flow (S^n, R), $n \geq 2$, does exist, it must be highly non-equicontinuous and highly nondistal. Examples 3.3 and 3.4 show that some reduction from $k = n$ in the dimension of the group R^k acting point-transitively on S^n is indeed possible. It is also evident that, in addition to the property of unicoherence, the existence of fixed points also influences the dynamical behavior of a transformation group.

To summarize our results as they apply to questions posed in this paper, we state a final corollary to Theorem 2.15.

COROLLARY 3.6. *Let (X, T) be a point-transitive flow where X is a sphere, real or complex projective space, or lens space (of dimension greater than one), and T is a connected abelian Lie group. Then (X, T) satisfies all conclusions of Theorem 2.15 and Proposition 3.5.*

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Connectivity properties in hyperspaces and product spaces

by

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Abstract. In this paper connectedness, local connectedness, and point-wise local connectedness and connectedness im kleinen in hyperspaces of product spaces and product spaces of hyperspaces are investigated and the relationships between these connectivity properties in hyperspaces of product spaces and product spaces of hyperspaces are determined. In order to include as many spaces as possible, the results in this paper are stated and proved for R_0 and R_1 topological spaces.

1. Introduction. One of the earliest results about connectivity properties in hyperspaces, due to Wojdyslawski [7] in 1939, is that for a metric continuum (X, T), $(2^X, E(X))$ is locally connected (l.c.) iff (X, T) is l.c. Since 1939 mathematicians have continued the investigation of connectivity properties in hyperspaces. In this paper connectivity properties in hyperspaces of product spaces and product spaces of hyperspaces are investigated. In order to include as many spaces as possible, the results in this paper are stated and proved for weak topological spaces. Listed below are definitions and theorems that will be utilized in this paper.

DEFINITION 1.1. A space (X, T) is R_0 iff for each $0 \in T$ and $x \in 0$, $\overline{\{x\}} \subset 0$ [1].

DEFINITION 1.2. A space (X, T) is R_1 iff for each pair $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [1].

DEFINITION 1.3. Let (X, T) be a space, let $A \subset X$, and define 2^X, $C(X)$, $K(X)$, $S(A)$, and $I(A)$ as follows: $2^X = \{F \subset X \mid F \text{ is nonempty and closed}\}$, $C(X) = \{F \in 2^X \mid F \text{ is connected}\}$, $K(X) = \{F \in 2^X \mid F \text{ is compact}\}$, $S(A) = \{F \in 2^X \mid F \subset A\}$, and $I(A) = \{F \in 2^X \mid F \cap A \neq \emptyset\}$. Then the Vietoris topology on 2^X, denoted by $E(X)$, is the smallest topology on 2^X which satisfies the conditions that if $G \in T$, then $S(G) \in E(X)$ and $I(G) \in E(X)$ [6].

THEOREM 1.1. *The product of an arbitrary family of nonempty topological spaces is R_0 iff each factor space is R_0* [4].

THEOREM 1.2. *If (X,T) is R_1, then (X, T) is R_0* [5].

THEOREM 1.3. *If (X, T) is R_0, then the following are equivalent*: (a) *X is connected*, (b) *2^X is connected, and* (c) *$K(X)$ is connected* [2].