Universally measurable spaces: an invariance theorem and diverse characterizations

by

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Abstract. This article proposes two new developments: firstly, a proof of the invariance of the universal measurability (u.m.) property over all metrizations generating the same Borel structure; thus, while the property has hitherto only been stated for metric spaces, this invariance (Theorem 1) allows for its application to a large class of measurable spaces (those we shall term "separable"); secondly, we exhibit several characterizations of these u.m. separable spaces: Theorem 2 generalizes a result of Sazonov on perfect probabilities; Theorem 3 regards the behavior of measures on the product of a u.m. and a standard space; Theorem 4 involves the existence of certain types of conditional probabilities; finally, Theorems 5 and 6 address the problem of the existence of laws with given marginals.

By separable space we mean a measurable space \((X, \mathcal{A})\) with \(\mathcal{A}\) countably generated and containing singletons. We shall often suppress the notation of a \(\sigma\)-algebra, calling the space \(X\) alone and indicating its measurable structure with \(\mathcal{A} = \mathcal{A}(X)\). If \(A\) is a subset of a measurable space \((X, \mathcal{A})\), we shall always consider \(A\) as a measurable space with \(\mathcal{B}(A) = \{A \cap B: B \in \mathcal{A}\}\); under this convention, a subset of a separable space is again separable. Separability is also preserved under the taking of countable products.

If \(X\) is a separable metric space with Borel \(\sigma\)-algebra \(\mathcal{A}\), then \((X, \mathcal{A})\) is a separable space. Furthermore, there is a well-known technique due to Marczewski ([13] and [14]) by which one may introduce metrics on separable spaces compatible with the measurable structure:

**Lemma 1.** If \((X, \mathcal{A})\) is a separable space and \(\mathcal{F}\) is a countable subset of \(\mathcal{A}\), there is a metric \(d\) on \(X\) such that:
1) \((X, d)\) is a totally-disconnected metric space with compact completion \((X, d)\) is totally bounded and therefore separable,
2) \(\mathcal{A}\) is the Borel \(\sigma\)-algebra for \((X, d)\); we say "\(d\) is a metric for \((X, \mathcal{A})\)"
whenever this happens,
3) the elements of \(\mathcal{F}\) are "clopen" (both closed and open in \((X, d)\)), and
4) if \(\mathcal{G}\) generates \(\mathcal{A}\), then \(\mathcal{G}\) is a base for the topology of \((X, d)\).

A separable space \((X, \mathcal{A})\) is standard if there is a metric \(d\) for \(X\) such that...
(X, d) is a complete separable metric (i.e., Polish) space. The isomorphism types of standard spaces have been completely classified by cardinality: every standard space X is isomorphic either with a finite set, the integers or to the Cantor discontinuum according as the cardinality of X is finite, countably infinite, or uncountable. See Cohn [6], p. 275.

**Lemma 2.** Let X be a separable space; then:
1) if S ⊆ X with , then S ∈ , and
2) if X is standard, the standard subsets of X are precisely the elements of ,
3) the collection of standard subsets of X is closed under countable unions and intersections.

**Proof.** See Cohn [6], pp. 275–276.

Now for some basic terminology from elementary measure theory (details are to be found in many standard texts e.g. Halmos [8]). If (X, , P) is a finite measure space, denote inner and outer measure for P by P and P, respectively. If A ⊆ X, we use the notations A and A to indicate (not necessarily unique) elements of such that A = A ∩ A and P(A) = P(A) + P(A). A measure A is called a measurable cover for A. A subset A of X is P-measurable if P(A) = P(A); is universally measurable (u.m.) in (X, ) if it is P-measurable for every probability measure P on (X, ).

Suppose that f: X → Y is a measurable function between measurable spaces (X, ) and (Y, α) and that P is a finite measure on (X, ). We define the image measure f*(P) on (Y, α) by the rule f*(P)(A) = P(f⁻¹(A)). Given any A ⊆ X and a finite measure P on (X, ), define P, the measure induced by P on X by the rule P(A) = P(A ∩ X). If P is defined on a product X × Y, the measure P on X defined by P(A) = P(A × Y), A ∈ (X), is the marginal of P on X.

We shall use the terms “probability measure” and “law” interchangeably. If S is a Hausdorff topological space, a law P on (the Borel subsets of S) is tight if for every ε > 0, there is a compact K with P(K) > 1 − ε. A separable metric space (S, d) is universally measurable (u.m.) if it is universally measurable in its completion S. If the Borel structure on S is standard, Lemma 2 implies that S is Borel in S and so is u.m.

**Lemma 3.** A separable metric space (S, d) is u.m. if and only if every law P on S is tight.

**Proof.** Straightforward; for details see e.g. Varadarajan [20], p. 224.

Thus the u.m. property of (S, d) depends on the metric d only through its topology; in fact, rather more is true.

**Theorem 1.** Let X be a set and let d₁ and d₂ be separable metrics on X generating the same Borel σ-algebra. Let Y₁ and Y₂ be completions of X for these respective metrics; then X is u.m. in Y₁ if and only if it is u.m. in Y₂.

**Proof.** Assume that (X, d₁) is u.m. and consider the identity map from X to itself. By the Lavrentiev–Kuratowski Theorem (Kuratowski [10], p. 436), this extends to a Borel isomorphism g: E₁ → E₂ between Borel subsets E₁ of Y₁ and E₂ of Y₂, each containing X. Let h: E₁ → Y₂ be g with extended range (h(x) = g(x) for x ∈ E₁). Let P₂ be any law on Y₂; we claim that X is P₂-measurable in Y₂. If P₂(∅) = 0, this is evident; assuming P₂(∅) > 0, let P₂ be the restriction of P₂ to Borel subsets of E₂ (Q₃(B) = P₂(B) for B ∈ , where B ⊆ E₂) and let h(Q₃) be the image measure on Y₁. Since P₂(∅) > 0, P₂(E₂) > 0 and (Q₃) are laws on Y₁. Because (X, d₁) is u.m., X is completion measurable for this law in Y₁; thus there are Borel subsets A and B of Y₂ with A ⊆ E₁ and h⁻¹(A) = h⁻¹(B); finally, P₂(h⁻¹(A)) = P₂(h⁻¹(B)) = P₂(E₂) = P₂(E₂). Thus the u.m. property is invariant under choice of metric, depends only on Borel structure, and is therefore properly an attribute of separable measurable (not metric) spaces. We shall call a separable space X u.m. if there is a metric d for X for which (X, d) is u.m.; this property is preserved under the taking of countable products.

**Lemma 4.** A separable space X is u.m. if and only if for every law P on X, there is a set S ∈ (X) with S ∈ (S) standard and P(S) = 1.

**Proof.** Assume that X is u.m. Let d be a metric for X and let X be the completion of X for d. Given a law P on X, let P be the law induced by P on X. Since X is u.m. in X and P(S) = 1, there is a set S ∈ (X) with S ⊆ X and P(S) = P(S) = 1. The relative structure on S is standard by Lemma 2.

Conversely, if Q is a law on X, then either Q₁(X) = 0, and X is Q-measurable or P = Q₁(Q₁(X)) is a law on X. If S ∈ (X) with (S) standard and P(S) = 1, then by Lemma 2, S ∈ (X) and (S) = Q₁(X), and again X is Q-measurable.

One characterization of u.m. subsets of the reals known for a while: our invariance result (Theorem 1) allows us to generalize it to measurable spaces: say that a separable space X is P-perfect if P is a law on X such that for all real-valued Borel-measurable functions f: X → R, the image set f (X) is f(P)-measurable; equivalently (Sazonov [18], p. 222), whenever E = R (or with f⁻¹(∅) ∈ ); then E is f (P)-measurable.

**Theorem 2.** A separable space X is P-perfect for all laws P on X if and only if X is u.m.

**Proof.** Each separable X is Borel isomorphic with a subset X' of the real line R. By a result of Sazonov [18], p. 245, X' is Q-perfect for all laws Q on X if and only if X is u.m. in R; the former occurs if and only if X is P-perfect for all laws P on X, while the latter holds (via Theorem 1) if and only if X is a u.m. space.

Before embarking on other characterizations of u.m. spaces, we review a basic property of analytic and co-analytic spaces. A subset A of a separable space X is analytic if it is the measurable image of a standard space; a separable
space $X$ is co-analytic if it is isomorphic with the complement of an analytic set in some standard space.

**Lemma 5.** Let $A$ be a subset of a separable space $X$. If $(A, \mathcal{S}(A))$ is either analytic or co-analytic, it is a u.m. space and is universally measurable in $X$.

**Proof.** See Cohn [6], p. 281 (8.4.3).

The following reveals a Fubini-type theorem for u.m. spaces and will prove useful in other characterizations.

**Theorem 3.** Let $X$ be a fixed uncountable standard space; a separable space $Y$ is u.m. if and only if every law $P$ on $X \times Y$ has the following property: if $P_1$ is the marginal of $P$ on $X$, and $A \subset Y$ is such that $P_1(A) = 1$, then $P^*(A \times Y) = 1$.

**Proof.** Suppose that $Y$ is u.m.; taking $P_2$ as the marginal of $P$ on $Y$, we choose a standard subset $S \subset Y$ with $P_2(S) = P(X \times S) = 1$. Now given $U \supset A \times Y$, $U \in \mathcal{S}(X \times Y)$, the set $(\{X \times Y\} \setminus U) \cap (X \times S)$ is measurable in $X$ and so is standard; define $B = X \setminus \{X \times Y\} \setminus U \cap (X \times S)$, where $p: X \times Y \to X$ is projection onto the first co-ordinate; then $A \subset B$. Now $X \setminus B$ is the measurable image of a standard space, and so $B$ is co-analytic and, by Lemma 5, $P_1$-completion measurable in $X$. Choose $C \in \mathcal{S}(X)$ with $C \subset B$ and $P_1(C) = P^*(B) \geq P^*(A) = 1$; then since $C \times S \subset U$, $P(U) \geq P(C \times S) = P_1(C) = 1$, proving $P^*(A \times Y) = 1$ as desired.

Suppose that $Y$ is not u.m. and let $\bar{d}$ be a metric for $Y$ by definition $\bar{d}$. Now $X$ and $\bar{Y}$ are Borel-isomorphic (since $Y$ is not u.m., it must be uncountable), and we identify $X$ with $\bar{Y}$. Let $D = \{(y, y'): y \in Y, y \in \bar{Y}\}$, the graph of the identity map from $Y$ to $\bar{Y}$, hence $D \in \mathcal{S}(X \times Y)$. Also, $f: X \to \bar{Y} \times Y$ defined by $f(y) = (y, y)$ is a Borel isomorphism of $X$ onto $D$. Since $Y$ is not u.m. in $\bar{Y}$, there is a law $\bar{Q}$ on $\bar{Y}$ with $Q_2(\bar{Y}) < Q_2(Y)$; define a new law $Q_2$ on $\bar{Y}$ by $Q_2(B) = Q'(B \cap \{Y \times \bar{Y}\}) + Q_2(Y)$. Then $Q_2(Y) = 1$ and $Q_2(B) = 0$. Since $Q_2$ is a law on $\bar{Y}$, we may put $P = f(\bar{Q})$ and obtain a law on $X \times Y$ whose marginal $P_1$ on $X$ is just $Q_2$; taking $A = \bar{Y}$, $Y$ gives $P_2^*(A) = 1$, but $P^*(A \times Y) = 0$, since $(A \times Y) \cap D = D$.

We now examine the connection between u.m. spaces and the existence of certain conditional probabilities.

**Blackwell's Theorem.** Let $P$ be a law defined on a u.m. space $X$ and let $\mathcal{G}$ be a countably generated sub-$\sigma$-algebra of $\mathcal{S}(X)$. Then there is a real-valued function $P(x, B)$ defined for $x \in X$ and $B \in \mathcal{G}(X)$ such that

1. For each $B \in \mathcal{G}(X)$, $P(x, B)$ is a $\mathcal{G}$-measurable function of $x$,
2. For fixed $x \in X$, $P(x, \cdot)$ is a law on $\mathcal{G}$,
3. For every $A \in \mathcal{G}$, $B \in \mathcal{G}(X)$, $P(x, \cdot) = P(B \cap \{x\})$ for $x \in A \cap \mathcal{G}$, and
4. $\exists N \in \mathcal{G}$ with $P(N) = 0$ such that $P(x, A) = 1$ for $x \in A \setminus N$ and $A \in \mathcal{G}$.

**Proof.** One is to be found in Blackwell [3], Theorem 5; although stated there for analytic sets, the only property of these spaces made use of is that under any metrization, every law $P$ on $X$ is tight, something we know to be true for u.m. spaces (Lemma 3 and Theorem 1).

The function $P(x, B)$ satisfying conditions 1–4 of the preceding theorem we term a **proper conditional probability of $P$ given $\mathcal{G}$**; it is a strengthening of the notion of regular conditional probability in which only conditions 1–3 are assumed; information about regularity is to be found in several monographs: Breiman [5], Bauer [1], for example; condition 4 is treated in Blackwell and Ryll-Nardzewski [4] and Musiak [16]. If $f: X \to Y$ is a measurable function into another separable space $Y$, taking $\mathcal{G} = f^{-1}(\mathcal{S}(Y)) = \{f^{-1}(B): B \in \mathcal{S}(Y)\}$ gives a proper conditional probability of $P$ given $f$. We are now ready for another characterization of u.m. spaces:

**Theorem 4.** Let $X$ be an uncountable standard space. A separable space $Y$ is u.m. if and only if every law on $X \times Y$ has a proper conditional probability given $p: X \times Y \to X$, the projection map $p(x, y) = x$.

**Proof.** If $Y$ is u.m. then so is $X \times Y$ and Blackwell's theorem applies. Conversely, if $Y$ has the property described in the theorem, we shall see that Theorem 3 implies that $Y$ is u.m. so let $P$ be any law on $X \times Y$ with proper conditional probability $P(x, B)$ given $p$. For each fixed $x \in \mathcal{S}(X \times Y)$, $P(x, y)$, $y \in \mathcal{S}(Y)$-measurable, is constant on each set $\{x\} \times Y \times x \in X$, and so is a function of $x$ alone. We write $P(x, B) = P(x, y, B)$. If $P_1$ is the marginal of $P$ on $X$, and $A \subset X$ has $P_1(A) = 1$, we wish to prove $P^*(A \times Y) = 1$. If $x \in Y \setminus U \in \mathcal{S}(X \times Y)$, let $B = \{x\} \times U \setminus 1$; then $P \in \mathcal{S}(X)$. There is, according to part 4 of Blackwell's Theorem, a set $N \in \mathcal{S}(X)$ with $P(N \cup Y) = P_1(N) = 0$ and $P(x, C \times Y) = 1$ for $C \in \mathcal{S}(Y)$ and $x \in C \setminus N$, then $A \cap (X \setminus N) = 0$. Finally,

$$P(U) = P((X \setminus N) \times Y \setminus U) = \int_{X \setminus N} P(x, U) dP_1(x) \geq P(B \cap (X \setminus N)) = P_1(A \cap (X \setminus N)) = 1.$$

Finally, we come to the "marginal problem"; before beginning, information about universally null spaces is required. A law $P$ on a separable space $X$ is **continuous** if $P(x) = 0$ for each $x \in X$. A subset $N$ of a separable space $X$ is **null** if it for every continuous law $P$ on $X$, $P^*(N) = 0$. A separable space $X$ is universally null if there are no continuous laws on $X$. Clearly, universally null spaces are u.m. It is also easy to prove that a subset of a measurable $A$ of $X$ is universally null in $X$ if and only if $(A, \mathcal{S}(A))$ is a universally null space. The existence of such spaces was first established under the assumption of the continuum hypothesis by Lusin [12]. This assumption was removed by Marczewski and Sierpinski [15]. Compare also Darst [7] and Kuratowski [10], p. 502, where the set $Z$ in Theorem 7 is universally null.
Lemma 6. If \( Q \) is a continuous law on a separable space \( X \), then there is a subset \( H \subset X \) with \( Q(H) = 0 \) and \( Q^*(H) = 1 \).

Proof. By Lemma 1, we may assume that \( X \) is a subset of the unit interval \( I = [0,1] \) under the relative Borel structure; then the law \( Q \) induced by \( Q \) on \( I \) is again continuous. There is a Borel isomorphism \( h \) of \( I \) onto \( I \) such that \( h(Q) = \lambda \), where \( \lambda \) is Lebesgue measure on \( I \); this follows, for example, from Halmos and von Neumann [9], Theorem 2 or Royden [17], p. 337. Hence \( Q^*(h^{-1}(Y)) = \lambda^*(Y) \) for all \( Y \subset I \); if \( Y = h(X) \), then \( Q^*(h^{-1}(Y)) = \lambda^*(Y) \).

From Sierpinski [19], Théorème 2, there are disjoint subsets \( Y_1 \) and \( Y_2 \) such that \( Y_1 \cup Y_2 = h(X) \) and such that \( \lambda^*(Y_1) = \lambda^*(Y_2) = \lambda^*(h(X)) = 1 \). Hence \( Q^*(h^{-1}(Y_1)) = Q^*(h^{-1}(Y_2)) = Q^*(h^{-1}(X)) = 1 \); set \( H = h^{-1}(Y) \).

Say that a triple \((X, Y, Z)\) of separable spaces has property (V) if whenever \( P_{xy} \) and \( P_{yz} \) are laws on \( X \times Y \) and \( Y \times Z \), respectively, having a common marginal \( P_{xz} \) on \( Y \), there is a law \( P \) on \( X \times Y \times Z \) with marginals \( P_{xy} \) and \( P_{yz} \).

Lemma 7. If \( X, Y, \) and \( Z \) are all u.m., then \((X, Y, Z)\) has property (V).

Proof. Since \( X \times Y \times Z \) has properties as required, we can find a common law \( P \) on \( X \times Y \times Z \) with the required marginals.

Theorem 5. A triple \((X, Y, Z)\) has property (V) for all separable \( Y \) and \( Z \) if and only if \( X \) is u.m.

Proof. Suppose that \( X \) is u.m. and that the marginals satisfy property (V). Choose measures for \( Y \) and \( Z \) with completions \( \overline{Y} \) and \( \overline{Z} \). Let \( P_{xy} \), \( P_{yz} \) and \( P_{xz} \) be induced laws on \( X \times \overline{Y} \times \overline{Z} \) and \( \overline{Y} \times \overline{Z} \). Then \( P_{xy} \) and \( P_{yz} \) have a common marginal \( P_{xz} \) on \( \overline{Y} \) by Lemma 7. Then, there is a law \( P \) on \( X \times \overline{Y} \times \overline{Z} \) with \( P_{xy} \) and \( P_{yz} \) as required, so that \( P \) is a law on \( X \times Y \times Z \). It is easy to check that \( P_{xy} \) and \( P_{yz} \) are the marginals of \( P \).

Conversely, if \( X \) is not u.m., then choose a metric for \( X \) with completion \( \overline{X} \). As in the proof of Theorem 3, there is a law \( Q \) on \( Q^*(X) = 1 \). Also, if \( D_1 = \{(x, y) : x \in X \times \overline{X} \} \) and \( D_2 = \{(x, y) : x \in X^c \} \), then \( D_1 \in \mathcal{A}(X \times \overline{X}) \) and \( D_2 \in \mathcal{A}(X \times \overline{X}) \); if \( f_1 : X \rightarrow Y \times Z \) and \( f_2 : (X^c) \times Y \rightarrow Y \times Z \) are defined by the rule \( f_1(x) = (x, y) \), \( f_2(x) = (x, y) \), they are Borel isomorphisms onto their images \( D_1 \) and \( D_2 \). Put \( P = f_1 \circ P^* \times f_2 \circ P^* \) on \( D_1 \times \overline{X} \times \overline{X} \) and \( P_{xy} = f_1 \circ P^* \times f_2 \circ P^* \) on \( X \times (\overline{X} \times \overline{X}) \); then \( P_{xy} \) and \( P_{yz} \) have a common marginal \( Q \) on \( X \), but there is no law \( P \) on \( X \times Y \times Z \) with \( P_{xy} \) and \( P_{yz} \) as marginals: if there were, \( P(D_1 \times \overline{X} \times \overline{X}) = P(D_2) + 1 \) and \( P(U \times D_2) = P_{xy}(D_2) = 1 \), but \( D_1 \times \overline{X} \times \overline{X} \cap (U \times D_2) = \emptyset \).

Theorem 6. A triple \((X, Y, Z)\) has property (V) for all separable \( X \) and \( Z \) if and only if \( Y \) is universally null.

Proof. Assume that \((X, Y, Z)\) has property (V) for all \( X \) and \( Z \), and that \( Y \) is not universally null. By virtue of Lemma 6, there is a common law \( P \) on \( Y \) and a subset \( H \) of \( Y \) with \( P(H) = 1 \) and \( P_x(H) = 0 \). Put \( \mathcal{A} = \sigma(\mathcal{A}(Y), H) \), the \( \sigma \)-algebra generated by \( \mathcal{A}(Y) \) and the set \( H \); then \( \mathcal{A} \) consists of all sets of the form \( (A \cap H) \cup (A_2 \cap (Y \setminus H)) \), \( A_1, A_2 \in \mathcal{A}(Y) \), and the rule

\[
Q((A \cap H) \cup (A_2 \cap (Y \setminus H))) = a P_{1}(A_1) + (1 - a) P_{2}(A_2), \quad 0 \leq a \leq 1,
\]

defines an extension of \( P \) to a law \( Q \) on \( (Y, \mathcal{A}) \) (see Loé and Marczewski [11], Theorems 2 and 4). Suppose \( 0 \leq a_1 < a_2 \leq 1 \) and let \( Q_1 \) and \( Q_2 \) be extensions of this form taking \( a = a_1 \) for \( Q_1 \), and \( a = a_2 \) for \( Q_2 \). Now take \( X = Z = (Y, \mathcal{A}) \), and consider the product \( X \times Y \times Z \).

Define

\[
D_1 = \{(y, y) : y \in Y \} \subset X \times Y,
D_2 = \{(y, y) : y \in Y \} \subset Y \times Z,
D_3 = \{(y, y, y) : y \in Y \} \subset X \times Y \times Z
\]

and functions \( f_1 : (Y, \mathcal{A}) \rightarrow D_1, f_2 : (Y, \mathcal{A}) \rightarrow D_2 \) by \( f_1(y) = f_2(y) = (y, y) \) and \( f_3 : (Y, \mathcal{A}) \rightarrow D_3 \) by \( f_3(y) = (y, y, y) \). Then \( D_1 \) and \( D_2 \) are members of \( \mathcal{A} \), \( \mathcal{A}(Y) \), \( \mathcal{A}(X \times Y) \), and \( \mathcal{A}(X \times Y \times Z) \); these have a common marginal on \( Y \); namely, \( P \), but there is no law \( R \) on \( X \times Y \times Z \) with marginals \( f_1(Q_1) \) and \( f_2(Q_2) \); if there were, \( z_1 = Q_1(H) = f_1(Q_1)(H \times Y) = R(H \times Y \times Z) = R((H \times Y \times Z) \cap D_3) = R((H \times Y \times Z) \cap D_3) = (R(H \times Y \times Z) \cap D_3) = f_3(Q_3)(Y \times Z) = Q_3(H) = z_2 \), and so \( P \) \((X, \mathcal{A})\) has property (V) and \( Y \) is universally null.

Now assume that \( Y \) is universally null; then for any separable \( X \) and law \( P \) on \( X \), there is a countable set \( C \subset X \) such that if \( p(P) \) is the marginal of \( P \) on \( Y \), \( p(P) (C) = 1 \) and \( p(P)(y) > 0 \) for each \( y \in C \). Then if \( y \in Y \) and \( B \in \mathcal{A}(X \times Y) \),

\[
P_B(y) = \left\{ \frac{P\{B \cap (X \times \{y\})\}}{P\{X \times \{y\}\}} \right\} \quad \text{if } y \in C,
\]

defines a proper conditional probability of \( P \) given the projection map \( p_B : X \times Y \rightarrow Y \). As in the proof of Lemma 7, the existence of such a \( p(y, B) \) establishes property (V).
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References


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Correction to my paper “Topological contraction principle”
(Fundamenta Mathematicae 110 (1980), pp. 135–144)

by

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The implication (ii) \(\Rightarrow\) (iv) of Theorem 4.1 is false. The proof was taken from an erroneous argument of Meyers [15, p. 74]. Claiming a counterexample, Dr. F. Guinand first pointed out the error to me. Subsequently, the paper “A converse to the principle of contracting maps” by V. I. Opozitsev [Russian Math. Surveys 31 (1976), pp. 175-204] confirmed this and also afforded me a means of correction. We give a corrected version of Theorem 4.1.

Let \(X = (X, \mathcal{E})\) be a uniform space and let \(f: X \to X\). If \(u\) is a fixed point of \(f\), we say that \(u\) is stable if, for every \(V \in \mathcal{E}\), there exists \(V = V(U) \in \mathcal{E}\) such that \(f^n(V(U)) \subseteq U(U)\) for every \(n = 1, 2, 3, \ldots\) This means that the set \(\{f^n: n = 1, 2, 3, \ldots\}\) is equicontinuous at \(u\). Using the arguments of the first and second paragraphs of the proof of Lemma 2.1 (where the hypothesis is merely that \(f\) is contractive), we show easily that, if \(f\) is contractive, then the set \(\{f^n: n = 1, 2, 3, \ldots\}\) is uniformly equicontinuous, so every fixed point of \(f\) is stable. If \(u\) is a fixed point of \(f\) and \(\lim f^n(x) = u\) for all \(x \in X\), we say that \(u\) is iteratively realizable. The corrected version of Theorem 4.1 is the following:

4.1. Theorem. Let \(X\) be a compact Hausdorff space. For a continuous function \(f: X \to X\) the following statements are equivalent:

(i) \(f\) is an occasionally small contraction.

(ii) \(f\) has one and only one fixed point which is stable and iteratively realizable.

(iii) The filter with base \(\mathcal{F} = \{f^n(X): n = 1, 2, 3, \ldots\}\) converges.

Proof. (i) \(\Rightarrow\) (ii). This follows from Theorem 1.1 and the preceding remark.

(ii) \(\Rightarrow\) (iii). Let \(u\) be the fixed point of \(f\) and let \(\mathcal{F}\) be the filter on \(X\) generated by \(\mathcal{F}\). Using a refining of the argument of Opozitsev in his Lemma 2.2, p. 182, we will show that \(\mathcal{F} \to u\).