References

[B] J. B. Brown, *Baire category in spaces of probability measures*, Fund. Math. 96 (1977), pp. 189–193.

[B'] — *Lusin density and Ceder's differentiable restrictions of arbitrary real functions*, Fund. Math. 84 (1974), pp. 35–45.

[K] K. Kuratowski, *Topology* I, New York–London–Warszawa 1966.

[L] N. Y. Luther, *Locally compact spaces of measures*, Proc. Amer. Math. Soc. 25 (1970), pp. 541–547.

[P] Y. V. Prohorov, *Convergence of random processes and limit theorems in probability theory*, Theor. Prob. Appl. 1 (1956), pp. 157–214.

AUBURN UNIVERSITY
DEPARTMENT OF MATHEMATICS
Auburn, Alabama 36830
201 Hastings
Redlands, California 92373

Received 28 September 1981

On Michael's problem concerning the Lindelöf property in the Cartesian products

by

K. Alster* (Lubbock, Tex.)

Abstract. In this paper we present a negative solution of Michael's conjecture which says that if $Y \times X$ is Lindelöf, for every hereditarily Lindelöf space Y, then $Y \times X^{\omega}$ is Lindelöf, for every hereditarily Lindelöf space Y.

Introduction. It is known that if Y is a hereditarily Lindelöf space and X a metric separable space then $Y \times X$ and also $Y \times X^{\omega}$ are Lindelöf. Z. Frolík proved (see [F]) that if Y is a hereditarily Lindelöf and X is a Lindelöf and complete in the sense of Čech space then $Y \times X$ and also $Y \times X^{\omega}$ are Lindelöf. R. Telgarski showed (see [T]) that if Y is a hereditarily Lindelöf space and X a Lindelöf and scattered space then $Y \times X$ is Lindelöf. I have improved the result of Telgarski [Al_1], by showing that $Y \times X^{\omega}$ is Lindelöf. I think that these results were the motivation of Michael's conjecture which says that if the product $Y \times X$ is Lindelöf for every hereditarily Lindelöf space Y then $Y \times X^{\omega}$ is Lindelöf for every hereditarily Lindelöf space Y. In this paper we proved that the answer to the Michael's conjecture is a negative one.

Examples.

EXAMPLE 1. There exists Z such that, for every natural number n and for every hereditarily Lindelöf space Y, the product $Y \times Z^n$ is Lindelöf but Z^{ω} is not.

EXAMPLE 2. There exist a separable metric space M and a space X such that, for every Lindelöf space Y and every natural number n, the products $Y \times X^n$ and X^{ω} are Lindelöf but $M \times X^{\omega}$ is not.

It is easy to see that in order to obtain Example 1 it is enough to put $Z = M \times X$, where M and X are from Example 2.

* 1978. Mathematics Subject Classification. Primary 54 B 10.

Terminology and notation. Our topological terminology follows [E].

Let us recall that X is a P-space if every G_δ-subset of X is open. The symbol N stands for natural numbers and $D=\{0,1\}$ for the two-points set. Greek letters are used to denote ordinal numbers, in particular ω stands for the first infinite ordinal number and ω_1 for the first uncountable ordinal number. The symbol D^ω stands for the Cantor set and $B(i_0\ldots i_n)$, where $\{i_0,\ldots,i_n\}\subset D$, denotes the set $\{i_0\}\times\ldots\times\{i_n\}\times D\times D\times\ldots$ If α is an ordinal number then we shall identify it with the set of ordinal numbers less than α. If A is a set then the symbol $|A|$ stands for the cardinality of A.

Auxiliary lemmas.

LEMMA 1. *If $N_k=N\times\{k\}$, for $k\in D$, $h\colon N_0\oplus N_1\to D$ is a mapping such that $h(N_k)=k$, for $k\in D$, and B is an analytic subset of the Cantor set then there is a closed subset B_1 of $(N_0\oplus N_1)^\omega$ such that $f=h^\omega|B_1$ is a mapping from B_1 onto B.*

Proof. Let g be a mapping from N^ω onto B. Then $B'=\{(x,g(x))\colon x\in N^\omega\}$ is a closed subset of $N^\omega\times D^\omega$. Let g_k, for $k\in D$, be a mapping given by $g_k(n,k)=n$. Write $z=[(g_0\oplus g_1)\Delta h]^\omega$. It is easy to see that z is a homeomorphism from $(N_0\oplus N_1)^\omega$ onto $N^\omega\times D^\omega$. Now it is enough to put $B_1=z^{-1}(B')$.

Let us attach to every limit countable ordinal number α a monotonically increasing sequence $(\alpha(n))_{n=1}^\infty$ of non-limit ordinal numbers which converges to α in the order topology of ω_1. Let us put $A=\{a\in D^{\omega_1}\colon |a^{-1}(1)|<\omega\}\cup\{a_\alpha\in D^{\omega_1}\colon \alpha$ is a limit ordinal number less than ω_1 and $a_\alpha^{-1}(1)=\{\alpha(n)\colon n\in N\}\}$. The topology on A is induced by the sets of the form $B(a,\beta)=\{b\in A\colon b|\beta+1=a|\beta+1\}$.

LEMMA 2. *The space A has the Lindelöf property.*

The proof of Lemma 2 appeared in [P]. We shall give a sketch of it for the sake of completeness.

Proof. Let $\mathscr{U}$ be an arbitrary open covering of A. There is $\beta_0<\omega_1$ and $U\in\mathscr{U}$ such that $B(0,\beta_0)\subset U$, where $0=(0\ldots0\ldots)$. Let us put $\mathscr{U}_0=\{B(0,\beta_0)\}$. If $\mathscr{U}_i$ and β_i, for $i\leqslant n$, are defined then put $K_n=\{a\in A\colon a^{-1}(1)\subset\beta_n+1\}$. The set K_n is countable so there are $\beta_{n+1}>\beta_n$ such that $\mathscr{U}_{n+1}=\{B(a,\beta_{n+1})\colon a\in K_n\}$ refines $\mathscr{U}$. Put $\beta=\sup\{\beta_n\colon n\in N\}$. Notice that if $a\neq a_\beta$ then $a\in\bigcup\{\bigcup\mathscr{U}_n\colon n\in N\}$. Indeed, there is $n\in N$ such that $a^{-1}(1)\cap\beta=a^{-1}(1)\cap(\beta_n+1)$. Let a' be an element of K_n such that $a'|\beta=a|\beta$. Then $a\in B(a',\beta_{n+1})\in\mathscr{U}_{n+1}$.

LEMMA 3. *If β is a countable ordinal number not less than ω then there is one-to-one function $h_\beta\colon N\to\beta$ from N onto β such that for every limit ordinal number α not greater than β there are subsequences of natural numbers $(n_k^\alpha)_{k=1}^\infty$ and $(s_k^\alpha)_{k=1}^\infty$ such that the following conditions are satisfied:*

(a) *$h_\beta(n_k^\alpha)=\alpha(s_k^\alpha)$, for $k\in N$,*

(b) *for every $k\in N$ and for every $i\leqslant n_k^\alpha$ if $h_\beta(i)<\alpha$ then $h_\beta(i)<\alpha(s_k^\alpha)$.*

The sequence $(\alpha(s_k^\alpha))_{k=1}^\infty$ is a subsequence of $(\alpha(n))_{n=1}^\infty$, where $(\alpha(n))_{n=1}^\infty$ was defined in connection with Lemma 2.

Proof. We shall consider only the more complicated case when the set $\{\alpha\leqslant\beta\colon \alpha$ is a limit number$\}$ is infinite. Let $N=\bigcup\{N_j\colon j=0,1,2,\ldots\}$ be a decomposition of N such that elements of it are infinite and pairwise disjoint. Let $(\alpha_j)_{j=1}^\infty$ be the sequence consisting of all limit numbers not greater than β. For every $j\in N$, there is a subsequence $(c_k^{\alpha_j})_{k=1}^\infty$ of natural numbers such that if i and $i'\in N$ and $i\neq i'$ then $\{\alpha_i(c_k^{\alpha_i})\colon k\in N\}\cap\{\alpha_{i'}(c_k^{\alpha_{i'}})\colon k\in N\}=\emptyset$. Write $\{\beta_i\colon i\in N\}=\beta\setminus\{\alpha_j(c_k^{\alpha_j})\colon k,j\in N\}$. Let us put $(c_k^{\alpha_1})_{k=1}^\infty=(s_k^{\alpha_1})_{k=1}^\infty$ and $n_1^{\alpha_1}=\inf N_1$. If $n_1^{\alpha_1},\ldots,n_k^{\alpha_1}$ are defined then put $n_{k+1}^{\alpha_1}=\inf\{n\in N_1\colon n>n_k^{\alpha_1}\}$. Write

$$n_{\beta_1}=\begin{cases}\inf N_0, & \text{if } \beta_1\geqslant\alpha_1,\\ \inf\{n\in N_0\colon n>n_k^{\alpha_1}, \text{ where } k=\inf\{k'\in N\colon \beta_1<\alpha_1(s_{k'}^{\alpha_1})\}, & \text{if } \beta_1<\alpha_1\end{cases}$$

and $h_\beta(n_{\beta_1})=\beta_1$. Let us assume that $(n_k^{\alpha_1})_{k=1}^\infty,(s_k^{\alpha_1})_{k=1}^\infty,\ldots,(n_k^{\alpha_j})_{k=1}^\infty,(s_k^{\alpha_j})_{k=1}^\infty$ are defined and the function h_β^{-1} is described on the set $\{\beta_1,\ldots,\beta_j\}\cup\{\alpha_{j'}(c_k^{\alpha_{j'}})\colon k\in N$ and $j'\leqslant j\}$ in such a way that the conditions (a) and (b) are satisfied. Write

$$k_i=\begin{cases}0, & \text{if } \alpha_{j+1}>\alpha_i,\\ \inf\{k\in N\colon \alpha_{j+1}<\alpha_i(s_k^{\alpha_i})\}, & \text{if } \alpha_{j+1}<\alpha_i\end{cases}$$

for $i\leqslant j$, and

$k'=\inf\{t\in N\colon$ for every $i\leqslant j$ such that if $\alpha_i<\alpha_{j+1}$ then $\alpha_{j+1}(c_t^{\alpha_{j+1}})>\alpha_i$, if $T=\{\beta_1,\ldots,\beta_j\}\cup\{\{\alpha_i(c_k^{\alpha_i})\colon k\in N\}\setminus\{\alpha_i(s_k^{\alpha_i})\colon k\in N\}\colon i\leqslant j\}$, $\partial\in T$ and $\partial<\alpha_{j+1}$ then $\partial<\alpha_{j+1}(c_t^{\alpha_{j+1}})\}$.

Let us put $s_k^{\alpha_{j+1}}=c_{k+k'-1}^{\alpha_{j+1}}$, for $k\in N$ and $n_1^{\alpha_{j+1}}=\inf\{n\in N_{j+1}\colon$ for every $i\leqslant j$, if $k_i>0$ then $n_1^{\alpha_{j+1}}>n_{k_i}^{\alpha_i}\}$. If $n_1^{\alpha_{j+1}},\ldots,n_k^{\alpha_{j+1}}$ are defined then put $n_{k+1}^{\alpha_{j+1}}=\inf\{n\in N_{j+1}\colon n>n_k^{\alpha_{j+1}}\}$. If $\partial\in\{\beta_{j+1}\}\cup\{\alpha_{j+1}(c_k^{\alpha_{j+1}})\colon k<k'\}$ then write $n_\partial=\inf\{n\in N_0\colon h_\beta(n)$ is not defined, if $\partial<\alpha_i$, for $i\leqslant j+1$, then there is $t_i\in N$ such that $\alpha_i(s_{t_i}^{\alpha_i})>\partial$ and $n>n_{t_i}^{\alpha_i}\}$ and put $h_\beta(n_\partial)=\partial$. If the domain of h_β is equal to N' and $N'\neq N$ then it is enough to replace h_β by the composition $h_\beta\circ h$, where h is a one-to-one function from N onto N' preserving the order of natural numbers.

Construction of the space X and M from Example 2. Write $A = \{\alpha_\lambda\colon\ \lambda \in \{-1\} \cup \omega_1\}$ where

$$\alpha_\lambda = \begin{cases} 0, & \text{if } \lambda = -1,\\ \alpha_\theta + \omega^2, & \text{if } \lambda = \theta+1,\\ \sup\{\alpha_\theta\colon\ \theta < \lambda\}, & \text{if } \lambda \text{ is a limit number.}\end{cases}$$

If $\lambda = \theta+1$ then put $\alpha_\lambda(n) = \alpha_\theta + (n-1)\omega + 1$, for $n \in N$. If λ is a limit number then let us attach to it a monotonically increasing sequence $(\lambda(n))_{n=1}^{\infty}$ of non-limit ordinal numbers converging to λ in ω_1 and put $\alpha_\lambda(n) = \alpha_{\lambda(n)} + 1$. Let us take $A' = \{a \in D^{\omega_1}\colon\ |a^{-1}(1)| < \omega\} \cup \{a_{\alpha_\lambda} \in D^{\omega_1}\colon\ \alpha_\lambda \in A$ and $a_{\alpha_\lambda}^{-1}(1) = \{\alpha_\lambda(n)\colon\ n \in N\}\}$. If the sequences $(\beta(n))_{n=1}^{\infty}$ for $\beta \in A$ are defined as above then A' is a subset of A. Let us consider the set A' with the topology of the subspace of A and put $X = A'_0 \oplus A'_1$, where A'_0 and A'_1 are copies of A'. Notice that A' is a closed subspace of A and A is a Lindelöf P-space so by the Noble's theorem (see [N]) the product X^ω is Lindelöf. From the fact that X is a Lindelöf P-space it follows very easily that for every Lindelöf space Y and $n \in N$ the product $Y \times X^n$ has the Lindelöf property.

Put $M = C$, where C is a coanalytic subset of the Cantor set which is not a Borel set.

The description of an uncountable subset L of $C \times X^\omega$ without points of condensation is the difficult part of the construction of Example 2. If $m = (m(n))_{n=0}^{\infty} \in M$ then put $X_m = \mathop{\boldsymbol{P}}\limits_{n=0}^{\infty} A'_{m(n)}$. Let us notice that $P = \bigcup\{\{m\} \times X_m\colon\ m \in M\}$ is a closed subset of $M \times X^\omega$. It is enough to define an uncountable set $L = \{l_\lambda\colon\ \lambda \in \{-1\} \cup \omega_1\}$, where $l_\lambda = (m_\lambda, x_\lambda)$, $m_\lambda \in C$, $x_\lambda \in X_{m_\lambda}$ and $m_\lambda \neq m_\beta$, for $\lambda \neq \beta$, without points of condensation in P. The set L will be defined by the transfinite induction with respect to $\lambda \in \{-1\} \cup \omega_1$.

If $x \in X$ then there is $r(x) \in D$ such that $x \in A'_{r(x)}$. If $x \in X^n$, where $n \in N$, $x = (x(0), \ldots, x(n-1))$ then $r(x) = (r(x(0)), \ldots, r(x(n-1)))$. The symbols $a^0_{\alpha_\lambda}$, $a^1_{\alpha_\lambda}$, for $\lambda \in \omega_1$, will denote elements of A'_0 and A'_1 respectively which correspond to a_{α_λ} of A'.

In every step of induction we shall also define some conditions which will restrict our freedom of choice of $l_\lambda = (m_\lambda, x_\lambda)$ in the consecutive steps of induction. In the sequel these conditions will be called restrictions. The restrictions defined in the steps precedent to the ∂-step will ensure that every point $(m, x(n)_{n=0}^{\infty})$ of $M \times X^\omega$, where $m \in M$ and $x(n) \notin \{a^0_{\alpha_\partial}, a^1_{\alpha_\partial}\}$, for $n \in \omega$, will not be a point of condensation of L. Notice that $x(n) \notin \{a^0_{\alpha_\partial}, a^1_{\alpha_\partial}\}$ is equivalent to the fact that $\alpha_\partial \notin \overline{x(n)^{-1}(1) \cap \alpha_\partial}^{\omega_1}$, where the closure operation is taken with respect to the order topology of ω_1. The role of restrictions will play some Borel subsets of the Cantor set. These subsets will be denoted by the symbols $R(\alpha_\partial + n\omega, x)$, where $\partial \in \{-1\} \cup \omega_1$, $x \in X^n$ and $n \in N$. The set $R(\alpha_\partial + n\omega, x)$ will depend only on $x|\alpha_\partial + n\omega$. We shall say that the point (m, x'), where $m \in D^\omega$ and $x' \in X_m$ is consistent with the restriction $R(\alpha_\partial + n\omega, x)$ or that (m, x') satisfies the condition $R(\alpha_\partial + n\omega, x)$ if $(m, x') \in \{l_\beta\colon\ \beta \leqslant \partial\}$ or if $(x'|n)|\alpha_\partial + n\omega = x|\alpha_\partial + n\omega$ and $r(x'|n) = r(x)$ then $m \in R(\alpha_\partial + n\omega, x)$. The set $R(\alpha_\partial + n\omega, x)$ will be defined in the ∂-step of induction.

The points of L will be defined in such a way that they will be consistent with defined restrictions.

Write $B = D^\omega \setminus C$ and for $n \in N$ and $p = (p(0), \ldots, p(n-1)) \in (N_0 \oplus N_1)^n$ put $H(p) = f\big((\overline{\{p(0)\}} \times \ldots \times \{p(n-1)\} \times (N_0 \oplus N_1) \times \ldots \times (N_0 \oplus N_1) \times \ldots) \cap B_1\big)$ and $Z(p) = \overline{H(p)}$, where f and B_1 are from Lemma 1 and the closure operation is taken with respect to the topology of the Cantor set. Let us notice that

(1) if $p \in (N_0 \oplus N_1)^\omega$ and $Z(p|n)$ is not empty, for $n \in N$, then $f(p) \in B$.

The claim (1) follows from the fact that B_1 is a closed subset of $(N_0 \oplus N_1)^\omega$.

We shall apply the claim (1) in order to destroy points of condensation of L in $M \times X^\omega$.

If $x \in X^n$ then put

$$i_\lambda(x) = \begin{cases} \inf\{0 \leqslant j < n\colon\ x(j) \in \{a^0_{\alpha_\lambda}, a^1_{\alpha_\lambda}\}\}, & \text{if } \{0 \leqslant j < n\colon\ x(j) \in \{a^0_{\alpha_\lambda}, a^1_{\alpha_\lambda}\}\} \neq \emptyset,\\ -1, & \text{if } \{0 \leqslant j < n\colon\ x(j) \in \{a^0_{\alpha_\lambda}, a^1_{\alpha_\lambda}\}\} = \emptyset.\end{cases}$$

Let us put, for $n \in N$, $H_n = \{H(p)\colon\ p \in (N_0 \oplus N_1)^n$ and $H(p) \neq \emptyset\}$, $P_n = \{p \in (N_0 \oplus N_1)^n\colon\ H(p) \in H_n\}$ and $F_n = \{Z(p)\colon\ p \in P_n$ and $Z(p) = \overline{H(p)}^{D^\omega}\}$. If $x \in X^n$, $\partial \in \omega_1$ then ${}_\partial x$ will stand for an element of X^n such that $r({}_\partial x) = r(x)$, ${}_\partial x|\partial = x|\partial$ and $({}_\partial x(j))^{-1}(1) \subset \partial$, for $0 \leqslant j < n$. In the sequel we shall write $x|\partial = y|\partial$, for $\partial \in \omega_1$, $n \in N$ and $x, y \in X^n$ if $r(x) = r(y)$ and $x|\partial = y|\partial$.

For $n \in N$, $\partial \in \{-1\} \cup \omega_1$, $x \in X$, let us put $[x]_{n,\partial} = \{y \in X\colon\ y|\alpha_\partial = x|\alpha_\partial$ and for every $k \in N$ and greater than n we have $y^{-1}(1) \cap ((\alpha_\partial + k\omega) \setminus (\alpha_\partial + (k-1)\omega)) \neq \emptyset$ if and only if $x^{-1}(1) \cap ((\alpha_\partial + k\omega) \setminus (\alpha_\partial + (k-1)\omega)) \neq \emptyset\}$.

First step of induction. Let $l_{-1} = (m_{-1}, x_{-1})$, where m_{-1} is an arbitrary element of C and x_{-1} of $X_{m_{-1}}$.

Write $E_1 = \{x \in X\colon\ {}_{\alpha_0}x^{-1}(1) \subset \omega\}$ and let w_1 be a function $w_1\colon E_1 \to P_1$ from E_1 onto P_1 such that for every $p \in P_1$ the set $w_1^{-1}(p)$ is infinite and, for every $x \in X$, $w_1(x) \in N_{r(x)}$. Let us assume that

$(1_{r,-1})$ For every $x \in X$, $R(\omega, x) = R(\omega, {}_\omega x) = Z(w_1({}_\omega x))$

and

$$(2_{r,-1})\quad I^0(x) = \begin{cases} w_1(x), & \text{if } x \in E_1,\\ \emptyset, & \text{otherwise}\end{cases} \quad \text{for } x \in X.$$

Let us assume that, for $x \in X^n$, $R(n\omega, x)$ and $I^0(x) = (I_0^0(x), \ldots, I_{n-1}^0(x))$ are defined in such a way that the following conditions are satisfied:

$(3_{r,-1})$ For $0 \leqslant i < n$ and $x \in X^n$, $I_i^0(x) \in \bigcup\{P_j\colon j \in N$ and $j \leqslant i+1\} \cup \{\emptyset\}$.

$(4_{r,-1})$ For $x \in X^n$ and $n \geqslant 2$, $I^0(x)$ is an extension of $I^0(x|n-1)$.

$(5_{r,-1})$ If $j < n-1$, $x \in X^n$, $I_j^0(x) = p$, where $p \in P_j$, then

$$I_{j+1}^0(x) = \begin{cases} q, \text{ where } q \in \{q' \in P_{j'+1}\colon q'|j' = p,\ H(q') \cap R((j+1)\omega, \\ \qquad x|j+1) \cap B(r(x(0)) \ldots r(x(j+1))) \neq \emptyset\} \\ \qquad \text{if this set is not empty and } ({}_{\alpha_0}x(j'))^{-1}(1) \subset (j+2)\omega, \\ \emptyset \quad \text{otherwise.} \end{cases}$$

$(6_{r,-1})$ If $x \in X^n$ and for every $j < n-1$, $I_j^0(x) = \emptyset$ then

$$I_{n-1}^0(x) = \begin{cases} q, \text{ where } q \in \{q' \in P_1\colon H(q') \cap R((n-1)\omega, x|n-1) \cap \\ \qquad \cap B(r(x(0)) \ldots r(x(n-1))) \neq \emptyset\} \\ \qquad \text{if this set is not empty and } ({}_{\alpha_0}x(0))^{-1}(1) \subset n\omega, \\ \emptyset \quad \text{otherwise.} \end{cases}$$

$(7_{r,-1})$ For every $x \in X^n$, $R(n\omega, x) = R(n\omega, {}_{n\omega}x)$.

$(8_{r,-1})$ Let us assume that $n > 1$, $x \in X^n$, $I^0(x) \neq (\emptyset, \ldots, \emptyset)$, $\sup\{j < n\colon I_j^0(x) \neq \emptyset\} = j_0$, $I_{j_0}^0(x) = p$, where $p \in P_{j_0'}$, and put, for $i < j_0'$, $k_i = \sup\{s \in N\colon x(i)^{-1}(1) \cap (s\omega \setminus (s-1)\omega) \neq \emptyset\}$, $s_0 = k_0$, $s_1 = \sup\{(k_0+1), k_1\}, \ldots$ $\ldots, s_{j_0'-1} = \sup\{(s_{j_0'-2}+1), k_{j_0'-1}\}$. If $y \in X^n$ such that ${}_{\alpha_0}y|j_0' = {}_{\alpha_0}x|j_0'$ and ${}_{s_j\omega}y|s_j = {}_{s_j\omega}x|s_j$, for $j < j_0'$, then $I^0(y|j_0+1) = I^0(x|j_0+1)$.

$(9_{r,-1})$ If $x \in X^n$ then

$$R(n\omega, x) = \begin{cases} R((n-1)\omega, x|n-1) \cap B(r(x(0)) \ldots r(x(n-1))) \cap Z(p), \\ \qquad \text{if } p = I_{n-1}^0({}_{n\omega}x), \\ \emptyset, \quad \text{if } I_{n-1}^0({}_{n\omega}x) = \emptyset. \end{cases}$$

$(10_{r,-1})$ For every $x \in X^n$, $\overline{R(n\omega, x)}^{D^\omega} \subset \bigcap\{Z(p)\colon$ there is $i < n$ such that $I_i^0(x) = p\}$ if $I^0(x) \neq (\emptyset, \ldots, \emptyset)$.

$(11_{r,-1})$ $R(n\omega, x)$ is a Borel set in D^ω, for $x \in X^n$; in fact $R(n\omega, x)$ is compact.

$(12_{r,-1})$ Let n be greater than 1, $i < n$, $x = {}_{n\omega}x \in X^{n-1}$, $S_i(x) = \{y \in X^n\colon y(j) = x(j)$, for $j < i$, and $y(j) = x(j-1)$, for $j > i\}$. For every $y_0 = {}_{n\omega}y_0 \in S_i(x)$ and for every q such that $q \in P_1$ if $I^0(y_0|n-1) = (\emptyset, \ldots, \emptyset)$ or $q \in P_{j'+1}$, where $j_0 = \sup\{j < n-1\colon I_j^0(y_0|n-1) \neq \emptyset\}$, $I_{j_0}^0(y_0|n-1) = p \in P_{j'}$ and $q|j' = p$, and $R((n-1)\omega, y_0|n-1) \cap H(q) \cap B(r(y_0(0)) \ldots r(y_0(n-1))) \neq \emptyset$, the set

$$\{{}_{n\omega}y\colon y \in S_i(x),\ q = I_{n-1}^0({}_{n\omega}y),\ y(i) \in [y_0(i)]_{i+1,-1},$$
$$I^0({}_{j\omega}y|j) = I^0({}_{j\omega}y_0|j) \text{ and } R(j\omega, y|j) = R(j\omega, y_0|j), \text{ for } j \leqslant n-1\}$$

is infinite.

Let us notice that if we define $R((n+1)\omega, y)$ and $I^0(y)$, for $y \in X^{n+1}$ and $y = {}_{(n+1)\omega}y$, in such a way that the conditions $(3_{r,-1})$–$(12_{r,-1})$ will be satisfied then the conditions $(3_{r,-1})$–$(9_{r,-1})$ will determine $R((n+1)\omega, y)$ and $I^0(y)$ for the remained points of X^{n+1}.

Write $S = \{S_i'(x)\colon i \leqslant n,\ x \in X^n$ and ${}_{(n+1)\omega}x = x\}$ where $S_i'(x) = \{y \in S_i(x)\colon {}_{(n+1)\omega}y = y\}$. Let us notice that the set $\{y \in X\colon {}_{(n+1)\omega}y = y\}$ is countable so also S is countable and it consists of countable and infinite sets. If $S_i'(x)$, $S_{i'}'(x')$ belong to S and $S_i'(x) \neq S_{i'}'(x')$ then the intersection $S_i'(x) \cap S_{i'}'(x')$ is finite. Let us order $S = \{O_k\colon k \in N\}$. Let us assume that $R((n+1)\omega, y)$ and $I^0(y)$ are defined for $y \in \bigcup\{O_k\colon k \leqslant k'\}$. Let us assume that $y_0 \in O_{k'+1} = S_i'(x)$, where $i \leqslant n$ and $x \in X^n$. Then the set

$$D(y_0) = \{y \in S_i'(x) \setminus \bigcup\{O_k\colon k \leqslant k'\}\colon y(i) \in [y_0(i)]_{i+1,-1},$$
$$I^0({}_{j\omega}y|j) = I^0({}_{j\omega}y_0|j) \text{ and } R(j\omega, y|j) = R(j\omega, y_0|j), \text{ for } j \leqslant n\}$$

is infinite. If $i = n$ then it follows from the definition of $S_j'(x)$; let us recall that $S_i'(x) \cap (\bigcup\{O_k\colon k \leqslant k'\})$ is finite. If $i < n$ then it follows from the inductive assumption (see $(12_{r,-1})$ and $(8_{r,-1})$). Let us notice that from the definition of $D(y_0)$ and from the conditions $(5_{r,-1})$, $(6_{r,-1})$ and $(8_{r,-1})$ it follows that for every $y \in D(y_0)$

$$I^0({}_{(n+1)\omega}y|n) = I^0(y|n) = I^0({}_{(n+1)\omega}y_0|n) = I^0(y_0|n).$$

Write

$$P(y_0) = \{q \in \bigcup\{P_j\colon j \in N\}\colon R(n\omega, y_0|n) \cap H(q) \cap B(r(y_0(0)) \ldots r(y_0(n))) \neq \emptyset,$$
if $I^0(y_0|n) = (\emptyset, \ldots, \emptyset)$ then $q \in P_1$, if $I^0(y_0|n) \neq (\emptyset, \ldots, \emptyset)$ then $q \in P_{j'+1}$ and $q|j' = p$, where $p = I_{j_0}^0(y_0|n) \in P_{j'}$, and $j_0 = \sup\{j < n\colon I_j^0(y_0|n) \neq \emptyset\}\}$.

If $P(y_0) = \emptyset$ then $R((n+1)\omega, y) = I_n^0(y) = \emptyset$, for every $y \in D(y_0)$. Assume that $P(y_0) \neq \emptyset$. Let v be a function from $D(y_0)$ onto $P(y_0)$ such that for every $q \in P(y_0)$ the set $v^{-1}(q)$ is infinite. Let us put $I_n^0(y) = v(y)$ and $R((n+1)\omega, y) = R(n\omega, y|n) \cap B(r(y(0)) \ldots r(y(n))) \cap Z(v(y))$, for $y \in D(y_0)$. This completes the construction of $R((n+1)\omega, y)$ and $I^0(y)$, for $y \in X^{n+1}$.

Let $y = (y(n))_{n=0}^{\infty}$ be an element of X^ω such that $\alpha_0 \notin \overline{y(n)^{-1}(1) \cap \alpha_0}^{\omega_1}$ and $m \in C$. We shall show that (m, y) will not be a point of condensation of L. Write, for $n \in \omega$, $k_n = \sup\{s \in N\colon y(n)^{-1} \cap (s\omega \setminus (s-1)\omega) \neq \emptyset\}$ and put $s_0 = k_0$, $s_1 = \sup\{k_1, (s_0+1)\}, \ldots, s_n = \{k_n, (s_{n-1}+1)\}, \ldots$ If there is $n \in N$ such that $R(n\omega, y|n) = \emptyset$ then, according to the condition attached to $R(n\omega, y|n)$, the point (m, y) will not be a point of condensation of L. Let us assume that $R(n\omega, y|n) \neq \emptyset$ for every $n \in N$. From $(6_{r,-1})$ it follows that there is $p(0) \in P_1$ such that $I_{s_0-1}^0(y|s_0) = p(0)$. Let us assume that $p(0) \ldots p(n)$ are defined. Then by $(5_{r,-1})$ we infer that there is $p(n+1) \in N_{r(y(n+1))}$ such that

$p_{n+2} = (p(0), \ldots, p(n+1)) \in P_{n+2}$ and $I^0_{s_{n+1}-1}(y|s_{n+1}) = p_{n+2}$. From $(9_{r,-1})$ it follows that

$$R(s_n\omega, y|s_n) \subset Z(p_{n+1}) \cap B\big(r(y(0)) \ldots r(y(s_n-1))\big) \neq \emptyset.$$

By (1) we infer that $(r(y_n))_{n=0}^{\infty} = f(p(0), \ldots, p(n), \ldots) \in B$ so (m, y) will not be a point of condensation of L.

Let us assume that we have already defined $R(\alpha_\partial + n\omega, x)$, $I^{\partial+1}(x) = (I_0^{\partial+1}(x), \ldots, I_{n-1}^{\partial+1}(x))$ and $l_\partial = (m_\partial, x_\partial)$ of L, for $\partial < \beta$, $n \in N$, $x \in X^n$, in such a way that for every $\partial < \beta$ the point l_∂ is consistent with defined restrictions and the following conditions are satisfied, for $\partial < \beta$ and $n \in N$,

$(1_{r,\partial})$ If $x \in X^n$ and $i_\partial(x) \in \{-1, n-1\} \setminus \{0\}$ then $R(\alpha_\partial + n\omega, x) = R(\alpha_{\partial'} + n\omega, x)$, where

$$\partial' = \begin{cases} \sup\{\lambda < \partial:\ -1 < i_\lambda(x) < n\}, & \text{if } \{\lambda < \partial:\ -1 < i_\lambda(x) < n\} \neq \emptyset, \\ -1, & \text{if } \{\lambda < \partial:\ -1 < i_\lambda(x) < n\} = \emptyset \end{cases}$$

and $I^{\partial+1}(x) = (\emptyset, \ldots, \emptyset)$, let us recall that $\alpha_{-1} = 0$.

If $x \in X$ and $x \in \{a^0_{\alpha_\partial}, a^1_{\alpha_\partial}\}$ then $R(\alpha_\partial + \omega, x) = I^{\partial+1}(x) = \emptyset$.

$(2_{r,\partial})$ For every $i < n$ and $x \in X^n$ $I_i^{\partial+1}(x) \in \bigcup\{P_j:\ j \in N$ and $j \leqslant i+1\} \cup \{\emptyset\}$.

$(3_{r,\partial})$ For every $x \in X^n$ and $n \geqslant 2$ $I^{\partial+1}(x)$ is an extension of $I^{\partial+1}(x|n-1)$. If there exists $t \in N$ such that $t < n$ and $\{x(t-1), x(t)\} \subset \{a^k_{\alpha_\lambda}:\ k \in D,\ \lambda \leqslant \partial+1\}$ then $I^{\partial+1}_{n-1}(x) = \emptyset$.

$(4_{r,\partial})$ If $x \in X^n$, $-1 < i_\partial(x) < n-1$, $\{x(t-1), x(t)\} \not\subset \{a^k_{\alpha_\lambda}:\ k \in D,\ \lambda \leqslant \partial+1\}$ and $I^{\partial+1}(x|n-1) = (\emptyset, \ldots, \emptyset)$ then

$$I^{\partial+1}_{n-1}(x) \in \{q \in P_1:\ H(q) \cap R(\alpha_\partial + (n-1)\omega, x|n-1) \cap R(\alpha_{\partial'} + n\omega, x) \neq \emptyset\}$$

if this set is not empty, $({}_{\alpha_{\partial+1}}x(0))^{-1}(1) \subset \alpha_\partial + n\omega$ and $({}_{\alpha_{\partial+1}}x(0))^{-1}(1) \not\subset \alpha_\partial$, where ∂' is defined in the same way as in $(1_{r,\partial})$; $I^{\partial+1}_{n-1}(x) = I^{\partial''+1}_k(x)$, where $\partial'' = \sup\{\lambda < \partial:\ -1 < i_\lambda(x) < i_\partial(x)\}$ if $\{\lambda < \partial:\ -1 < i_\lambda(x) < i_\partial(x)\} \neq \emptyset$ or $\partial'' = -1$ otherwise, if $I^{\partial''+1}(x) \neq (\emptyset, \ldots, \emptyset)$, $({}_{\alpha_{\partial+1}}(x)(0))^{-1}(1) \subset \alpha_\partial$ and $i_\partial(x) \neq 0$, where $I^{\partial''+1}_k(x) = \{I^{\partial''+1}_0(x), \ldots, I^{\partial''+1}_{n-1}(x)\} \cap P_1$; $I^{\partial+1}_{n-1}(x) = \emptyset$ if none of the above cases hold.

$(5_{r,\partial})$ If $x \in X^n$, $I^{\partial+1}_{j-1}(x) = p$, where $p \in P_{j'}$ and $j \leqslant n-1$ then $I^{\partial+1}_j(x) \in \{q \in P_{j'+1}:\ q|j' = p,\ H(q) \cap R(\alpha_\partial + j\omega, x|j) \cap R(\alpha_{\partial'} + (j+1)\omega, x|j+1) \neq \emptyset\}$, where ∂' is defined in the same way as in $(1_{r,\partial})$, if this set is not empty, $({}_{\alpha_{\partial+1}}x(j'))^{-1}(1) \subset \alpha_\partial + (j+1)\omega$ and the following condition is satisfied:

$\{x(j-1), x(j)\} \not\subset \{a^k_{\alpha_\lambda}:\ k \in D,\ \lambda \leqslant \partial+1\}$,

$({}_{\alpha_{\partial+1}}x|j'+1 \neq {}_{\alpha_\partial}x|j'+1$ or $i_\partial(x) \leqslant j')$;

$I^{\partial+1}_j(x) = \{I^{\partial''+1}_0(x), \ldots, I^{\partial''+1}_j(x)\} \cap P_{j'+1}$, where ∂'' is defined in the same way as in $(4_{r,\partial})$, if ${}_{\alpha_{\partial+1}}x|j'+1 = {}_{\alpha_\partial}x|j'+1$ and $i_\partial(x) > j'$; $I^{\partial+1}_j(x) = \emptyset$ if none of the above cases hold.

Remark 1. Notice that if x and j' are taken from $(5_{r,\partial})$ and $\{x(j'-1), x(j')\} \subset \{a^k_{\alpha_\lambda}:\ k \in D,\ \lambda \leqslant \partial\}$ then $I^{\partial+1}_{j+1}(x) = \emptyset$.

$(6_{r,\partial})$ $R(\alpha_\partial + n\omega, x) = R(\alpha_\partial + n\omega, {}_{\alpha_\partial+n\omega}x)$, for $x \in X^n$ and $n \in N$.

$(7_{r,\partial})$ Let us assume that $x \in X^n$, $I^{\partial+1}(x) \neq (\emptyset, \ldots, \emptyset)$ and $(I^{\partial+1}_{i_1}(x), \ldots, I^{\partial+1}_{i_j}(x))$ is a subsequence of $I^{\partial+1}(x)$ consisting of all its non-empty elements. If $y \in X^n$, ${}_{\alpha_{\partial+1}}y|j = {}_{\alpha_{\partial+1}}x|j$ and ${}_{\alpha_\partial+(i_s+1)\omega}y|i_s+1 = {}_{\alpha_\partial+(i_s+1)\omega}x|i_s+1$, for $s \leqslant j$, then $I^{\partial+1}(y|i_j+1) = I^{\partial+1}(x|i_j+1)$.

$(8_{r,\partial})$ If $x \in X^n$, $n \geqslant 2$ and $i_\partial(x) < n-1$ then

$$R(\alpha_\partial + n\omega, x) = R\big(\alpha_\partial + (n-1)\omega, x|n-1\big) \cap R(\alpha_{\partial'} + n\omega, x) \cap Z(p),$$

where $p = I^{\partial+1}_{n-1}({}_{\alpha_\partial+n\omega}x)$ and ∂' is defined in the same way as in $(1_{r,\partial})$, if $I^{\partial+1}_{n-1}({}_{\alpha_\partial+n\omega}x) \neq \emptyset$; $R(\alpha_\partial + n\omega, x) = R(\alpha_{\partial'} + n\omega, x)$ if ${}_{\alpha_\partial+n\omega}x|j+1 = {}_{\alpha_\partial}x|j+1$, where $j = 0$ if $\{I^{\partial+1}_0(x), \ldots, I^{\partial+1}_{n-1}(x)\} \cap P_1 = \emptyset$ or $j = \sup\{s \in N:\ \{I^{\partial+1}_0(x), \ldots, I^{\partial+1}_{n-1}(x)\} \cap P_s \neq \emptyset\}$ otherwise, and $i_\partial(x) > j$; $R(\alpha_\partial + n\omega, x) = \emptyset$ if none of the above cases hold.

Remark 2. Notice that from $(5_{r,\lambda})$ and $(8_{r,\lambda})$, for $\lambda \leqslant \partial$, it follows that if

$$j = \begin{cases} -1, & \text{if } \{I^{\partial+1}_0(x), \ldots, I^{\partial+1}_{n-1}(x)\} \cap P_1 = \emptyset, \\ \sup\{s \in n:\ I^{\partial+1}_s(x) \neq \emptyset\}, & \text{otherwise} \end{cases}$$

is less than $n-1$, where $n \in N$ and $x \in X^n$, and

$$j' = \begin{cases} 0, & \text{if } j = -1, \\ s, \quad \text{where } I^{\partial+1}_j(x) \in P_s, & \text{if } j \neq 0 \end{cases}$$

then $R\big(\alpha_\partial + (j+2)\omega, x|j+2\big) = \emptyset$ if $j' \neq 0$ and $\{x(j'-1), x(j')\} \subset \{a^k_{\alpha_\lambda}:\ k \in D,\ \lambda \leqslant \partial\}$.

$(9_{r,\partial})$ The closure of $R(\alpha_\partial + n\omega, x)$ with respect to the topology of the Cantor set is included in $\bigcap\{Z(p):$ there is $i \in n$ such that $I^{\partial+1}_i(x) = p\}$, for $x \in X^n$, if $I^{\partial+1}(x) \neq (\emptyset, \ldots, \emptyset)$.

$(10_{r,\partial})$ $R(\alpha_\partial + n\omega, x)$ is a Borel subset of the Cantor set, for $x \in X^n$.

$(11_{r,\partial})$ If $n > 2$, $i \in n$, $x = {}_{\alpha_\partial+n\omega}x \in X^{n-1}$, $0 \leqslant i_\partial(x) \leqslant i-1$ and $i_\lambda(x) \neq 0$, for $\lambda \leqslant \partial$, then put $S_i(x) = \{y \in X^n:$ for $j \in i$ $y(j) = x(j)$, for $i < j \leqslant n-1$, $y(j) = x(j-1)$ and $y(i) \notin \{a^k_{\alpha_\lambda}:\ \lambda \leqslant \partial,\ k \in D\}$ and for every $y_0 \in S_i(x)$, $y_0 = {}_{\alpha_\partial+n\omega}y_0$, write

$$j(y_0) = \begin{cases} 0, \quad \text{if } \{I^{\partial+1}_0({}_{\alpha_\partial+n\omega}y_0(0)), \ldots, I^{\partial+1}_{n-2}({}_{\alpha_\partial+n\omega}y_0(n-2))\} \cap P_1 = \emptyset, \\ \sup\{s \in N:\ \{I^{\partial+1}_0({}_{\alpha_\partial+n\omega}y_0(0)), \ldots, I^{\partial+1}_{n-2}({}_{\alpha_\partial+n\omega}y_0(n-2))\} \cap P_s \neq \emptyset\}, \\ \hfill \text{otherwise.} \end{cases}$$

If $i_\partial(y_0) \leqslant j(y_0)$ or ${}_{\alpha_\partial}y_0|j(y_0)+1 \neq {}_{\alpha_\partial+n\omega}y_0|j(y_0)+1$ then for every $q \in P_{j(y_0)+1}$ such that q is an extension of

$$\{I^{\partial+1}_0({}_{\alpha_\partial+n\omega}y_0(0)), \ldots, I^{\partial+1}_{n-2}({}_{\alpha_\partial+n\omega}y_0(n-2))\} \cap P_{j(y_0)}, \quad \text{if} \quad j(y_0) > 0,$$

and

$$R(\alpha_\partial+(n-1)\omega,\ y_0|n-1)\cap H(q)\cap R(\alpha_{\partial'}+n\omega,\ y_0)\neq\emptyset,$$

where ∂' is defined in the same way as in $(1_{r,\partial})$, the set

$$\begin{aligned}\{{}_{\alpha_\partial+n\omega}y:\ & y\in S_i(x),\ y(i)\in[y_0(i)]_{i+1,\partial},\ q=I^{\partial+1}_{n-1}({}_{\alpha_\partial+n\omega}y),\\ & I^{\partial+1}({}_{\alpha_\partial+j\omega}y|j)=I^{\partial+1}({}_{\alpha_\partial+j\omega}y_0|j) \text{ and } R(\alpha_\partial+j\omega,\ y|j)\\ & = R(\alpha_\partial+j\omega,\ y_0|j),\ \text{for } 1\leqslant j<n\}\end{aligned}$$

is infinite.

Remark 3. Let us notice that if $y_0\in X^n$ and $(11_{r,\partial})$ does not apply to it then the definition of $R(\alpha_\partial+n\omega,\ y_0)$ and $I^{\partial+1}_{n-1}(y)$ follows from $(8_{r,\partial})$, $(1_{r,\partial})$, $(4_{r,\partial})$ $(5_{r,\partial})$ and $(7_{r,\partial})$.

Let $y=(y(n))_{n=0}^{\infty}$ be an element of X^ω such that $\alpha_\beta\notin\overline{y(n)^{-1}(1)\cap\alpha_\beta}^{\omega_1}$, it is equivalent to the fact that $y(n)\notin\{a^0_{\alpha_\beta},\ a^1_{\alpha_\beta}\}$, and $m\in C$. We shall show that $(m,\ y)$ will not be a point of condensation of L. If there is $n\in N$ and $\partial<\beta$ such that $R(\alpha_\partial+n\omega,\ y|n)=\emptyset$ then, according to the condition attached to $R(\alpha_\partial+n\omega,\ y|n)$ the point $(m,\ y)$ will not be a point of condensation of L. Let us assume that, for $n\in N$ and $\partial<\beta$, $R(\alpha_\partial+n\omega,\ y|n)\neq\emptyset$. From (1) and the definition of f it follows that it is enough to show that there are sequences $(p_n)_{n=0}^{\infty}$, $(\partial_n)_{n=0}^{\infty}$ and $(s_n)_{n=0}^{\infty}$ such that $p_n\in P_{n+1}$, $p_{n+1}|n+1=p_n$, $\partial_n\in\{-1\}\cup\beta$, $s_n\in N$, and $R(\alpha_{\partial_n}+s_n\omega,\ y|s_n)\subset Z(p_n)\cap B\big(r(y(0))\ldots r(y(s_n-1))\big)$. By $(1_{r,\partial})$ for $\partial<\beta$ we infer that the set $({}_{\alpha_\beta}y(0))^{-1}(1)$ is finite. Put $\Gamma=\{\partial\in\beta$: there is $s\in\omega$ such that $y(s)\in\{a^0_{\alpha_\partial},\ a^1_{\alpha_\partial}\}\}$. If $\Gamma\neq\beta$ then by the inductive assumption $(m,\ y)$ is not a point of condensation of L. Let us assume that $\Gamma=\beta$. Write $j_0=\inf\{i\in N$: there is $\partial_0\in\beta$ and $s\in N$ such that $y(i)\in\{a^0_{\alpha_{\partial_0}},\ a^1_{\alpha_{\partial_0}}\}$ and $({}_{\alpha_\beta}y(0)^{-1}(1))\subset\alpha_{\partial_0}+s\omega\}$. If $({}_{\alpha_\beta}y(0))^{-1}(1)\not\subset\alpha_{\partial_0}$ then put

$$s'_0=\inf\{s\in N:\ {}_{\alpha_\beta}y(0)^{-1}(1)\subset\alpha_{\partial_0}+s\omega\}.$$

If $y(0)^{-1}(1)\subset\alpha_{\partial_0}$ then put

$$\begin{aligned}W_0=\{\partial\in\Gamma:\ & \text{there are } i<j_0 \text{ and } s\in N \text{ such that } y(i)\in\{a^0_{\alpha_\partial},\ a^1_{\alpha_\partial}\},\\ & y(0)^{-1}(1)\cap\{\lambda<\omega_1:\ \alpha_\partial\leqslant\lambda<\alpha_\partial+s\omega\}\neq\emptyset \text{ and does not exist}\\ & \partial<\lambda'<\beta \text{ such that } i_{\lambda'}(y|i+1)\neq-1\},\end{aligned}$$

$$\theta_0=\begin{cases}\sup W_0, & \text{if } W_0\neq\emptyset,\\ -1, & \text{if } W_0=\emptyset\end{cases}$$

and

$$s'_0=\inf\{s\in N:\ ({}_{\alpha_{\theta_0+1}}y(0))^{-1}(1)\subset\alpha_{\theta_0}+s\omega\}.$$

Let us put $s_0=\sup\{s'_0,\ j_0+2\}$. Using $(4_{r,\partial_0})$ and $(5_{r,\partial_0})$ one can prove that $I^{\partial_0+1}_{s_0-1}(y|s_0)\in P_1$. Put $I^{\partial_0+1}_{s_0-1}(y|s_0)=p_0$. Let us notice that for every $\partial_0\leqslant\partial<\beta$ and $s\in N$

$$I^{\partial+1}_s(y|s)\in\{q\in\bigcup\{P_j:\ j\in N\}:\ q|1=p_0\}\cup\{\emptyset\}.$$

Let us assume that $j_0,\ldots,j_{n-1}$, $s_0,\ldots,s_{n-1}$, $\partial_0,\ldots,\partial_{n-1}$ and $p_0,\ldots,p_{n-1}$ are defined. Put $j_n=\inf\{i\in N:\ i\geqslant j_{n-1}$ and there are $\partial_n\in\Gamma$ and $s\in N$ such that $y(i)\in\{a^0_{\alpha_{\partial_n}},\ a^1_{\alpha_{\partial_n}}\}$ and $({}_{\alpha_\beta}y(n))^{-1}(1)\subset\alpha_{\partial_n}+s\omega\}$. If $({}_{\alpha_\beta}y(n))^{-1}(1)\not\subset\alpha_{\partial_n}$ then put

$$s'_n=\inf\{s\in N:\ ({}_{\alpha_\beta}y(n))^{-1}(1)\subset\alpha_{\partial_n}+s\omega\}.$$

If $({}_{\alpha_\beta}y(n))^{-1}(1)\subset\alpha_{\partial_n}$ then put

$$\begin{aligned}W_n=\{\partial\in\Gamma:\ & \text{there are } i<j_n \text{ and } s\in N \text{ such that}\\ & y(i)\in\{a^0_{\alpha_\partial},\ a^1_{\alpha_\partial}\},\ ({}_{\alpha_\beta}y(n))^{-1}(1)\cap\{\lambda<\omega_1:\ \alpha_\partial<\lambda<\alpha_\partial+s\omega\}\neq\emptyset,\\ & \partial>\sup\{\lambda<\beta:\ i_\lambda(y|i)>-1\},\ \text{if } ({}_{\alpha_\beta}y(n-1))^{-1}(1)\not\subset\alpha_{\partial_{n-1}}\\ & \text{then } \partial\geqslant\partial_{n-1},\ \text{if } ({}_{\alpha_\beta}y(n-1))^{-1}(1)\subset\alpha_{\partial_{n-1}} \text{ then } \partial\geqslant\theta_{n-1}\},\end{aligned}$$

if $({}_{\alpha_\beta}y(n))^{-1}(1)$ is finite; $W_n=\{\partial\}$, if $y(n)\in\{a^0_{\alpha_\partial},\ a^1_{\alpha_0}\}$

$$\theta_n=\begin{cases}\sup W_n, & \text{if } W_n\neq\emptyset,\\ \partial_{n-1}, & \text{if } W_n=\emptyset \text{ and } ({}_{\alpha_\beta}y(n-1))^{-1}(1)\not\subset\alpha_{\partial_{n-1}},\\ \theta_{n-1}, & \text{if } W_n\neq\emptyset \text{ and } ({}_{\alpha_\beta}y(n-1))^{-1}(1)\subset\alpha_{\partial_{n-1}}\end{cases}$$

and

$$s'_n=\inf\{s\in\omega:\ ({}_{\alpha_{\theta_n+1}}y(n))^{-1}(1)\subset\alpha_{\theta_n}+s\omega\}.$$

Put $s_n=\sup\{s'_n,\ j_n+n+2,\ s_{n-1}+1\}$. Using $(4_{r,\partial_n})$ and $(5_{r,\partial_n})$ one can prove that $I^{\partial_n+1}_{s_n-1}(y|s_n)=p_n\in P_{n+1}$ and $p_n|n=p_{n-1}$. Let us notice that, for every $\partial_n\leqslant\partial<\beta$ and $s\in N$, $I^{\partial+1}_{s-1}(y|s)\in\{q\in\bigcup\{P_i:\ i\in N\}$: there is $j\in n+1$ such that $q=p_j$ or $q|n+1=p_n\}\cup\{\emptyset\}$. From $(8_{r,\partial_n})$, for $n\in\omega$, and $(9_{r,-1})$ it follows that the sequences $(p_n)_{n=0}^{\infty}$, $(\partial_n)_{n=0}^{\infty}$ and $(s_n)_{n=0}^{\infty}$ have required properties.

We shall show that there are $m_\beta\in C\setminus\{m_\partial:\ \partial\in\{-1\}\cup\beta\}$ and $x_\beta\in X_{m_\beta}$ such that the point $(m_\beta,\ x_\beta)$ of $C\times X^\omega$ is consistent with $R(\alpha_\partial+n\omega,\ x)$, where $\partial\in\{-1\}\cup\beta$, $n\in N$ and $x\in X^n$. Let p be an arbitrary element of P_2. By $(1_{r,-1})$, $(2_{r,-1})$, $(12_{r,-1})$ and $(9_{r,-1})$ we infer that there is $y_p\in X^2$ such that $y_p(0)={}_\omega y_p(0)$, $y_p(1)\in\{a^0_{\alpha_\beta},\ a^1_{\alpha_\beta}\}$, $I^0_1({}_{2\omega}y_p)=p$ and $Z(p)=R(2\omega,\ y_p)\supset H(p)$. From $(8_{r,-1})$ it follows that we can assume, without loss of generality that p is an extension of $I^0_j({}_{3\omega}y_p)$, for $j\in D$.

In order to prove that there are $m_\beta\in C\setminus\{m_\partial:\ \partial\in\{-1\}\cup\beta\}$ and $x_\beta\in X_{m_\beta}$ such that $(m_\beta,\ x_\beta)$ is consistent with already defined restrictions, it is enough to show that there is a Borel subset $C(\alpha_\beta,\ y_p)$ of the Cantor set such that $C(\alpha_\beta,\ y_p)\supset H(p)$ and for every $m\in C(\alpha_\beta,\ y_p)$ there is $x\in X_m$ satisfying the following conditions: $x|2=y_p$ and $(m,\ x)$ is consistent with already defined restrictions.

We shall omit the proof of the case of $\beta<\omega$. In fact the proof of this case is included in the proof of the case $\beta\geqslant\omega$. If $\beta<\omega$ then we do not need Lemma 3. Let us assume that $\beta\geqslant\omega$ and $h=h_\beta$ be a one-to-one function from N onto β, given from Lemma 3. Write $W_1 = \{x \in X:\ x =_{3\omega} x\}$. If W_{n-1} is defined then put

$$W_n = \{x \in X:\ x^{-1}(1) \subset (2n+1)\omega \cup \bigcup \{\partial < \omega_1:\ \alpha_{h(j)} \leqslant \partial < \alpha_{h(j)} + (2n+1)\omega,\ j < n\}\}.$$

Let us put

$$Y_n = \begin{cases} \{y_p(n)\}, & \text{if } n \in 2, \\ W_j, & \text{if } 2 \leqslant n = 2j, \\ \{a^0_{\alpha_{h(j)}},\ a^1_{\alpha_{h(j)}}\}, & \text{if } 2 < n = 2j+1. \end{cases}$$

For every $\partial \in \beta$, let $n(\partial)$ be the unique natural number such that, for every y of $\mathbf{P}_{i=0}^{\infty} Y_i$, $y(n(\partial)) \in \{a^0_{\alpha_\partial}, a^1_{\alpha_\partial}\}$. Notice that if $x \in W_j$, for $j \in N$, then $x^{-1}(1)$ is finite. Let y be a point of $\mathbf{P}_{i=0}^{n(0)} Y_i$. From the definition of Y_i, for $i \in \omega$, it follows that $I^0_{n(0)}({}_{(n(0)+1)\omega}y) \in P_{n(0)+1}$, if $R((n(0)+1)\omega,\ y) \neq \emptyset$. Let us put $p_y = I^0_{n(0)}({}_{(n(0)+1)\omega}y)$ and $C(\alpha_0, y) = Z(p_y)$. We shall show that for every m of $C(\alpha_0, y)$ there is $z \in (\mathbf{P}_{i=0}^{\infty} Y_i) \cap X_m$ such that $z|n(0)+1 = y$ and (m, z) is consistent with $R(n\omega, x)$, for $n \in N$ and $x \in X^n$. From the definition of Y_i, for $i \in \omega$, and $p_y = I^0_{n(0)}({}_{(n(0)+1)\omega}y) \in P_{n(0)+1}$ it follows that $R((n(0)+1)\omega,\ {}_{(n(0)+1)\omega}y) = Z(p_y)$. Write $t_j = n(0)+2j$, for $j \in N$. Let $m = (m(n))_{n=0}^{\infty}$ be an arbitrary element of $Z(p_y)$. By the definition of $(\alpha_0(n))_{n=1}^{\infty}$ one can infer, applying $(12_{r,-1})$ two times, that there is $z_1 \in \mathbf{P}_{i=0}^{t_1} Y_i$ such that $z_1|n(0)+1 = y$, $I^0_{t_1}({}_{(t_1+1)\omega}z_1) = p_y$ and $R((t_1+1)\omega,\ z_1) = Z(p_y) \cap B(m(0)\ldots m(t_1))$. If z_k is defined then one can show, in a similar way as above, that there is $z_{k+1} \in \mathbf{P}_{i=0}^{t_{k+1}} Y_i$ such that $z_{k+1}|t_k+1 = z_k$, $I^0_{t_{k+1}}({}_{(t_{k+1}+1)\omega}z_{k+1}) = p_y$ and $R((t_{k+1}+1)\omega,\ z_{k+1}) = Z(p_y) \cap B(m(0)\ldots m(t_{k+1}))$. The point z which is defined by $(z_k)_{k=1}^{\infty}$ has the required properties.

From the reasoning presented above it follows that for $k \geqslant n(0)$ there is $x \in \mathbf{P}_{i=0}^{k} Y_i$ such that for every $n(0) \leqslant j \leqslant k$

$$I^0_j({}_{(j+1)\omega}x|j+1) = I^0_{n(0)}({}_{(n(0)+1)\omega}x|n(0)+1).$$

By the definition of Y_i, for $i \in \omega$, and $(9_{r,-1})$ we infer that $R((k+1)\omega,\ x) = Z(p_{x|n(0)+1}) \cap B(r(x(0))\ldots r(x(k)))$, if $R((n(0)+1)\omega,\ x|n(0)+1) \neq \emptyset$. Put $T_k = \{x \in \mathbf{P}_{i=0}^{k} Y_i:\ I^0_j({}_{(j+1)\omega}x|j+1) = I^0_{n(0)}({}_{(n(0)+1)\omega}x|n(0)+1),\ \text{for } n(0) \leqslant j \leqslant k\}$, $B(C(\alpha_0, x|n(0)+1),\ x) = Z(p_{x|n(0)+1}) \cap B(r(x(0))\ldots r(x(k)))$ for $x \in T_k$ and $B(C(\alpha_0, x|n(0)+1),\ x) = \emptyset$ if $x \notin T_k$, where $p_{x|n(0)+1} = I^0_{n(0)}({}_{(n(0)+1)\omega}x|n(0)+1)$.

Let us notice that for every $m \in B(C(\alpha_0, x|n(0)+1),\ x)$ there is $x' \in (\mathbf{P}_{i=0}^{\infty} Y_i) \cap X_m$ such that $x'|k+1 = x$ and (m, x') is consistent with $R(n\omega, z)$, for $z \in X^n$ and $n \in N$, and

$$B(C(\alpha_0, x|n(0)+1),\ x) = \bigcup\{B(C(\alpha_0, x|n(0)+1),\ x''):\ x'' \in \mathbf{P}_{i=0}^{k+1} Y_i,\ \ x''|k+1 = x\}.$$

Notice that $C(\alpha_0, x|n(0)+1)$, and $p_{x|n(0)+1}$ depend only on $(x|n(0)+1)|\alpha_0$ and $B(C(\alpha_0, x|n(0)+1),\ x)$ on $x|\alpha_0$.

Let us assume that, for $\partial' < \partial$, where $\partial \leqslant \beta$, and $x \in \mathbf{P}_{i=0}^{\infty} Y_i$, we have defined $p_{x|n(\partial')+1} \in P_{n(\partial')+1}$, $C(\alpha_{\partial'}, x|n(\partial')+1)$ and $B(C(\alpha_{\partial'}, x|n(\partial')+1),\ x|i+1)$, for $i \geqslant n(\partial')$, in such a way that the following conditions are satisfied:

$(1_{c,\partial'})$ For $z \in \mathbf{P}_{i=0}^{k} Y_i$ and $k \geqslant n(\partial')$, $p_{z|n(\partial')+1}$ and $C(\alpha_{\partial'}, z|n(\partial')+1)$ depend only on $(z|n(\partial')+1)|\alpha_{\partial'}$ and $B(C(\alpha_{\partial'}, z|n(\partial')+1),\ z)$ depends only on $z|\alpha_{\partial'}$.

$(2_{c,\partial'})$ Let us assume that $y \in \mathbf{P}_{i=0}^{n(\partial')} Y_i$ and

$$\eta = \begin{cases} \sup\{\lambda < \partial':\ n(\lambda) < n(\partial')\}, & \text{if } \{\lambda < \partial':\ n(\lambda) < n(\partial')\} \neq \emptyset, \\ -1, & \text{otherwise.} \end{cases}$$

If $\eta \neq -1$ $(\eta = -1)$ and $B(C(\alpha_\eta, y|n(\eta)+1),\ y) \cap R(\alpha_\eta + (n(\partial')+1)\omega,\ y) = \emptyset$ $(R((n(\partial')+1)\omega,\ y) = \emptyset)$ then $C(\alpha_{\partial'}, y) = \emptyset$ and p_y is an arbitrary element of $P_{n(\partial')+1}$.

$(3_{c,\partial'})$ If $y \in \mathbf{P}_{i=0}^{n(\partial')} Y_i$, $\eta = -1$ (see $(2_{c,\partial'})$) and $R((n(\partial')+1)\omega,\ y) \neq \emptyset$ then put $p_y = I^0_{n(\partial')}({}_{(n(\partial')+1)\omega}y) \in P_{n(\partial')+1}$.

$(4_{c,\partial'})$ Let us assume that $y \in \mathbf{P}_{i=0}^{n(\partial')} Y_i$, $\eta \neq -1$ and $B(C(\alpha_\eta, y|n(\eta)+1),\ y) \cap R(\alpha_\eta + (n(\partial')+1)\omega,\ y) \neq \emptyset$. Put

$$[y] = \{y' \in \mathbf{P}_{i=0}^{n(\partial')} Y_i:\ y'|\alpha_\eta = y|\alpha_\eta,\ I^{\eta+1}({}_{\alpha_\eta+(j+1)\omega}y'|j+1) = I^{\eta+1}({}_{\alpha_\eta+(j+1)\omega}y|j+1)$$
$$\text{and } R(\alpha_\eta + (j+1)\omega,\ y'|j+1) = R(\alpha_\eta + (j+1)\omega,\ y|j+1),\ \text{for } j \leqslant n(\partial')\},$$

$$Q(y) = \{q \in P_{n(\partial')+1}:\ B(C(\alpha_\eta, y|n(\eta)+1),\ y) \cap H(q) \cap R(\alpha_\eta + (n(\partial')+1)\omega,\ y) \neq \emptyset,$$
$$\text{if } j = \sup\{i \leqslant n(\eta):\ {}_{\alpha_\eta}y|i+1 = {}_{\alpha_\eta+(n(\partial')+1)\omega}y|i+1\} \text{ then}$$
$$q|j+1 = p_{y|n(\eta)+1}|j+1,\ \text{if } I_i^{\eta+1}({}_{\alpha_\eta+(n(\partial')+1)\omega}y) \neq \emptyset,\ \text{for } i \leqslant n(\partial'),$$
$$\text{then } q \text{ is an extension of } I_i^{\eta+1}({}_{\alpha_\eta+(n(\partial')+1)\omega}y)\}.$$

If $Q(y) = \emptyset$ then $C(\alpha_{\partial'}, y) = \emptyset$ and p_y is an arbitrary element of $P_{n(\partial')+1}$. Let us assume that $Q(y) \neq \emptyset$. Then for every $y' \in [y]$, for every even natural number i

greater than $n(\eta)$ and less than $n(\partial')$ and for every $q \in Q(y)$ the set $\{z \in [y]: y'(j) = z(j)$ if $j \neq i$ and $p_z = q\}$ is not empty, $p_{y'} \in Q(y)$ and if z and z' belong to $[y]$ and ${}_{\alpha_\eta+(n(\partial')+1)\omega}z = {}_{\alpha_\eta+(n(\partial')+1)\omega}z$ then $p_z = p_{z'}$.

($5_{c,\partial'}$) For every $y \in \mathop{\boldsymbol{P}}_{i=0}^{n(\partial')} Y_i$, $C(\alpha_{\partial'}, y)$ is a Borel subset of the Cantor set including $H(p_y) \cap R\big((n(\partial')+1)\omega, y\big)$, if $\eta = -1$, or $H(p_y) \cap B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap R\big(\alpha_\eta+(n(\partial')+1)\omega, y\big)$, if $\eta \neq -1$ (for the definition of η see ($2_{c,\partial'}$)).

($6_{c,\partial'}$) For every y of $\mathop{\boldsymbol{P}}_{i=0}^{n(\partial')} Y_i$ and $m \in C(\alpha_{\partial'}, y)$ there is z of $(\mathop{\boldsymbol{P}}_{i=0}^{\infty} Y_i) \cap X_m$ such that the point (m, z) of $C \times X^\omega$ is consistent with $R(\alpha_\lambda+n\omega, x)$, for $\lambda < \partial'$, $n \in N$ and $x \in X^n$, and $z|n(\partial')+1 = y$.

($7_{c,\partial'}$) For every $k \geqslant n(\partial')$ and $z \in \mathop{\boldsymbol{P}}_{i=0}^{k} Y_i$ the set $B\big(C(\alpha_{\partial'}, z|n(\partial')+1), z\big)$ $= \{m \in C(\alpha_{\partial'}, z|n(\partial')+1)$: there is $z' \in (\mathop{\boldsymbol{P}}_{i=0}^{\infty} Y_i) \cap X_m$ such that $z'|k+1 = z$ and the point (m, z') is consistent with $R(\alpha_\lambda+n\omega, x)$, for $\lambda < \partial'$, $n \in N$ and $x \in X^n\}$ is a Borel subset of the Cantor set which depends only on $z|\alpha_{\partial'}$, and

$$B\big(C(\alpha_{\partial'}, z|n(\partial')+1), z\big) = \bigcup\{B\big(C(\alpha_{\partial'}, z|n(\partial')+1), z''\big): z'' \in \mathop{\boldsymbol{P}}_{i=0}^{k+1} Y_i,\ z''|k+1 = z\}.$$

Let y be a point of $\mathop{\boldsymbol{P}}_{i=0}^{n(\partial)} Y_i$. There are two cases:

(a) there is $\lambda < \partial$ such that $n(\lambda) < n(\partial)$,

(b) there is no $\lambda < \partial$ such that $n(\lambda) < n(\partial)$.

The proofs of this cases are similar but the second one is a little bit simpler than the first one so we shall consider only the first case.

Put $\eta = \sup\{\lambda < \partial: n(\lambda) < n(\partial)\}$. Let us assume that

$$B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap R\big(\alpha_\eta+(n(\partial)+1)\omega, y\big) \neq \emptyset,$$

$$[y] = \{y' \in \mathop{\boldsymbol{P}}_{i=0}^{n(\partial)} Y_i:\ y'|\alpha_\eta = y|\alpha_\eta,\ I^{\eta+1}({}_{\alpha_\eta+(j+1)\omega}y'|j+1) = I^{\eta+1}({}_{\alpha_\eta+(j+1)\omega}y|j+1)$$

$$\text{and } R\big(\alpha_\eta+(j+1)\omega, y'|j+1\big) = R\big(\alpha_\eta+(j+1)\omega, y|j+1\big), \text{ for } j \leqslant n(\partial)\}.$$

The set $Q(y)$ is defined in a similar way as in ($4_{c,\partial'}$). Let us assume that $Q(y) \neq \emptyset$.

Let i be an arbitrary even number greater than $n(\eta)$ and less than $n(\partial)$ and z an element of $[y]$ satisfying ${}_{\alpha_\eta+(n(\partial)+1)\omega}z = z$. Write $A(z, i) = \{z' \in [y]: z' = {}_{\alpha_\eta+(n(\partial)+1)\omega}z'$ and $z'(j) = z(j)$, for $j \neq i\}$. The family of all sets of the form $A(z, i)$ is countable so let us assume that it is equal to $\{A_s: s \in N\}$. From the definition of Y_j, for $j > n(\eta)$, and ($11_{r,\eta}$) it follows that A_s is infinite and

from the definition of A_s that it is countable, for $s \in N$. Write $A_1 = \{z_n: n \in N\}$ and define p_{z_n} in such a way that the condition ($4_{c,\partial}$) is satisfied, for $z \in A_1$. Let us assume that p_z is defined for $z \in A_1 \cup \ldots \cup A_k$. The set $A_{k+1} \cap (A_1 \cup \ldots \cup A_k)$ is finite so $A_{k+1} \setminus (A_1 \cup \ldots \cup A_k)$ is infinite. We can define p_z for $z \in A_{k+1} \setminus (A_1 \cup \ldots \cup A_k)$ in such a way that required condition is satisfied, for $z \in A_1 \cup \ldots \cup A_{k+1}$. If z is an arbitrary element of $[y]$ then put

$$p_z = p_{z'}, \quad \text{where } z' = {}_{\alpha_\eta+(n(\partial)+1)\omega}z.$$

Case 1 ($\partial = \lambda+1$). Put $K_1(y) = \{y\}$ and $v(1) = n(\partial)$. Let us assume that a subset $K_j(y)$ of $\mathop{\boldsymbol{P}}_{i=0}^{v(j)} Y_i$, where $v(j)$ is an odd number not less than $n(\partial)$, is defined. Then write

$$v(j+1) = \begin{cases} v(j)+2, & \text{if } n(\lambda) \leqslant v(j), \\ \inf\{s \in N: s > v(j) \text{ and there is } \lambda' \text{ such that} \\ \qquad \partial > \lambda' > \sup\{\lambda'' < \partial: n(\lambda'') \leqslant v(j)\} \text{ and } n(\lambda') = s\}, \\ & \text{if } n(\lambda) > v(j), \end{cases}$$

$$K_{j+1}(y) = \{z \in K_j(y) \times \mathop{\boldsymbol{P}}_{i=v(j)+1}^{v(j+1)} Y_i: \text{ if } n(\lambda) > v(j) \text{ then } p_{z|v(j)+1} = p_z|v(j)+1$$

$$\text{and } Q(z) \neq \emptyset, \text{ if } n(\lambda) \leqslant v(j) \text{ then, for every } j' \leqslant v(j+1)$$

$$\text{such that } I_{j'}^{\lambda+1}(z) \neq \emptyset, p_y \text{ is an extension of } I_{j'}^{\lambda+1}(z)\}.$$

Let us put

$$j_0 = \begin{cases} 1, & \text{if } n(\lambda) < n(\partial), \\ j, \text{ where } v(j) = n(\lambda), & \text{if } n(\lambda) > n(\partial) \end{cases}$$

and

$$J_0(y) = \begin{cases} \emptyset, & \text{if } n(\lambda) < n(\partial), \\ B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap \\ \quad \cap R\big(\alpha_\eta+(n(\partial)+1)\omega, y\big) \setminus \bigcup\{C(\alpha_\lambda, z): z \in K_{j_0}(y)\}, \\ & \text{if } n(\lambda) > n(\partial). \end{cases}$$

If $J_s(y)$ is defined then put

$$J_{s+1}(y) = \bigcup\{B\big(C(\alpha_\lambda, z|n(\lambda)+1), z\big) \cap R\big(\alpha_\lambda+(v(j_0)+2s+1)\omega, z\big) \setminus$$
$$\setminus \big(\bigcup\{R\big(\alpha_\lambda+(v(j_0)+2(s+1)+1)\omega, z'\big): z' \in K_{j_0+s+1}(y) \text{ and}$$
$$z'|v(j_0)+2s+1 = z\}\big): z \in K_{j_0+s}(y)\}.$$

Write $J(y) = \bigcup\{J_s(y): s \in \omega\}$ and put

$$C(\alpha_\partial, y) = \Big(B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap R\big(\alpha_\eta+(n(\partial)+1)\omega, y\big)\Big) \setminus J(y).$$

Let z be a point of $\mathop{\boldsymbol{P}}\limits_{i=0}^{t} Y_i$, where $t \geqslant n(\partial)$. If $t < v(j)$, where j is such that $v(j) = n(\lambda)$ and $j' = \inf\{s \in N\colon t \leqslant v(s)\}$ then

$$B\big(C(\alpha_\partial, z|n(\partial)+1), z\big) = \begin{cases} \emptyset, & \text{if there is no } z' \text{ of } K_{j'}\big(z|n(\partial)+1\big) \\ & \quad \text{such that } z'|t+1 = z', \\ \bigcup\{C(\alpha_\lambda, z'')\colon z'' \in K_j\big(z|n(\partial)+1\big) \text{ and} \\ \quad z''|t+1 = z\} \setminus J\big(z|n(\partial)+1\big), & \text{otherwise.} \end{cases}$$

If $t \geqslant v(j)$,

$$t' = \begin{cases} t, & \text{if } t \text{ is an odd number,} \\ t+1, & \text{otherwise,} \end{cases}$$

and j' is such that $t' = v(j')$ then

$$B\big(C(\alpha_\partial, z|n(\partial)+1), z\big) = \begin{cases} \emptyset, & \text{if there is no } z' \text{ of } K_{j'}\big(z|n(\partial)+1\big) \\ & \quad \text{such that } z'|t+1 = z, \\ \bigcup\{B\big(C(\alpha_\lambda, z|n(\lambda)+1), z\big) \cap R\big(\alpha_\lambda + (t'+1)\omega, z''\big) \setminus \\ \setminus J\big(z|n(\partial)+1\big)\colon z'' \in K_{j'}\big(z|n(\partial)+1\big) \text{ and } z''|t+1 = z\}, & \\ & \text{otherwise.} \end{cases}$$

Case 2 (∂ is a limit number). Put $K_1(y) = \{y\}$ and $v(1) = n(\partial)$. Let us assume that $K_j(y) \subset \mathop{\boldsymbol{P}}\limits_{i=0}^{v(j)} Y_i$ is defined, where $v(j)$ is an odd number not less than $n(\partial)$. Write $v(j+1) = \inf\{s \in N\colon s > v(j)\}$ and there is $\lambda < \partial$ such that $n(\lambda) = s$ and $\lambda > \sup\{\lambda' < \partial\colon n(\lambda') \leqslant v(j)\}$ and put

$$K_{j+1}(y) = \{z \in K_j(y) \times \mathop{\boldsymbol{P}}\limits_{i=0}^{v(j+1)} Y_i\colon p_z|n(\partial)+1$$
$$= p_{z|v(j)+1}|n(\partial)+1 = p_y \text{ and } Q(z) \neq \emptyset\}.$$

Put $s'_0 = (n(\partial)-1)/2$. If s'_k is defined then write $s'_{k+1} = \inf\{s \in \{n^{\partial}_{k'}\colon k' \in N\}\colon s > s'_k\}$ (see Lemma 3), $\partial(n)$ was defined in connection with the definition of a_{α_∂}, and $s_k = 2s'_k + 1$. This is the unique place where we need Lemma 3.

Let $j(s_k)$ and η_k be such that $v(j(s_k)) = s_k$ and $s_k = n(\eta_k)$. Put

$$J_0(y) = B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap R\big(\alpha_\eta + (n(\partial)+1)\omega, y\big) \setminus \big(\bigcup\{C(\alpha_{\eta_1}, z)\colon$$
$$\text{where } z \in K_{j(s_1)}(y)\}\big).$$

If $J_k(y)$ is defined then put

$$J_{k+1}(y) = \bigcup\{C(\alpha_{\eta_{k+1}}, z) \setminus \big(\bigcup\{C(\alpha_{\eta_{k+2}}, z')\colon z' \in K_{j(s_{k+2})}(y)$$
$$\text{and } z'|v\big(j(s_{k+1})\big)+1 = z\}\big)\colon z \in K_{j(s_{k+1})}(y)\}.$$

Put $J(y) = \bigcup\{J_k(y)\colon k \in \omega\}$ and

$$C(\alpha_\partial, y) = B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap R\big(\alpha_\eta + (n(\partial)+1)\omega, y\big) \setminus J(y).$$

Let z be a point of $\mathop{\boldsymbol{P}}\limits_{i=0}^{t} Y_i$, where $t \geqslant n(\partial)$. Write $k = \inf\{k' \in \omega\colon t \leqslant s_{k'}\}$ then

$$B\big(C(\alpha_\partial, z|n(\partial)+1), z\big) = \begin{cases} \emptyset, & \text{if there is no } z' \in K_{j(s_k)}\big(z|n(\partial)+1\big) \\ & \quad \text{such that } z'|t+1 = z, \\ \big(\bigcup\{C(\alpha_{\eta_k}, z'')\colon z'' \in K_{j(s_k)}\big(z|n(\partial)+1\big) & \\ \quad \text{and } z''|t+1 = z\}\big) \setminus J\big(z|n(\partial)+1\big), & \\ & \text{otherwise.} \end{cases}$$

From the definition of the constructed objects and the inductive assumption it follows that they depend on $(y|n(\partial)+1)|\alpha_\partial$ or on $y|\alpha_\partial$, for $y \in \mathop{\boldsymbol{P}}\limits_{i=0}^{k} Y_i$, where $k \geqslant n(\partial)$. From the last fact and the inductive assumption we infer that $C(\alpha_\partial, y|n(\partial)+1)$ and $B\big(C(\alpha_\partial, y|n(\partial)+1), y\big)$ are Borel sets, for $y \in \mathop{\boldsymbol{P}}\limits_{i=0}^{k} Y_i$ where $k \geqslant n(\partial)$. The condition $(7_{c,\partial})$ follows from the definition of $J\big(z|n(\partial)+1\big)$, $(1_{r,\lambda'})$, $(4_{r,\lambda'})$, $(5_{r,\lambda'})$, $(8_{r,\lambda'})$, for $\lambda' < \partial$, and from the inductive assumption. In order to finish the proof that the conditions $(1_{c,\partial})-(7_{c,\partial})$ hold it is enough to show that the intersection $J(y) \cap H(p_y)$ is empty, for $y \in \mathop{\boldsymbol{P}}\limits_{i=0}^{n(\partial)} Y_i$. The proof of this fact in the case of limit ordinal numbers is more or less the same as in the case of non-limit numbers so we shall give only the proof of the first case.

In order to show that the intersection $J(y) \cap H(p_y)$ is empty it is enough to prove that if

$$m \in H(p_y) \cap B\big(C(\alpha_\eta, y|n(\eta)+1), y\big) \cap R\big(\alpha_\eta + (n(\partial)+1)\omega, y\big)$$

or

$$m \in H(p_y) \cap C(\alpha_{\eta_k}, z), \text{ for } z \in K_{j(s_k)}(y), \text{ where } k \in N,$$

then there is $z' \in K_{j(s_1)}(y)$ or $z' \in K_{j(s_{k+1})}(y)$ and $z'|s_k+1 = z$ such that $m \in C(\alpha_{\eta_1}, z')$ or $m \in C(\alpha_{\eta_{k+1}}, z')$. The consideration of these cases is similar but the second case is a little bit more complicated so in the sequel we shall assume that $m \in H(p_y) \cap C(\alpha_{\eta_k}, z)$. Let q be an element of $P_{s_{k+1}+1}$ such that q is an extension of p_y and $m \in H(q)$. Put $t_{i'} = j(s_k)+i'$, where $i' \in N$ and $i' \leqslant j(s_{k+1}) - j(s_k)$. From the fact that $m \in C(\alpha_{\eta_k}, z)$ it follows that there is $z'_1 \in \mathop{\boldsymbol{P}}\limits_{i=0}^{v(t_1)} Y_i$ such that $z'_1|s_k+1 = z$ and $m \in B\big(C(\alpha_{\eta_k}, z), z'_1\big)$. From the definition of η_k and a_{α_∂} it follows that $z\big(n(\partial)\big)^{-1}(1) \cap \big((\alpha_{\eta_k}+\omega) \setminus \alpha_{\eta_k}\big) \neq \emptyset$ so by $(4_{r,\eta_k})$, $(5_{r,\eta_k})$, $(8_{r,\eta_k})$, $(11_{r,\eta_k})$

and the definition of $Y_{k'}$ for $k \in \omega$, we infer that the set

$$V = \{x \in \mathop{\mathbf{P}}_{i=0}^{v(t_1)} Y_i\colon\ x|\alpha_{\eta_k} = z'_1|\alpha_{\eta_k},\ x|s_k+1 = z \text{ and if } I_i^{\eta_k+1}(_{\alpha_{\eta_k}+(i+1)\omega}x) \neq \emptyset$$

$$\text{then } q_1 = q|v(t_1)+1 \text{ is an extension of } I_i^{\eta_k+1}(_{\alpha_{\eta_k}+(i+1)\omega}x), \text{ for } i \leqslant v(t_1)\}$$

is not empty. Notice that if $x \in V$ then $q_1 \in Q(x)$. By $(4_{c,\lambda})$, where λ is such that $n(\lambda) = v(t_1)$, we infer that there is $z_1 \in V$ such that $p_{z_1} = q_1$. It is easy to see that $m \in C(\alpha_\lambda, z_1)$. After $\big(j(s_{k+1})-j(s_k)\big)$ steps we shall find $z_{(j(s_{k+1})-j(s_k))} = z'$ which has the required properties.

From $(1_{c,\beta})-(7_{c,\beta})$ it follows that $C(\alpha_\beta, y_p)$ is a required set.

The construction of $R(\alpha_\beta+n\omega, x)$, for $n \in N$, $x \in X^n$ is similar to the construction of $R(n\omega, x)$. This completes the definition of L so we conclude that X and M have properties mentioned in Example 2.

Comments.

Remark 4. If X' is the derivative of X, where X is from Example 2, then one can show (see [Al$_2$]) that $X' = \bigcup\{X_n\colon n \in N\}$, where X_n is a Lindelöf scattered space so from [Al$_1$] it follows that the product $Y \times (X')^\omega$ is Lindelöf, for every hereditarily Lindelöf space Y.

Remark 5. In [Al$_2$] we proved, in some sense, a dual result to Example 2. We showed that if X is a Lindelöf P-space and M is a separable metric space which admits a complete metric space M' such that $M' \supset M$ and $M' \setminus M$ does not contain uncountable compact subsets then the product $M \times X^\omega$ is Lindelöf.

Let me finish this paper with some problems related to the Michael's conjecture.

PROBLEM 1. Let us assume that the product $Y \times X$ is Lindelöf, for every Lindelöf space Y. Is it true that X^ω is a Lindelöf space?

PROBLEM 2. Let us assume that $Y \times X$ has the Lindelöf property, for every hereditarily Lindelöf space Y. Is it true that X^2 is a Lindelöf space?

PROBLEM 3. One can ask similar questions for other covering properties. I do not know, for example, whether the product $Y \times X^\omega$ is paracompact, where X is a space having only one non-isolated point and Y is a perfect paracompact space?

Notice that it is not enough to assume that Y is a hereditarily paracompact space.

References

[Al$_1$] K. Alster, *A class of spaces whose Cartesian product with every hereditarily Lindelöf space is Lindelöf*, Fund. Math. 114 (1981), pp. 173–181.

[Al$_2$] K. Alster, *Some remarks on Michael's question concerning Cartesian product of Lindelöf spaces*, Colloq. Math. 47 (1982), pp. 23–29.

[E] R. Engelking, *General Topology*, Warszawa 1977.

[F] Z. Frolík, *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. 8 (1960), pp. 747–750.

[N] N. Noble, *Products with closed projection II*, Trans. Amer. Math. Soc. 160 (1971), pp. 169–183.

[P] R. Pol, *A function space C(X) which is weakly Lindelöf but not weakly compactly generated*, Studia Math. 64 (1979), pp. 279–285.

[T] R. Telgarski, *C-scattered and paracompact spaces*, Fund. Math. 73 (1971), pp. 59–74.

Present address:
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Warszawa, Poland

Received 12 October 1981
in revised form 1 July 1982