On Michael’s problem concerning the Lindelöf property in the Cartesian products

by

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Abstract. In this paper we present a negative solution of Michael’s conjecture which says that if $Y \times X$ is Lindelöf, for every hereditarily Lindelöf space $Y$, then $Y \times X^\omega$ is Lindelöf, for every hereditarily Lindelöf space $Y$.

Introduction. It is known that if $Y$ is a hereditarily Lindelöf space and $X$ a metric separable space then $Y \times X$ and also $Y \times X^\omega$ are Lindelöf. Z. Frolik proved (see [F]) that if $Y$ is a hereditarily Lindelöf and $X$ is a Lindelöf and complete in the sense of Čech space then $Y \times X$ and also $Y \times X^\omega$ are Lindelöf. R. Telgarski showed (see [T]) that if $Y$ is a hereditarily Lindelöf space and $X$ a Lindelöf and scattered space then $Y \times X$ is Lindelöf. I have improved the result of Telgarski [A1], showing that $Y \times X^\omega$ is Lindelöf. I think that these results were the motivation of Michael’s conjecture which says that if the product $Y \times X$ is Lindelöf for every hereditarily Lindelöf space $Y$ then $Y \times X^\omega$ is Lindelöf for every hereditarily Lindelöf space $Y$. In this paper we proved that the answer to the Michael’s conjecture is a negative one.

Examples.

Example 1. There exists $Z$ such that, for every natural number $n$ and for every hereditarily Lindelöf space $Y$, the product $Y \times Z^n$ is Lindelöf but $Z^n$ is not.

Example 2. There exist a separable metric space $M$ and a space $X$ such that, for every Lindelöf space $Y$ and every natural number $n$, the products $Y \times X^n$ and $X^n$ are Lindelöf but $M \times X^n$ is not.

It is easy to see that in order to obtain Example 1 it is enough to put $Z = M \times X$, where $M$ and $X$ are from Example 2.

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Terminology and notation. Our topological terminology follows [E].

Let us recall that X is a P-space if every Gδ-subset of X is open. The symbol N stands for natural numbers and D = (0, 1] for the two-points set. Greek letters are used to denote ordinal numbers, in particular α stands for the first infinite ordinal number and ω1 for the first uncountable ordinal number. The symbol Dα stands for the Cantor set and B(0, 1) = D, where |c0| × ... × |c1| × D × D × ... denotes the set {c0} × ... × {c1} × D × D × ... If α is an ordinal number then we shall identify it with the set of ordinal numbers less than α. If A is a set then the symbol |A| stands for the cardinality of A.

Auxiliary lemmas.

Lemma 1. If Ν, N = N × |k|, for k ∈ D, k: N ⊕ D N → D is a mapping such that h(Nk) = k, for k ∈ D, and B is an analytic subset of the Cantor set then there is a closed subset B1 of (N ⊕ N) such that f = h|B1 is a mapping from B1 onto B.

Proof. Let g be a mapping from Nk onto B. Then B' = [z, g(z): z ∈ Nk] is a closed subset of Nk × D. Let gα, for k ∈ D, be a mapping given by 〈gα(n1, n2) = n2. Write z = 〈gα ⊕ gR, Dk〉. It is easy to see that z is a homeomorphism from (N ⊕ N) onto D. Now it is enough to put B1 being an inverse of B'.

Let us attach to every limit countable ordinal number x a monotonical increasing sequence (a(n)n) of non-limit ordinal numbers which converges to α in the order topology of ω1. Let us put A = {a ∈ D+: a < ω}; B = {b ∈ D+: b > 0}. The topology on A is induced by the sets of the form B(a, b) = {b ∈ B: a + b < 1}. Let us put A = {a ∈ D+: a ∈ [0, 1]} and B = {b ∈ D+: b > 1}. The topology on A is induced by the sets of the form B(a, b) = {b ∈ B: a < b}.

Lemma 2. The space A has the Lindelöf property.

The proof of Lemma 2 appeared in [P]. We shall give a sketch of it for the sake of completeness.

Proof. Let be an arbitrary open covering of A. There is b < a1 and C such that B(0, b) ⊊ C, where 0 = 0, ... 0. Let us put A = {a ∈ D+: a < b}. If b < a and b is defined then put K = {a ∈ A: a < b}. Then K is countable so that b > a and b is defined. Let B(a, b + 1) = {x ∈ D+: a + b < 1}. The topology on A is induced by the sets of the form B(a, b + 1) = {x ∈ D+: a + b < 1}.

Lemma 3. If b is a countable ordinal number not less than α then there is one-to-one function h: N → B from N onto B such that for every limit ordinal number α not greater than b there are subsequences of natural numbers (n(i)n) and (n(i)n) such that the following conditions are satisfied:

(a) h(n) = n, for k ∈ N;
(b) for every i ∈ N and for every i < n if h(i) < α then h(i) < a;

The sequence (z(z)n) is a subsequence of (z(n)n) where (z(n)n) was defined in connection with Lemma 2.

Proof. We shall consider only the more complicated case when the set [a < b: a is a limit number] is infinite. Let N = N ∪ {Nj: j = 0, 1, 2, ...} be a decomposition of N such that elements of it are infinite and pairwise disjoint. Let (z(z)) be the sequence consisting of all limit numbers not greater than b. For every j ∈ N, there is a sequence (z(z)) of natural numbers such that if i and j ∈ N and i ≠ j then [z(z) ⊊ N] ∩ [z(z) ⊊ N] = 0. Write [z(z) ∈ N] = β (z(z) ⊊ N); k, j ∈ N. Let us put (z(z)) = (z(z)) and n = inf N1. If n1, ... n are defined then put n+1 = inf n ∈ N1: n > n+1. Write

n+1 = inf n ∈ N1: n > n1.

If b < a and b is defined then put h(b) = b. Let us assume that (z(z)) = (z(z)), (z(z)) = (z(z)), (z(z)) are defined and the function h is described on the set (z(z)) = (z(z)) in N and N. In this way the conditions (a) and (b) are satisfied. Write

k1 = {0, if a > b, if a > a1, k1 = inf k ∈ N: a > a1 (z(z))}, k1 = 0 if a > a1, k1 = inf k ∈ N: a > a1 (z(z))}.

For i < j, and

K = inf k ∈ N: for every i < j such that a < a1 then a > a1 (z(z)) > a;

if T = (z(z)) then i ∈ j, if i < j, if h(i) = 0 then n(i) = inf n ∈ Nj: n > n(i);

Let us put (z(z)) = (z(z)) for k ∈ N and n = inf n ∈ Nj: for every i < j, if h(i) = 0 then n(i) = inf n ∈ Nj: n > n(i). If h(i) = 0 then n(i) = inf n ∈ Nj: n > n(i). If h(i) = 0 then n(i) = inf n ∈ Nj: n > n(i). If h(i) = 0 then n(i) = inf n ∈ Nj: n > n(i).
Construction of the space $X$ and $M$ from Example 2. Write $A = \{a_\lambda : \lambda \in \{-1\} \cup \omega_1\}$ where

$$a_\lambda = \begin{cases} 0, & \text{if } \lambda = -1, \\ a_\lambda + \omega^2, & \text{if } \lambda = \theta + 1, \\ \sup \{a_\beta : \beta < \lambda\}, & \text{if } \lambda \text{ is a limit number.} \end{cases}$$

If $\lambda = \theta + 1$ then put $a_\lambda(n) = a_\lambda + (n-1)(\omega+1)$, for $n \in N$. If $\lambda$ is a limit number then let us attach to it a monotonically increasing sequence $(\lambda(n))_{n=1}^\infty$ of non-limit ordinal numbers converging to $\lambda$ in $\omega_1$ and put $a_\lambda(n) = a_{\lambda(n)} + 1$.

Let us take $A' = \{a \in D^\omega : \{\lambda \in \omega_1: a_\lambda \neq 0\} \text{ is a subset of } A\}$ and $A'' = \{a \in A' : \text{for all } \beta \in A, a_\beta \neq 0\}$.

Notice that $A'$ is a closed subspace of $A$ and $A''$ is a Lindelöf $P$-space as well as the Cantor set $[0, 1]$.

Put $M = C$, where $C$ is a coanalytic subset of the Cantor set which is not a Borel set.

The description of an uncountable subset $L$ of $C \times X^*$ without points of condensation is the difficult part of the construction of Example 2.

If $m = (m(n))_{n=0}^\infty \in M$ then put $X_m = \bigcap_{n=0}^\infty A_{m_n}$. Let us notice that $P = \bigcup_{m \in M} X_m$ is a closed subset of $M \times X^*$. It is enough to define an uncountable subset $L = \{l_m : m \in M\}$, where $l_m = (m_0, x_0), m_0 \in C, x_0 \in X_{m_0}, m_0 \neq x_0, \lambda \neq \beta$, without points of condensation in $P$.

The set $L$ will be defined by the transfinite induction with respect to $\lambda \in \omega_1$.

If $x \in X^*$ then there is $\rho \in D^\omega$ such that $x \in A_\rho$. If $x \in X^*$, then for $n \in N$, $x \in (x(0), \ldots, x(n-1))$ then $x(n) = \{x(0), \ldots, x(n-1)\}$. The symbols $a_\rho^0, a_\rho^1$ for $\lambda \in \omega_1$, will denote elements of $A_\rho$ and $A_{\rho_1}$ respectively which correspond to $a_\rho^0$ of $A'$.

In every step of induction we shall also define some conditions which will restrict our freedom of choice of $l_m = (m_0, x_0)$ in the consecutive steps of induction. In the sequel these conditions will be called restrictions.

The restrictions defined in the steps precede to the $\omega$-step to ensure that every point $(m, x(\lambda_{-1}) \in M \times X^*$, where $m \in M$ and $x(\lambda_{-1}) \in A_\lambda$, for $\lambda \in \omega_1$ will not be a point of condensation of $L$. Notice that $x(\lambda_{-1}) \in A_\lambda$, for $\lambda \in \omega_1$, is equivalent to the fact that $a_\lambda \in x(\lambda_{-1}) \cap A_\lambda$, with respect to the topology of $A_\lambda$. The role of restrictions will play some Borel subsets of the Cantor set. These subsets will be denoted by the symbols $R(x_\lambda + n, n)$, $x \in X^*$, and $n \in N$. The set $R(x_\lambda + n, n)$ will depend only on $x_\lambda x_\lambda + n, n)$. We shall say that the point $(m, x)$, where $m \in M$ and $x \in X^*$, consistent with the restriction $R(x_\lambda + n, n)$ or that $(m, x)$ satisfies the condition $R(x_\lambda + n, n)$ if $(m, x) \in \bigcup_{n} \{1 \leq \beta \leq \delta\}$ or if $(x(\lambda_{-1}) x_\lambda x_\lambda + n, n) = x_\lambda x_\lambda + n, n)$ and $x(\lambda_{-1}) = x(\lambda_{-1})$ (and $R(m, x)$ is a set $R(x_\lambda + n, n)$). The set $R(x_\lambda + n, n)$ will be defined in the $\omega$-step of induction.

The points of $L$ will be defined in such a way that they will be consistent with defined restrictions.

Write $B = D^\omega \cup C$ and for $n \in N$ and $p = (p(0), \ldots, p(n-1)) \in (N_0 \cap N_1)^n$, put $H(p) = \bigcup_{n=0}^N (p(n), \ldots, p(n)) \times (N_0 \cap N_1)^{n+1} \times (N_0 \cap N_1)^{n+1}$, and $Z(p) = H(p)$, where $f$ and $B_1$ are from Lemma 1 and the closure operation is taken with respect to the topology of the Cantor set. Let us notice that

$$1. \text{if } p \in (N_0 \cap N_1)^n \text{ and } Z(p) \neq \emptyset, \text{then } n \in N, \text{then } f(p) \in B.$$
Let us assume that, for \( x \in X^* \), \( R(n_0, x) \) and \( I^p(x) = (I^p_1(x), \ldots, I^p_{n_1}(x)) \) are defined in such a way that the following conditions are satisfied:

\((3, \neg1)\) For \( 0 < i < n \) and \( x \in X^* \), \( I^p_i(x) \in \bigcup \{P_i : j \in N \land j \leq i + 1 \} \cup \{\emptyset\} \).

\((4, \neg1)\) For \( x \in X^* \) and \( n \geq 2 \), \( I^p(x) \) is an extension of \( I^p(x_{n-1}) \).

\((5, \neg1)\) If \( j < n - 1 \), \( x \in X^* \), \( I^p_j(x) = p \), where \( p \in P_j \), then

\[
I^p_{j+1}(x) = \begin{cases} 
q, & \text{where } q \in \{q' \in P_{j+1} : q' \neq p, H(q') \cap R((n+1-\omega) \circ x_{n-1}) \cap \bigcap B[r(x(0)) \ldots r(x(n-1))] \neq \emptyset \} \\
\emptyset & \text{otherwise }
\end{cases}
\]

\((6, \neg1)\) If \( x \in X^* \) and for every \( j < n - 1 \), \( I^p_j(x) = \emptyset \) then

\[
I^p_{n-1}(x) = \begin{cases} 
q, & \text{where } q \in \{q' \in P_{n-1} : H(q') \cap R((n-1) \circ x_{n-1}) \cap \bigcap B[r(x(0)) \ldots r(x(n-1))] \neq \emptyset \} \\
\emptyset & \text{otherwise }
\end{cases}
\]

\((7, \neg1)\) For every \( x \in X^* \), \( R(n_0, x) = R(n_0, x_{n-1}) \).

\((8, \neg1)\) Let us assume that \( n > 1 \), \( x \in X^* \), \( I^p(x) \neq \emptyset \), \( p \in P_n \), and \( j < n - 1 \) such that

\[
s_j = \sup \{s \in N : x(s) \neq x(n-1) \} \cap \{s \in N : I^p_s(x_{n-1}) \neq \emptyset \}, s_0 = k_0, s_1 = \sup \{s_0, \ldots, s_j \}, \ldots, s_{n-2} = \sup \{s_0, \ldots, s_{n-2} \} \}
\]

\((9, \neg1)\) If \( x \in X^* \) then

\[
R(n_0, x) = \begin{cases} 
R((n-1) \circ x_{n-1}) \cap \bigcap B[r(x(0)) \ldots r(x(n-1)) \cap Z(p)], & \text{if } p \neq p_{n-1}, x_{n-1} \\
\emptyset & \text{otherwise }
\end{cases}
\]

\((10, \neg1)\) For every \( x \in X^* \), \( R(n_0, x) \) is a Borel set in \( D^p \), for \( x \in X^* \); in fact \( R(n_0, x) \) is compact.

\((11, \neg1)\) Let \( n > 1 \), \( x \in X^* \), \( I^p(x) \neq \emptyset \), \( p \in P_n \), and \( q \in P_{n-1} \), where \( j_q = \sup \{j < n - 1 : I^p_j(x_{n-1}) \neq \emptyset \} \), \( I^p_q(x_{n-1}) = p \in P_j \), and \( q' = p \), and

\[
R((n-1) \circ x_{n-1}) \cap H(q) \cap \bigcap B[r(x(0)) \ldots r(x(n-1)) \cap Z(p)] \neq \emptyset.
\]

\((12, \neg1)\) Assume that \( R(n_0, x) \neq \emptyset \). Let \( x \in X^* \), \( y = x_{n-1} \), and \( y \in R(n_0, y_{n-1}) \cap H(q) \cap \bigcap B[r(x(0)) \ldots r(x(n-1)) \cap Z(p)] \neq \emptyset \).

\[
I^p(y_{n-1}) = I^p(y_{n-2}) \cap R((n-1) \circ x_{n-1}) \cap H(q) \cap \bigcap B[r(x(0)) \ldots r(x(n-1)) \cap Z(p)] \neq \emptyset.
\]

Let us notice that if we define \( R((n+1) \circ x) \) and \( I^p(x) \) for \( x \in X^{*+1} \) in such a way that the conditions \((3, \neg1) - (12, \neg1)\) will be satisfied then the conditions \((3, \neg1) - (9, \neg1)\) will determine \( R((n+1) \circ x) \) and \( I^p(x) \) for the remaining points of \( X^{*+1} \).

Write \( S = \{S_i(x) : i < n, x \in X^* \} \) and \( (n+1) \circ x = x \). Let us notice that the set \( \{y \in S_i(x) : (n+1) \circ y = y \} \) is countable so also \( S \) is countable and it consists of countable and infinite sets. If \( S_i(x) \) and \( S_j(x) \) belong to \( S \) and \( S_i(x) \neq S_j(x) \) then the intersection \( S_i(x) \cap S_j(x) \) is finite. Let us order \( S = \{O_i : i \in N \} \). Let us assume that \( R(n+1, x) \) and \( I^p(x) \) are defined for \( y \in \bigcup \{O_i : k \leq i \} \). Let us assume that \( y_0 = O_{k+1} \neq S_i(x) \), where \( i < n \) and \( x \in X^* \). Then the set

\[
D(y_0) = \{y_0 \in S_i(x) : (n+1) \circ y = y_0 \}, y_0(0) = y_0(1), \ldots, y_0(n_0-1)
\]

is infinite. If \( i = n \) then it follows from the definition of \( S_i(x) \); if \( i < n \) then it follows from the inductive assumption (see \((12, \neg1) - (8, \neg1)\)). Let us notice that from the definition of \( D(y_0) \) and from the conditions \((5, \neg1) - (6, \neg1)\) it follows that for every \( y \in D(y_0) \)

\[
I^p_{(n+1) \circ y_0}(n) = I^p(y_0(n)) = I^p_{(n+1) \circ y_0}(n+1).
\]

Write

\[
P(y_0) = \{q \in \bigcup \{P_j : j \in N \} : R(n_0, y_0(n)) \cap \bigcap B[r(x(0)) \ldots r(x(n-1)) \cap Z(p)] \neq \emptyset, \text{ if } p = p_{n-1}, x_{n-1} \}
\]

If \( q \in P_{n-1} \), then \( q' = p_0 \), if \( q \in P_{n-1} \), then \( q' = p_{n-1} \), and \( q' = p_0 \).

Let us put \( y_0(1) = y_0(2) = \ldots, y_0(n_0-1) = y_0(n_0) \).

Let us assume that \( R(n_0, x) \neq \emptyset \) for every \( x \in D(y_0) \). Assume that \( P(y_0) = \emptyset \). Let \( x \) be a function from \( D(y_0) \) onto \( P(y_0) \) such that for every \( q \in P(y_0) \) the set \( q^{-1}(q) \) is infinite. Let us put \( x_0 = y_0(1) = y_0(2) = \ldots, y_0(n_0-1) = y_0(n_0) \).

Let \( y = (y_0)_{n_0}^{n_0+1} \) be an element of \( X^{n_0+1} \) such that \( q_0 \in y_0(1) \cap \ldots, q_{n_0-1} \in y_0(n_0) \).

Write

\[
S_i(x) = \{y_0 \in D(y_0) : y_0(0) = y_0(1) = \ldots, y_0(n_0-1) = y_0(n_0) \}, y_0(0) = y_0(1) = \ldots, y_0(n_0-1) = y_0(n_0) \}
\]

is infinite.
If \(x_j \in eX^*\) and \(\{i(x_j)\} \subseteq \{p \in P \mid j \notin \{p \} \} = \emptyset\), then \(I^{r+1}(x_j) = \emptyset\). If \(x_j \not\in eX^*\), then \(I^{r+1}(x_j) \subseteq \emptyset\).

**Remark 1.** Notice that if \(x = f(p, \ldots, p(n+1))\), then \(I^{r+1}(x) = \emptyset\).

**Remark 2.** Notice that from \((S+1)\) and \((S+2)\), it follows that if \(x_j \not\in eX^*\), then \(I^{r+1}(x_j) \subseteq \emptyset\).
Let us assume that \( j_0, \ldots, j_n-1, s_0, \ldots, s_{n-1}, \hat{c}_0, \ldots, \hat{c}_{n-1} \) and \( p_0, \ldots, p_{n-1} \) are defined. Put \( j_i = \inf \{ i \in N : j_i \geq j_n-1 \} \) and there are \( \hat{c}_0 \in F \) and \( s \in N \) such that \( y(i) \in \{ a_{\hat{c}0}, a_{\hat{c}1}, \ldots, a_{\hat{c}k} \} \) and \( (y(p))^{-1} \{ 1 \} \subset a_{s0} + s0 \). If \( (y(p))^{-1} \{ 1 \} \subset a_{s0} + s0 \), then put

\[
s_i = \inf \{ s \in N : (y(p))^{-1} \{ 1 \} \subset a_{s0} + s0 \}.
\]

If \( (y(p))^{-1} \{ 1 \} \subset a_{s0} + s0 \), then put

\[
W_i = \{ \hat{c} \in \hat{F} : \text{there are } i < j_0 \text{ and } s \in N \text{ such that } y(i) \in \{ a_{\hat{c}0}, a_{\hat{c}1}, \ldots, a_{\hat{c}k} \}, \quad y(i)^{-1} \{ 1 \} \subset \{ \hat{c} \} \}.
\]

Then put \( W_i = \{ \hat{c} \in \hat{F} : \text{there are } i < j_0 \text{ and } s \in N \text{ such that } y(i) \in \{ a_{\hat{c}0}, a_{\hat{c}1}, \ldots, a_{\hat{c}k} \}, \quad y(i)^{-1} \{ 1 \} \subset \{ \hat{c} \} \} \). We notice that, for every \( \hat{c}_0 \leq \hat{c} \leq \hat{c}_0 \) and \( \hat{c}_0 \in \hat{F} \) we infer that the set \( \{ \hat{c} \} \{ y(p) \} \) is finite. Put \( \Gamma = \{ \hat{c} \} \hat{F} \); there is \( \hat{c} \in \hat{F} \) such that \( y(i) \in \{ a_{\hat{c}0}, a_{\hat{c}1}, \ldots, a_{\hat{c}k} \} \). If \( \hat{c} \in \hat{F} \) then by the inductive assumption (m, y) is not a point of condensation of \( L \). We assume that, for \( n \in \mathbb{N} \) and \( \hat{c} < \hat{c} \), there is a point of condensation of \( L \) for the \( n \)-th \( \hat{c} \). Write \( \theta_n = \inf \{ i \in N : \text{there is } \hat{c} \in \hat{F} \text{ such that } y(i) \in \{ a_{\hat{c}0}, a_{\hat{c}1}, \ldots, a_{\hat{c}k} \} \} \).

Let us put \( s_0 = \sup \{ s_0, j_0 + 2, s_0 \} \). Using \( (s_0, s_0) \) and \( (s_0, s_0) \) one can prove that

\[
(\psi_{s0})^{-1} \{ 1 \} \subset a_{s0} + s0.
\]

Put \( \theta_n = \inf \{ s \in N : (\psi_{s0})^{-1} \{ 1 \} \subset a_{s0} + s0 \} \).
given from Lemma 3. Write \( W_i = \{ x \in X : x = 3x \} \). If \( W_{n-1} \) is defined then put
\[
W_n = \{ x \in X : x^{-1}(1) < (2n+1)\alpha \cup
\begin{align*}
\cup \{(\hat{\beta} < 0) & : a_{\hat{\beta}} \leq \hat{\beta} < a_{\hat{\beta}} + (2n+1)\alpha, j < n\}\end{align*}
\]
Let us put
\[
Y_i = \begin{cases} \{ y_{i}(n) \} & \text{if } n \in \mathbb{Z}, \\
W_j & \text{if } 2 \leq n = 2j, \\
\{a_{\hat{\beta}} \leq \hat{\beta} < a_{\hat{\beta}} + (2n+1)\alpha \} & \text{if } 2 < n = 2j+1.
\end{cases}
\]
For every \( \hat{\beta} \in \mathbb{B} \), let \( n(\hat{\beta}) \) be the unique natural number such that, for every \( y \) of \( \bar{P} Y_i \), \( y(n(\hat{\beta})) \in \{a_{\hat{\beta}} \leq \hat{\beta} < a_{\hat{\beta}} + (2n+1)\alpha \} \). Notice that if \( x \in W_j \), for \( j \in \mathbb{N} \), then \( x^{-1}(1) \) is finite.

Let \( y \) be a point of \( \bar{P} Y_i \). From the definition of \( Y_i \), for \( i \in \mathbb{N} \), it follows that
\[
I_{P}(x_0, x_{n(\hat{\beta})}) \in P_{x_0 + 1}, \quad \text{if } \mathcal{R}(x_{n(\hat{\beta})} + (2n)\alpha, y) = \varnothing.
\]
Let us put \( p_y = I_{P}(x_0, x_{n(\hat{\beta})}) \) and \( C(x_0, y) = Z(p_y) \). We shall show that for every \( m \) of \( C(x_0, y) \) there is
\[
z \in \{ \bar{P} Y_i \} \cap X_\alpha \text{ such that } z(n(\hat{\beta}) + 1) = y \text{ and } (m, z) \text{ is consistent with } R(\mathbb{R}(\hat{\beta}), x_0).
\]
Let \( m = (m(n))_{n=0}^{\infty} \) be an arbitrary element of \( Z(p_y) \). By the definition of \( (a_{\hat{\beta}} \leq \hat{\beta} < a_{\hat{\beta}} + (2n+1)\alpha) \) one can infer, applying \((12, \ldots, 1)\) two times, that there is \( z \in \bar{P} Y_i \) such that
\[
z = Z(p_y) \cap B(m(0) \ldots m(t_1)).
\]
If \( z \) is defined then one can show, in a similar way as above, that there is \( z_{k+1} = p_{z_{k+1}} \) such that \( z_{k+1} = z_{k+1} \cap B(m(0) \ldots m(t_k)) \).

The point \( z \) which is defined by \( (a_{\hat{\beta}} \leq \hat{\beta} < a_{\hat{\beta}} + (2n+1)\alpha) \) has the required properties.

From the reasoning presented above it follows that for \( k \geq n(0) \) there is
\[
x \in \bar{P} Y_i \text{ such that for every } n(0) \leq j \leq k
\]
\[
I_{P}(x_0, x_{n(0)+1}) = I_{P}(x_0, x_{n(0)+1}) \cap B(m(0) + 1).
\]

By the definition of \( Y_i \), for \( i \in \mathbb{N} \), and \( (9, \ldots, 1) \) we infer that \( R((k+1)\alpha, x) \cap B(r(x_0) \ldots r(x(k))) \), if \( \mathcal{R}((n(0)+1)\alpha, x \cap B(m(0) + 1) = \varnothing \). Put \( T_0 = \{ x \in \bar{P} Y_i : I_{P}(x_0, x_{n(0)+1}) \cap B(m(0) + 1) \cap \mathbb{N} \}, \) for \( n(0) \leq j \leq k \), \( B(x_0, x_{n(0)+1}), x \in \bar{P} Y_i \cap B(r(x_0) \ldots r(x(k))) \) for \( x \in T_0 \) and \( B(x_0, x_{n(0)+1}), x \in \varnothing \) if \( x \notin T_0 \), where \( P_{x_0 + 1} = I_{P}(x_0, x_{n(0)+1}) \cap B(m(0) + 1) \).

Let us notice that for every \( m \in \mathbb{B} \) \( B(C(x_0, x_{n(0)+1}), x) \) there is \( x \in \bar{P} Y_i \) such that \( x_{n(0)+1} = x \) and \( (m, x) \) is consistent with \( R(\mathbb{R}(\hat{\beta}), x_0) \) for \( x \in X_\alpha \) and \( (n \in \mathbb{N} \), and
\[
B(C(x_0, x_{n(0)+1}, x) \in \mathcal{R}(x_{n(0)+1} + 1) \cap \mathbb{N} \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), we have defined
\[
P_{x_{n(0)+1} + 1} \cap B(C(x_0, x_{n(0)+1}, x) \in \mathcal{R}(x_{n(0)+1} + 1) \cap \mathbb{N} \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \), where \( \hat{\beta} \in \mathbb{B} \) and \( x \in \bar{P} Y_i \).
greater than \( n(\eta) \) and less than \( n(\bar{\theta}) \) and for every \( q \in Q(y) \) the set \( \mathcal{Y} \in [y] \): 
\( y'(j) = z(j) \) if \( j \neq 1 \) and \( p_\eta = q \) is not empty, \( p_\eta \in Q(y) \) and if \( m \) and \( z \) belong to 
\( \mathcal{Y} \) and \( x_\eta + (n(\bar{\theta}) + 1)z_\eta = x_\eta + (n(\eta) + 1)z_\eta \) then \( p_\eta \in \mathcal{Y} \).

\( (\eta, x) \) For every \( y \in \prod_{\alpha \in \Gamma} Y_\alpha \), \( C(x, y) \) is a Borel subset of the Cantor set including 
\( H(y) \cap R((n(\bar{\theta}) + 1)\omega, y) \), if \( \eta = \eta_1 \), or 
\( H(y) \cap B(C(x_\eta, y(n(\eta) + 1), y) \cap R(x_\eta + (n(\bar{\theta}) + 1)\omega, y), \eta) \), if \( \eta \neq \eta_1 \) (for the definition of \( \eta \) see 
\( (\eta, x) \)).

\( (\gamma, \alpha) \) For every \( y \in \prod_{\alpha \in \Gamma} Y_\alpha \) and \( m \in C(x, \gamma) \), \( y \) there is \( z \) of 
\( \prod_{\alpha \in \Gamma} Y_\alpha \) such that the point \( (m, z) \) of \( C \times X^\alpha \) is consistent with 
\( R(x_\eta + n, x) \), \( \lambda < \gamma \), \( n \in N \) and \( x \in X^\alpha \), and \( \nu(n(\bar{\theta}) + 1) = y. \)

\( (\gamma, \alpha) \) For every \( k \geq n(\bar{\theta}) \) and \( z \in \prod_{\alpha \in \Gamma} Y_\alpha \), the set \( B(C(x_\eta, z(n(\bar{\theta}) + 1)), z) = \{ m \in C(x, \gamma) \cap X_\eta \} \times X_\eta \) such that \( z \in \prod_{\alpha \in \Gamma} Y_\alpha \) and the point \( (m, z) \) is consistent with \( R(x_\eta + n, x) \), \( \lambda < \gamma \), \( n \in N \) and \( x \in X^\alpha \) is a Borel subset of the Cantor set which depends only on \( x_\eta \), and 
\( B(C(x_\eta, z(n(\bar{\theta}) + 1)), z) = \bigcup [B(C(x_\eta, z(n(\bar{\theta}) + 1)), z')] \cap \prod_{\alpha \in \Gamma} Y_\alpha, z' \in X_\eta, z' \in X_\eta, z' \in X_\eta. \)

Let \( y \) be a point of \( \prod_{\alpha \in \Gamma} Y_\alpha \). There are two cases:

(a) \( \lambda < \gamma \) such that \( n(\lambda) \neq n(\bar{\theta}) \),
(b) \( \lambda = \gamma \) such that \( n(\lambda) = n(\bar{\theta}) \).

The proofs of these cases are similar but the second one is a little bit simpler than the first one so we shall consider only the first case.

Put \( \eta = \sup \{ \lambda < \gamma : n(\lambda) \neq n(\bar{\theta}) \} \). Let us assume that 
\( B(C(x_\eta, z(n(\bar{\theta}) + 1)), y) \cap R(x_\eta + (n(\bar{\theta}) + 1)\omega, y) \neq \emptyset \).

\( [y] = \{ y \in \prod_{\alpha \in \Gamma} Y_\alpha : y_\alpha = y_\alpha, \prod_{\alpha \in \Gamma} (x_\alpha + (n(\bar{\theta}) + 1)\omega, \omega, y_\alpha) \neq \emptyset \}, \)

\( \mathcal{Y} = \{ y \in \prod_{\alpha \in \Gamma} Y_\alpha : y_\alpha = y_\alpha, \prod_{\alpha \in \Gamma} (x_\alpha + (n(\bar{\theta}) + 1)\omega, \omega, y_\alpha) \neq \emptyset \}. \)

The set \( Q(y) \) is defined in a similar way as in \( (\lambda, \eta) \).

Let us assume that 
\( Q(y) \neq \emptyset \).

Let \( t \) be an integer even number greater than \( n(\eta) \) and less than \( n(\bar{\theta}) \) and an element of \( [y] \) satisfying 
\( x_\eta + (n(\bar{\theta}) + 1)z_\eta = x_\eta + (n(\eta) + 1)z_\eta \). Write \( A(z, 0) = \{ z' \in [y] : 
\)
\( z' = x_\eta + (n(\bar{\theta}) + 1)z' \) and \( z(j) = z(j), \) for \( j \neq 1 \). The family of all sets of the form \( A(z, 0) \) is countable so let us assume that it is equal to \( \{ A_\sigma : s \in N \}. \)

From the definition of \( Y_\alpha \), \( y > n(\eta) \), and \( (11, \alpha) \) it follows that \( A_s \) is infinite and
Let \( \mathbf{P} \) be a point of \( \hat{\mathbf{p}}, \mathbf{P} \), where \( r \geq n(\check{\eta}) \). If \( t < v(j) \), where \( j \) is such that \( v(j) = n(\lambda) \) and \( f = \inf \{s \in \mathbb{N} : t < v(s) \} \) then

\[
B(C(a_\eta, z[n(\check{\eta})+1], z) = \begin{cases} 
\mathcal{O}, & \text{if there is no } z' \text{ of } K_f(z[n(\check{\eta})+1]) \text{ such that } z'[t] = z \\
\bigcup \{C(a_{\eta}', z') : z' \in K_{f}(z[n(\check{\eta})+1]) \text{ and } z'[t] = z \} \cup \{z[n(\check{\eta})+1] \} \text{, otherwise.}
\end{cases}
\]

If \( t \geq v(j) \),

\[
t' = \begin{cases} 
t, & \text{if } t \text{ is an odd number,} \\
t + 1, & \text{otherwise,}
\end{cases}
\]

and \( f' \) is such that \( f' = v(j) \) then

\[
B(C(a_\eta, z[n(\check{\eta})+1], z) = \begin{cases} 
\mathcal{O}, & \text{if there is no } z' \text{ of } K_{f}(z[n(\check{\eta})+1]) \text{ such that } z'[t] = z \\
\bigcup \{B(C(a_{\eta}', z[n(\check{\eta})+1], z) \cap R(a_{\eta} + (t' + 1) \omega, z') : z'[t] = z \} \cup \{z[n(\check{\eta})+1] \} \text{, otherwise.}
\end{cases}
\]

Case 2 (\( \check{\eta} \) is a limit number). Put \( K_1(y) = \{y\} \) and \( v(1) = n(\check{\eta}) \). Let us assume that \( K_f(y) = \mathbb{N} \). If \( v(j) \) is defined, where \( v(j) \) is an odd number less than \( n(\check{\eta}) \).

Write \( v(j + 1) = \inf \{s \in \mathbb{N} : s > v(j)\} \) and there is \( \lambda < \check{\eta} \) such that \( n(\lambda) = s \) and \( \lambda > \sup \{\check{\eta} < \check{\eta} : n(\check{\eta}) \leq v(j)\} \) and put

\[
K_{j+1}(y) = \{z \in K_j(y) \times \prod_{i=0}^{n(j+1)} Y_i : p_i[n(j)+1] = p_i[n(j)+1] + 1 \text{ and } (z) \in \mathcal{Q} \}.
\]

Put \( s' = n(j+1)/2 \). If \( s' \) is defined then write \( s' = \inf \{s' : k \in \mathbb{N} : s' > s'_k \} \) (see Lemma 3), \( n(\check{\eta}) \) was defined in connection with the definition of \( a_{\eta} \), and \( s = 2s' + 1 \). This is the unique place where we need Lemma 3.

Let \( j(\eta) \) and \( \eta_\eta \) be such that \( v(j(\eta)) = s' \) and \( \eta_\eta = n(\eta_\eta) \). Put

\[
J_0(y) = B(C(a_\eta, y[n(\check{\eta})+1], y) \cap R(a_{\eta} + (n(j+1) + 1) \omega, y) \cup \{z \in K_{j+1}(y) : z \in \mathcal{Q} \}.
\]

If \( J_0(y) \) is defined then put

\[
J_{k+2}(y) = \{z \in K_{j+2}(y) \} \text{, where } z \in K_{j+1}(y)\).
\]

Put \( J_0(y) = \bigcup \{J_k(y) : k \in \mathbb{N} \} \text{ and } C(a_\eta, y) = B(C(a_\eta, y[n(\check{\eta})+1], y) \cap R(a_{\eta} + (n(j+1) + 1) \omega, y) \cup \{z \in K_{j+1}(y) : z \in \mathcal{Q} \}.
\]

Let \( z \) be a point of \( \hat{\mathbf{p}}, \mathbf{P} \), where \( r \geq n(\check{\eta}) \). If \( t < v(j) \), where \( j \) is such that \( v(j) = n(\lambda) \) and \( f = \inf \{s \in \mathbb{N} : t < v(s) \} \) then

\[
B(C(a_\eta, z[n(\check{\eta})+1], z) = \begin{cases} 
\mathcal{O}, & \text{if there is no } z' \text{ of } K_f(z[n(\check{\eta})+1]) \text{ such that } z'[t] = z \\
\bigcup \{C(a_{\eta}', z') : z' \in K_{f}(z[n(\check{\eta})+1]) \text{ and } z'[t] = z \} \cup \{z[n(\check{\eta})+1] \} \text{, otherwise.}
\end{cases}
\]

If \( t \geq v(j) \),

\[
t' = \begin{cases} 
t, & \text{if } t \text{ is an odd number,} \\
t + 1, & \text{otherwise,}
\end{cases}
\]

and \( f' \) is such that \( f' = v(j) \) then

\[
B(C(a_\eta, z[n(\check{\eta})+1], z) = \begin{cases} 
\mathcal{O}, & \text{if there is no } z' \text{ of } K_{f}(z[n(\check{\eta})+1]) \text{ such that } z'[t] = z \\
\bigcup \{B(C(a_{\eta}', z[n(\check{\eta})+1], z) \cap R(a_{\eta} + (t' + 1) \omega, z') : z'[t] = z \} \cup \{z[n(\check{\eta})+1] \} \text{, otherwise.}
\end{cases}
\]

From the definition of the constructed objects and the inductive assumption it follows that they depend on \( (y[n(\check{\eta})+1] \), \( \omega \))", and \( C(a_\eta, y) = B(C(a_\eta, y[n(\check{\eta})+1], y) \cap R(a_{\eta} + (n(j+1) + 1) \omega, y) \cup \{z \in K_{j+1}(y) : z \in \mathcal{Q} \}.
\]

Let \( z \) be a point of \( \hat{\mathbf{p}}, \mathbf{P} \), where \( r \geq n(\check{\eta}) \). Write \( k = \inf \{k' \in \omega : t < s_k \} \) then

\[
B(C(a_\eta, z[n(\check{\eta})+1], z) = \begin{cases} 
\mathcal{O}, & \text{if there is no } z' \text{ of } K_{f}(z[n(\check{\eta})+1]) \text{ such that } z'[t] = z \\
\bigcup \{B(C(a_{\eta}', z[n(\check{\eta})+1], z) \cap R(a_{\eta} + (t' + 1) \omega, z') : z'[t] = z \} \cup \{z[n(\check{\eta})+1] \} \text{, otherwise.}
\end{cases}
\]

From the definition of the constructed objects and the inductive assumption it follows that they depend on \( (y[n(\check{\eta})+1] \), \( \omega \))", and \( C(a_\eta, y) = B(C(a_\eta, y[n(\check{\eta})+1], y) \cap R(a_{\eta} + (n(j+1) + 1) \omega, y) \cup \{z \in K_{j+1}(y) : z \in \mathcal{Q} \}.
\]
and the definition of $Y_{k}$ for $k \in \omega$, we infer that the set
\[ V = \{ x \in P \mid \exists z \in F_{k}, z = x \mid \forall k \} \]
for $i \leq v(t_{i})$ is not empty. Notice that if $x \in V$ then $q_{i} \in Q(x)$. By $(4,3)$, where $\lambda$ is such that
\[ n(\lambda) = v(t_{i}), \]
we infer that there is $z_{i} \in V$ such that $p_{i} = q_{i}$. It is easy to see that
\[ n \in C_{i}(z_{i}). \]
After $(j(t_{i+1}) - j(t_{i}))$ steps we shall find $z_{i+1} = x'$ which has the required properties.

From $(1,4)-(7,9)$ it follows that $C_{i}(z_{i}, y_{i})$ is a required set.

The construction of $R_{i}(x_{i} + n_{0}, x)$, for $i \in N$, $x \in X^{*}$ is similar to the construction of $R_{i}(n_{0}, x)$. This completes the definition of $L$ so we conclude that $X$ and $M$ have properties mentioned in Example 2.

**Comments.**

**Remark 4.** If $X$ is the derivative of $X$, where $X$ is from Example 2, then one can show (see [Al1]) that $X = \bigcup \{ X_{n} : n \in N \}$, where $X_{n}$ is a Lindelöf scattered space so from [Al1] it follows that the product $Y \times (X)^{n}$ is Lindelöf, for every hereditarily Lindelöf space $Y$.

**Remark 5.** In [Al1] we proved, in some sense, a dual result to Example 2. We showed that if $X$ is a Lindelöf $P$-space and $M$ is a separable metric space which admits a complete metric space $M'$ such that $M' \supset M$ and $M' \setminus M$ does not contain uncountable compact subsets then the product $M \times X^{n}$ is Lindelöf.

Let me finish this paper with some problems related to the Michael's conjecture.

**Problem 1.** Let us assume that the product $Y \times X$ is Lindelöf, for every Lindelöf space $Y$. Is it true that $X^{n}$ is a Lindelöf space?

**Problem 2.** Let us assume that $Y \times X$ has the Lindelöf property, for every hereditarily Lindelöf space $Y$. Is it true that $X^{2}$ is a Lindelöf space?

**Problem 3.** One can ask similar questions for other covering properties. I do not know, for example, whether the product $Y \times X^{n}$ is paracompact, where $X$ is a space having only one non-isolated point and $Y$ is a perfect paracompact space?

Notice that it is not enough to assume that $Y$ is a hereditarily paracompact space.

**References**