Universally measurable spaces: an invariance theorem and diverse characterizations

by

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Abstract. This article proposes two new developments: firstly, a proof of the invariance of the universal measurability (u.m.) property over all metrizations generating the same Borel structure; thus, while the property has hitherto only been stated for metric spaces, this invariance (Theorem 1) allows for its application to a large class of measurable spaces (those we shall term "separable"); secondly, we exhibit several characterizations of these u.m. separable spaces: Theorem 2 generalizes a result of Sazonov on perfect probabilities; Theorem 3 regards the behaviour of measures on the product of a u.m. and a standard space; Theorem 4 involves the existence of certain types of conditional probabilities; finally, Theorems 5 and 6 address the problem of the existence of laws with given marginals.

By *separable space* we mean a measurable space $(X, \mathscr{B})$ with $\mathscr{B}$ countably generated and containing singletons. We shall often suppress the notation of a σ-algebra, calling the space X alone and indicating its measurable structure with $\mathscr{B} = \mathscr{B}(X)$. If A is a subset of a measurable space $(X, \mathscr{B})$, we shall always consider A as a measurable space with $\mathscr{B}(A) = \{A \cap B: B \in \mathscr{B}\}$; under this convention, a subset of a separable space is again separable. Separability is also preserved under the taking of countable products.

If X is a separable metric space with Borel σ-algebra $\mathscr{B}$, then $(X, \mathscr{B})$ is a separable space. Furthermore, there is a well-known technique due to Marczewski ([13] and [14]) by which one may introduce metrics on separable spaces compatible with the measurable structure:

LEMMA 1. *If $(X, \mathscr{B})$ is a separable space and $\mathscr{C}$ is a countable subset of $\mathscr{B}$, there is a metric d on X such that:*

1) (X, d) is a totally-disconnected metric space with compact completion ((X, d) is totally bounded and therefore separable),

2) $\mathscr{B}$ is the Borel σ-algebra for (X, d); we say "d is a metric for $(X, \mathscr{B})$" whenever this happens,

3 the elements of $\mathscr{C}$ are "clopen" (both closed and open in (X, d)), and

4) if $\mathscr{C}$ generates $\mathscr{B}$, then $\mathscr{C}$ is a base for the topology of (X, d).

A separable space $(X, \mathscr{B})$ is *standard* if there is a metric d for X such that

(X, d) is a complete separable metric (i.e. Polish) space. The isomorphism types of standard spaces have been completely classified by cardinality: every standard space X is isomorphic either with a finite set, the integers or to the Cantor discontinuum according as the cardinality of X is finite, countably infinite, or uncountable. See Cohn [6], p. 275.

LEMMA 2. *Let X be a separable space; then:*

1) *if $S \subset X$ with $(S, \mathscr{B}(S))$ standard, then $S \in \mathscr{B}(X)$,*

2) *if X is standard, the standard subsets of X are precisely the elements of $\mathscr{B}(X)$, and*

3) *the collection of standard subsets of X is closed under countable unions and intersections.*

Proof. See Cohn [6], pp. 275–276.

Now for some basic terminology from elementary measure theory (details are to be found in many standard texts e.g. Halmos [8]). If $(X, \mathscr{B}, P)$ is a finite measure space, denote inner and outer measure for P by P_* and P^*, respectively. If $A \subset X$, we use the notations A_* and A^* to indicate (not necessarily unique) elements of $\mathscr{B}$ such that $A_* \subset A \subset A^*$ and $P(A_*) = P_*(A)$, $P(A^*) = P^*(A)$; A^* is called a *measurable cover* for A. A subset A of X is *P-completion measurable* if $P_*(A) = P^*(A)$; A is *universally measurable (u.m.) in* $(X, \mathscr{B})$ if it is P-completion measurable for every probability measure P on $(X, \mathscr{B})$.

Suppose that $f\colon X \to Y$ is a measurable function between measurable spaces $(X, \mathscr{B})$ and $(Y, \mathscr{A})$ and that P is a finite measure on $(X, \mathscr{B})$. We define the *image measure* $f(P)$ on $(Y, \mathscr{A})$ by the rule $f(P)(A) = P(f^{-1}(A))$. Given any $A \subset X$ and a finite measure P on $(A, \mathscr{B}(A))$, define $\bar{P}$, the *measure induced by P on X* by the rule $\bar{P}(B) = P(B \cap A)$. If P is defined on a product $X \times Y$, the measure P_1 on X defined by $P_1(A) = P(A \times Y)$, $A \in \mathscr{B}(X)$, is the *marginal of P on X*.

We shall use the terms "probability measure" and "law" interchangeably. If S is a Hausdorff topological space, a law P on (the Borel subsets of) S is *tight* if for every $\varepsilon > 0$, there is a compact K with $P(K) > 1-\varepsilon$. A separable metric space (S, d) is *universally measurable (u.m.)* if it is universally measurable in its completion $\bar{S}$. If the Borel structure on S is standard, Lemma 2 implies that S is Borel in $\bar{S}$ and so is u.m.

LEMMA 3. *A separable metric space (S, d) is u.m. if and only if every law P on S is tight.*

Proof. Straightforward; for details see e.g. Varadarajan [20], p. 224.

Thus the u.m. property of (S, d) depends on the metric d only through its topology; in fact, rather more is true.

THEOREM 1. *Let X be a set and let d_1 and d_2 be separable metrics on X generating the same Borel σ-algebra. Let Y_1 and Y_2 be completions of X for these respective metrics; then X is u.m. in Y_1 if and only if it is u.m. in Y_2.*

Proof. Assume that (X, d_1) is u.m. and consider the identity map from X to itself. By the Lavrentiev–Kuratowski Theorem (Kuratowski [10], p. 436), this extends to a Borel isomorphism $g\colon E_2 \to E_1$ between Borel subsets E_1 of Y_1 and E_2 of Y_2, each containing X. Let $h\colon E_2 \to Y_1$ be g with extended range ($h(x) = g(x)$ for $x \in E_2$). Let P_2 be any law on Y_2: we claim that X is P_2-completion measurable in Y_2. If $P_2^*(X) = 0$, this is evident; assuming $P_2^*(X) > 0$, let Q_2 be the restriction of P_2 to Borel subsets of E_2 ($Q_2(B) = P_2(B)$ for $B \in \mathscr{B}(Y_2)$, $B \subset E_2$) and let $h(Q_2)$ be the image measure on Y_1. Since $P_2^*(X) > 0$, $P_2(E_2) > 0$ and $h(Q_2)/P_2(E_2)$ is a law on Y_1. Because (X, d_1) is u.m., X is completion measurable for this law in Y_1; thus there are Borel subsets A and B of Y_1 with $A \subset X \subset B$ and $h(Q_2)(A) = h(Q_2)(B)$. Then $h^{-1}(A)$ and $h^{-1}(B)$ are Borel subsets of Y_2 with $h^{-1}(A) \subset X \subset h^{-1}(B)$; finally, $P_2(h^{-1}(A)) = Q_2(h^{-1}(A)) = h(Q_2)(A) = h(Q_2)(B) = Q_2(h^{-1}(B)) = P_2(h^{-1}(B) \cap E_2) = P_2(h^{-1}(B))$. ∎

Thus the u.m. property is invariant under choice of metric, depends only on Borel structure, and is therefore properly an attribute of separable measurable (not metric) spaces. We shall call a separable space X u.m. if there is a metric d for X for which (X, d) is u.m.; this property is preserved under the taking of countable products.

LEMMA 4. *A separable space X is u.m. if and only if for every law P on X, there is a set $S \in \mathscr{B}(X)$ with $(S, \mathscr{B}(S))$ standard and $P(S) = 1$.*

Proof. Assume that X is u.m., let d be a metric for X and let $\bar{X}$ be the completion of X for d. Given a law P on X, let $\bar{P}$ be the law induced by P on $\bar{X}$. Since X is u.m. in $\bar{X}$ and $\bar{P}^*(S) = 1$, there is a set $S \in \mathscr{B}(\bar{X})$ with $S \subset X$ and $\bar{P}(S) = P(S) = 1$. The relative structure on S is standard by Lemma 2.

Conversely, if Q is a law on $\bar{X}$, then either $Q^*(X) = 0$, and X is Q-completion measurable or $P = Q^*/Q^*(X)$ is a law on X. If $S \in \mathscr{B}(X)$ with $(S, \mathscr{B}(S))$ standard and $P(S) = 1$, then by Lemma 2, $S \in \mathscr{B}(\bar{X})$ and $Q(S) = Q^*(S) = Q^*(X)$, and again X is Q-completion measurable. ∎

One characterization of u.m. subsets of the reals has been known for a while: our invariance result (Theorem 1) allows us to generalize it to measurable spaces: say that a separable space X is *P-perfect* if P is a law on X such that for all real-valued Borel-measurable functions $f\colon X \to \boldsymbol{R}$, the image set $f(X)$ is $f(P)$-completion measurable; equivalently (Sazonov [18], p. 222), whenever $E \subset \boldsymbol{R}$ with $f^{-1}(E) \in \mathscr{B}(X)$, then E is $f(P)$-measurable.

THEOREM 2. *A separable space X is P-perfect for all laws P on X if and only if X is u.m.*

Proof. Each separable X is Borel isomorphic with a subset X' of the real line $\boldsymbol{R}$. By a result of Sazonov [18], p. 245, X' is Q-perfect for all laws Q on X' if and only if X' is u.m. in $\boldsymbol{R}$; the former occurs if and only if X is P-perfect for all laws P on X, while the latter holds (*via* Theorem 1) if and only if X is a u.m. space. ∎

Before embarking on other characterizations of u.m. spaces, we review a basic property of analytic and co-analytic spaces. A subset A of a separable space X is *analytic* if it is the measurable image of a standard space; a separable space X is *co-analytic* if it is isomorphic with the complement of an analytic set in some standard space.

LEMMA 5. *Let A be a subset of a separable space X. If $(A, \mathscr{B}(A))$ is either analytic or co-analytic, it is a u.m. space and is universally measurable in X.*

Proof. See Cohn [6], p. 281 (8.4.3).

The following reveals a Fubini-type theorem for u.m. spaces and will prove useful in other characterizations.

THEOREM 3. *Let X be a fixed uncountable standard space; a separable space Y is u.m. if and only if every law P on $X\times Y$ has the following property: if P_1 is the marginal of P on X, and $A\subset X$ is such that $P_1^*(A)=1$, then $P^*(A\times Y)=1$.*

Proof. Suppose that Y is u.m.; taking P_2 as the marginal of P on Y, we choose a standard subset $S\subset Y$ with $P_2(S)=P(X\times S)=1$. Now given $U\supset A\times Y$, $U\in\mathscr{B}(X\times Y)$, the set $((X\times Y)\setminus U)\cap(X\times S)\in\mathscr{B}(X\times S)$ and so is standard; define $B=X\setminus p(((X\times Y)\setminus U)\cap(X\times S))$, where $p\colon X\times Y\to X$ is projection onto the first co-ordinate; then $A\subset B$. Now $X\setminus B$ is the measurable image of a standard space, and so B is co-analytic and, by Lemma 5, P_1-completion measurable in X. Choose $C\in\mathscr{B}(X)$ with $C\subset B$ and $P_1(C)=P_1^*(B)\geqslant P_1^*(A)=1$; then since $C\times S\subset U$, $P(U)\geqslant P(C\times S)=P_1(C)=1$, proving $P^*(A\times Y)=1$ as desired.

Suppose that Y is not u.m. and let d be a metric for Y with completion $\bar{Y}$. Now X and $\bar{Y}$ are Borel-isomorphic (since Y is not u.m., it must be uncountable), and we identify X with $\bar{Y}$. Put $D=\{(y, y)\colon y\in Y\}\subset\bar{Y}\times Y$, the graph of the identity map from Y to $\bar{Y}$; hence $D\in\mathscr{B}(\bar{Y}\times Y)$. Also, $f\colon Y\to\bar{Y}\times Y$ defined by $f(y)=(y, y)$ is a Borel isomorphism of Y onto D. Since Y is not u.m. in $\bar{Y}$, there is a law Q on $\bar{Y}$ with $Q_*(Y)<Q^*(Y)$; define a new law Q_0 on $\bar{Y}$ by $Q_0(B)=Q(B\cap(Y^*\setminus Y_*))/Q(Y^*\setminus Y_*)$; then $Q_0^*(Y)=1$ and $(Q_0)_*(Y)=0$. Since Q_0^* is a law on Y, we may put $P=f(Q_0^*)$ and obtain a law on $\bar{Y}\times Y$ whose marginal P_1 on $\bar{Y}$ is just Q_0; taking $A=\bar{Y}\setminus Y$ gives $P_1^*(A)=1$, but $P^*(A\times Y)=0$, since $(A\times Y)\cap D=\emptyset$. ■

We now examine the connexion between u.m. spaces and the existence of certain conditional probabilities.

BLACKWELL'S THEOREM. *Let P be a law defined on a u.m. space X and let $\mathscr{A}$ be countably generated sub-σ-algebra of $\mathscr{B}(X)$. Then there is a real-valued function $P(x, B)$ defined for $x\in X$ and $B\in\mathscr{B}(X)$ such that*

1) *for fixed $B\in\mathscr{B}(X)$, $P(x, B)$ is an $\mathscr{A}$-measurable function of x,*

2) *for fixed $x\in X$, $P(x,\cdot)$ is a law on $\mathscr{B}$,*

3) *for every $A\in\mathscr{A}$, $B\in\mathscr{B}(X)$, $\int_A P(x, B)\,dP(x)=P(A\cap B)$, and*

4) *there is a set $N\in\mathscr{A}$ with $P(N)=0$ such that $P(x, A)=1$ for $x\in A\setminus N$ and $A\in\mathscr{A}$.*

Proof. One is to be found in Blackwell [3], Theorem 5; although stated there for analytic sets, the only property of these spaces made use of is that under any metrization, every law P on X is tight, something we know to be true for u.m. spaces (Lemma 3 and Theorem 1). ■

The function $P(x, B)$ satisfying conditions $1-4$ of the preceding theorem we term a *proper conditional probability of P given $\mathscr{A}$*; it is a strengthening of the notion of *regular* conditional probability in which only conditions $1-3$ are assumed; information about regularity is to be found in several monographs: Breiman [5], Bauer [1], for example; condition 4 is treated in Blackwell and Ryll-Nardzewski [4] and Musiał [16]. If $f\colon X\to Y$ is a measurable function into another separable space Y, taking $\mathscr{A}=f^{-1}(\mathscr{B}(Y))=\{f^{-1}(B)\colon B\in\mathscr{B}(Y)\}$ gives a proper conditional probability of P *given* f. We are now ready for another characterization of u.m. spaces:

THEOREM 4. *Let X be an uncountable standard space. A separable space Y is u.m. if and only if every law on $X\times Y$ has a proper conditional probability given $p\colon X\times Y\to X$, the projection map $p(x, y)=x$.*

Proof. If Y is u.m., then so is $X\times Y$ and Blackwell's theorem applies. Conversely, if Y has the property described in the theorem, we shall see that Theorem 3 implies that Y is u.m. So let P be any law on $X\times Y$ with proper conditional probability $P((x, y), B)$ given p. For each fixed $B\in\mathscr{B}(X\times Y)$, $P((x, y), B)$ is $p^{-1}(\mathscr{B}(X))$-measurable, is constant on each set $\{x\}\times Y$, $x\in X$, and so is a function of x alone. We write $P(x, B)=P((x, y), B)$. If P_1 is the marginal of P on X, and $A\subset X$ has $P_1^*(A)=1$, we wish to prove $P^*(A\times Y)=1$. If $A\times Y\subset U\in\mathscr{B}(X\times Y)$, let $B=\{x\colon P(x, U)=1\}$; then $B\in\mathscr{B}(X)$. There is, according to part 4 of Blackwell's Theorem, a set $N\in\mathscr{B}(X)$ with $P(N\times Y)=P_1(N)=0$ and $P(x, C\times Y)=1$ for $C\in\mathscr{B}(X)$ and $x\in C\setminus N$; then $A\cap(X\setminus N)\subset B$. Finally,

$$P(U)=P\big(((X\setminus N)\times Y)\cap U\big)=\int_{X\setminus N}P(x, U)\,dP_1(x)$$
$$\geqslant P_1\big(B\cap(X\setminus N)\big)\geqslant P_1^*\big(A\cap(X\setminus N)\big)=1. \quad ■$$

Finally, we come to the "marginal problem"; before beginning, information about universally null spaces is required. A law P on a separable space X is *continuous* if $P(\{x\})=0$ for each $x\in X$. A subset N of a separable space X is *universally null in X* if for every continuous law P on X, $P^*(N)=0$. A separable space X is *universally null* if there are no continuous laws on $(X, \mathscr{B}(X))$. Clearly, universally null spaces are u.m. It is also easy to prove that a subset A of X is universally null in X if and only if $(A, \mathscr{B}(A))$ is a universally null space. The existence of such spaces was first established under the assumption of the continuum hypothesis by Lusin [12]. This assumption was removed by Marczewski and Sierpiński [15]. Compare also Darst [7] and Kuratowski [10], p. 502, where the set Z in Theorem 7 is universally null.

LEMMA 6. *If Q is a continuous law on a separable space X, then there is a subset $H \subset X$ with $Q_*(H) = 0$ and $Q^*(H) = 1$.*

Proof. By Lemma 1, we may assume that X is a subset of the unit interval $I = [0, 1]$ under the relative Borel structure; then the law $\bar{Q}$ induced by Q on I is again continuous. There is a Borel isomorphism h of I onto I such that $h(\bar{Q}) = \lambda$, where λ is Lebesgue measure on I; this follows, for example, from Halmos and von Neumann [9], Theorem 2 or Royden [17], p. 337. Hence $\bar{Q}^*(h^{-1}(Y)) = \lambda^*(Y)$ for all $Y \subset I$; if $Y \subset h(X)$, then $Q^*(h^{-1}(Y)) = \lambda^*(Y)$.

From Sierpiński [19], Théorème 2, there are disjoint subsets Y_1 and Y_2 such that $Y_1 \cup Y_2 = h(X)$ and such that $\lambda^*(Y_1) = \lambda^*(Y_2) = \lambda^*(h(X)) = 1$. Hence $Q^*(h^{-1}(Y_1)) = Q^*(h^{-1}(Y_2)) = Q^*(X) = 1$; set $H = h^{-1}(Y_1)$. ∎

Say that a triple (X, Y, Z) of separable spaces has property (V) if whenever P_{xy} and P_{yz} are laws on $X \times Y$ and $Y \times Z$, respectively, having a common marginal P_y on Y, there is a law P on $X \times Y \times Z$ with marginals P_{xy} and P_{yz}.

LEMMA 7. *If X, Y, and Z are all u.m., then (X, Y, Z) has property* (V).

Proof. Since $X \times Y$ and $Y \times Z$ are u.m. there are respective proper conditional probabilities P_1 and P_2 of P_{xy} and P_{yz} given the projection maps $p_1\colon X \times Y \to Y$ and $p_2\colon Y \times Z \to Y$ (Blackwell's Theorem). Then a law P on $X \times Y \times Z$ may be defined with

$$P(A \times B \times C) = \int_B P_1(y, A \times Y) P_2(y, Y \times C)\, dP_y(y)$$

for $A \in \mathscr{B}(X)$, $B \in \mathscr{B}(Y)$ and $C \in \mathscr{B}(Z)$: (cf. Berkes and Philipp [2], Lemma A.1; for the case of an arbitrary (finite) number of factors, Vorob'ev [21] has results for discrete distributions). ∎

Another characterization of u.m. spaces is forthcoming, this time using property (V):

THEOREM 5. *A triple (X, Y, Z) has property* (V) *for all separable Y and Z if and only if X is u.m.*

Proof. Suppose that X is u.m. and that the prospective marginals P_{xy}, P_{yz} and P_y are given. Choose metrics for Y and Z with completions $\bar{Y}$ and $\bar{Z}$. Let $\bar{P}_{xy}$, $\bar{P}_{yz}$ and $\bar{P}_y$ be the induced laws on $X \times \bar{Y}$, $\bar{Y} \times \bar{Z}$ and $\bar{Y}$. Then $\bar{P}_{xy}$ and $\bar{P}_{yz}$ have a common marginal $\bar{P}_y$ on $\bar{Y}$. By Lemma 7, there is a law $\bar{P}$ on $X \times \bar{Y} \times \bar{Z}$ with marginals $\bar{P}_{xy}$ and $\bar{P}_{yz}$. Now $\bar{P}^*_{yz}(Y \times Z) = 1$, so that Theorem 3 applies to show that $\bar{P}^*(X \times Y \times Z) = 1$. Thus $Q = \bar{P}^*$ is a law on $X \times Y \times Z$. It is easy to check that P_{xy} and P_{yz} are the marginals of Q.

Conversely, if X is not u.m., then choose a metric for X with completion $\bar{X}$. As in the proof of Theorem 3, there is a law Q on $\bar{X}$ with $Q^*(X) = 1$ and $Q_*(X) = 0$; also, if $D_1 = \{(x, x)\colon x \in X\} \subset X \times \bar{X}$ and $D_2 = \{(x, x)\colon x \in \bar{X} \setminus X\} \subset \bar{X} \times (\bar{X} \setminus X)$, then $D_1 \in \mathscr{B}(X \times \bar{X})$ and $D_2 \in \mathscr{B}(\bar{X} \times (\bar{X} \setminus X))$; if $f_1\colon X \to X \times \bar{X}$ and $f_2\colon (\bar{X} \setminus X) \to \bar{X} \times (\bar{X} \setminus X)$ are defined by the rule $f_1(x) = (x, x)$, $f_2(x) = (x, x)$, they are Borel isomorphisms onto their images D_1 and D_2. Put $P_{xy} = f_1(Q^*)$ on $X \times \bar{X}$ and $P_{yz} = f_2(Q^*)$ on $\bar{X} \times (\bar{X} \setminus X)$; then P_{xy} and P_{yz} have a common marginal Q on $\bar{X}$, but there is no law P on $X \times \bar{X} \times (\bar{X} \setminus X)$ with P_{xy} and P_{yz} as marginals: if there were, $P(D_1 \times (\bar{X} \setminus X)) = P_{xy}(D_1) = 1$ and $P(X \times D_2) = P_{yz}(D_2) = 1$, but $(D_1 \times (\bar{X} \setminus X)) \cap (X \times D_2) = \emptyset$. ∎

THEOREM 6. *A triple (X, Y, Z) has property* (V) *for all separable X and Z if and only if Y is universally null.*

Proof. Assume that (X, Y, Z) has property (V) for all X and Z, but that Y is not universally null. By virtue of Lemma 6, there is a continuous law P on Y and a subset H of Y with $P^*(H) = 1$ and $P_*(H) = 0$. Put $\mathscr{A} = \sigma(\mathscr{B}(Y), H)$, the σ-algebra generated by $\mathscr{B}(Y)$ and the set H; then $\mathscr{A}$ consists of all sets of the form $(A_1 \cap H) \cup (A_2 \cap (Y \setminus H))$, $A_1, A_2 \in \mathscr{B}(Y)$, and the rule

$$Q\big((A_1 \cap H) \cup (A_2 \cap (Y \setminus H))\big) = \alpha P(A_1) + (1-\alpha) P(A_2), \quad 0 \leqslant \alpha \leqslant 1,$$

defines an extension of P to a law Q on $(Y, \mathscr{A})$, (see Łoś and Marczewski [11], Theorems 2 and 4). Suppose $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$ and let Q_1 and Q_2 be extensions of this form taking $\alpha = \alpha_1$ for Q_1 and $\alpha = \alpha_2$ for Q_2. Now take $X = Z = (Y, \mathscr{A})$, and consider the product $X \times Y \times Z$.

Define

$$D_1 = \{(y, y)\colon y \in Y\} \subset X \times Y,$$
$$D_2 = \{(y, y)\colon y \in Y\} \subset Y \times Z,$$
$$D_3 = \{(y, y, y)\colon y \in Y\} \subset X \times Y \times Z$$

and functions $f_1\colon (Y, \mathscr{A}) \to D_1$, $f_2\colon (Y, \mathscr{A}) \to D_2$ by $f_1(y) = f_2(y) = (y, y)$ and $f_3\colon (Y, \mathscr{A}) \to D_3$ by $f_3(y) = (y, y, y)$. Then D_1 and D_2 are members of $\mathscr{A} \times \mathscr{B}(Y)$, $\mathscr{B}(Y) \times \mathscr{A}$ and f_1, f_2 and f_3 are isomorphisms of $(Y, \mathscr{A})$ onto D_1, D_2 and D_3. Consider also the laws $f_1(Q_1)$ on $X \times Y$ and $f_2(Q_2)$ on $Y \times Z$: these have a common marginal on Y, namely P, but there is no law R on $X \times Y \times Z$ with marginals $f_1(Q_1)$ and $f_2(Q_2)$: if there were, $\alpha_1 = Q_1(H) = f_1(Q_1)(H \times Y) = R(H \times Y \times Z) = R((H \times Y \times Z) \cap D_3) = R((X \times Y \times H) \cap D_3) = R(X \times Y \times H) = f_2(Q_2)(Y \times H) = Q_2(H) = \alpha_2$, a contradiction.

Now assume that Y is universally null; then for any separable X and law P on $X \times Y$, there is a countable set $C \subset Y$ such that if $p(P)$ is the marginal of P on Y, $p(P)(C) = 1$ and $p(P)\{y\} > 0$ for each $y \in C$. Then if $y \in Y$ and $B \in \mathscr{B}(X \times Y)$,

$$P(y, B) = \begin{cases} P(B \cap (X \times \{y\}))/P(X \times \{y\}) & \text{if } y \in C, \\ P(B) & \text{if } y \notin C \end{cases}$$

defines a proper conditional probability of P given the projection map $p\colon X \times Y \to Y$. As in the proof of Lemma 7, the existence of such a $P(y, B)$ establishes property (V). ∎

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Correction to my paper "Topological contraction principle" (Fundamenta Mathematicae 110 (1980), pp. 135–144)

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The implication (ii) $\Rightarrow$ (iv) of Theorem 4.1 is false. The proof was taken from an erroneous argument of Meyers [15, p. 74]. Claiming a counterexample, Dr. F. Guénard first pointed out the error to me. Subsequently, the paper "A converse to the principle of contracting maps" by V. I. Opoitsev [Russian Math. Surveys 31 (1976), pp. 175–204] confirmed this and also afforded me a means of correction. We give a corrected version of Theorem 4.1.

Let $X = (X, \mathscr{U})$ be a uniform space and let $f\colon X \to X$. If u is a fixed point of f, we say that u is *stable* if, for every $U \in \mathscr{U}$, there exists $V = V(U) \in \mathscr{U}$ such that $f^n(V[u]) \subseteq U[u]$ for every $n = 1, 2, 3, \ldots$ This means that the set $\{f^n\colon n = 1, 2, 3, \ldots\}$ is equicontinuous at u. Using the arguments of the first and second paragraphs of the proof of Lemma 2.1 (where the hypothesis is merely that f is contractive), we show easily that, if f is contractive, then the set $\{f^n\colon n = 1, 2, 3, \ldots\}$ is uniformly equicontinuous, so every fixed point of f is stable. If u is a fixed point of f and $\lim_n f^n(x) = u$ for all $x \in X$, we say that u is *iteratively realizable*. The corrected version of Theorem 4.1 is the following:

4.1. Theorem. *Let X be a compact Hausdorff space. For a continuous function $f\colon X \to X$ the following statements are equivalent*:

(i) *f is an occasionally small contraction.*

(ii) *f has one and only one fixed point which is stable and iteratively realizable.*

(iii) *The filter with base $\mathscr{B} = \{f^n(X)\colon n = 1, 2, 3, \ldots\}$ converges.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 1.1 and the preceeding remark.

(ii) $\Rightarrow$ (iii). Let u be the fixed point of f and let $\mathscr{F}$ be the filter on X generated by $\mathscr{B}$. Using a refinement of the argument of Opoitsev in his Lemma 2.2, p. 182, we will show that $\mathscr{F} \to u$.