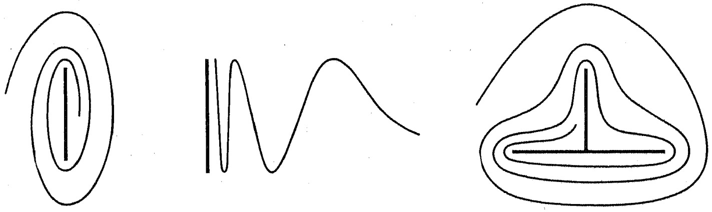
may also think that X is not a quasi retract of a disk. This is not so. The continuum X can be embedded as the $\sin 1/x$ curve as shown in Figure 5b. By Theorem 12, the continuum in Figure 5b is a quasi retract of a disk.



(a) (b) (c)

Fig.5

It is also easy to show that being a quasi retract of a disk is a topological property, i.e., it does not depend on the embedding. Hence the continuum in Figure 5a is a quasi retract of the disk. It is not known if the triod with a spiral shown in Figure 5c is a quasi retract of a disk.

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Baire category in spaces of probability measures, II

by

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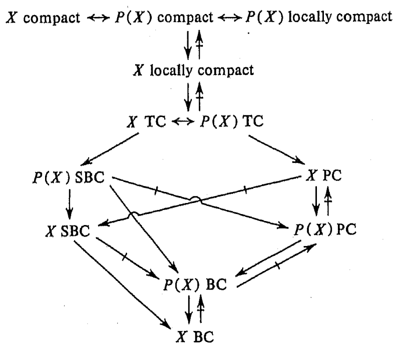
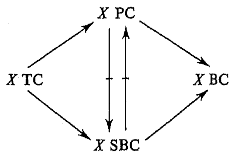
Abstract. Completeness relationships for a space X, and its space of probability measures $P(X)$ are compared. All implications between X and $P(X)$ and between compactness, local compactness, topological completeness, pseudo completeness, Baire completeness, and strong Baire completeness are resolved. The continuum hypothesis has been assumed when needed.

1. Introduction. In [B], completeness relationships between a separable metric space (X, d) and the space of probability measures on X endowed with the separable metric of weak convergence, $(P(X), \varrho)$ were investigated. It was shown that $X\,\mathrm{PC} \to P(X)\,\mathrm{PC} \to P(X)\,\mathrm{BC} \to X\,\mathrm{BC}$ and none of the implications are reversible. Here, as in [B], TC means topologically complete, PC means pseudo complete (i.e., contains a dense TC subspace), and BC means Baire complete (i.e., is a Baire space). We also denote strongly Baire complete by SBC. A space is SBC if every closed subspace is BC.

Based upon results of Prohorov [P] and Luther [L], we know that X compact $\leftrightarrow P(X)$ compact $\leftrightarrow P(X)$ locally compact, and also that $X\,\mathrm{TC} \leftrightarrow P(X)\,\mathrm{TC}$. The purpose of this paper is to resolve the following diagram.



That $P(X)\,\mathrm{SBC} \to X\,\mathrm{SBC}$ follows from the fact that if F is closed in X, $\{\mu \in P(X)\colon\ \mu(x) = 1$ for some $x \in F\}$ is homeomorphic to F and is closed in $P(X)$. As it is well known that



the only remaining items to be shown are (Theorem 4) $X\,\mathrm{SBC} \nrightarrow P(X)\,\mathrm{BC}$, and (Theorem 5) $P(X)\,\mathrm{SBC} \nrightarrow P(X)\,\mathrm{PC}$.

As a tool for obtaining these theorems, we prove two results related to the theory of totally imperfect spaces (cf. [K]); namely we provide a characterization of property C'' and have shown which of the "Lusin-type" universal null spaces can be SBC and which cannot.

2. Results concerning totally imperfect spaces. For the proof of Theorem 4, we will require a space which is SBC and a β space (one of universal measure zero or equivalently, one which supports atomic measures only). Therefore it was desirable to determine just which of the spaces in the hierarchy of "Lusin-type" totally imperfect spaces discussed in Section 40 of [K] and elsewhere in the literature (e.g., L_1 is defined in [B′])

$$\text{countable} \to L \to \nu \to L_1 \to P \begin{smallmatrix} \nearrow \mathrm{CON} \searrow \\ \searrow C'' \nearrow \end{smallmatrix} C \to \beta \to \text{totally imperfect},$$

can have property SBC. A space X (assumed to be embedded in some space Y) has *property* CON (relative to Y) if there exists a countable subset M of Y about which it is concentrated, i.e., such that if Q is an open subset of Y which contains M, then $X \setminus Q$ is countable. A space has *property P* if it is concentrated about a countable subset of itself. The statement that X has *property C''* means that if $\{U(x, n)\colon\ x \in X,\ n = 0, 1, 2, \ldots\}$ is a family of open subsets of X such that $x \in U(x, n)$ for each x and n, then there exists a sequence $x_0, x_1, x_2, \ldots$ such that $X = \bigcup_{n=0}^{\infty} U(x_n, n)$. A space has *property C* if it is true that for every sequence $f_0, f_1, f_2, \ldots$ of positive numbers, there exists a sequence $x_0, x_1, x_2, \ldots$ of elements of X such that $X = \bigcup_{n=0}^{\infty} N(x_n, f_n)$, where $N(x, f)$ denotes the f-neighborhood of x. In order to facilitate the construction given in the proof of Theorem 3, we first give the following C-like characterization of property C''.

THEOREM 1. *A space X has property C'' if and only if it is true that for every sequence $f_0, f_1, f_2, \ldots$ of positive valued continuous functions with domain X, there exists a sequence $x_0, x_1, x_2, \ldots$ of elements of X such that $X = \bigcup_{n=0}^{\infty} N(x_n, f_n(x_n))$.*

Proof. It is obvious that property C'' implies the latter property, so suppose that X satisfies the latter property. Let $\{U(x, n)\colon\ x \in X,\ n = 0, 1, 2, \ldots\}$ be a family of open sets such that $x \in U(x, n)$ for each x and n. For each n, let g_n be the function defined by $g_n(x) = \sup\{\varepsilon\colon$ there exist $y \in X$ such that $N(x, \varepsilon)$ is a subset of $U(y, n)\}$, for each $x \in X$. (If this sup fails to exist for infinitely many n, the conclusion follows, so we may assume that the sup exists in every case.) The functions $g_0, g_1, g_2, \ldots$ are continuous and positive valued on X. Let $f_n = g_n/2$ for each n, and let $x_0, x_1, x_2, \ldots$ be a sequence such that $X = \bigcup_{n=0}^{\infty} N(x_n, f_n(x_n))$. For each n, let y_n be an element of X such that $N(x_n, f_n(x_n))$ lies in $U(y_n, n)$. Then $X = \bigcup_{n=0}^{\infty} U(y_n, n)$.

THEOREM 2. *There exists no uncountable space which is* SBC *and has property* CON.

Proof. Let X be an uncountable subset of the space Y, and assume that X is concentrated about the subset $M = \{m_0, m_1, m_2, \ldots\}$ of Y. Let Q_0 be an open subset of Y containing m_0 such that $X \setminus Q_0$ is uncountable. Let K_0 be the range of a nonrepeating sequence $k_0, k_1, k_2, \ldots$ of elements of and condensation points of $X \setminus (M \cup Q_0)$ such that $k_1, k_2, k_3, \ldots$ converges to k_0. Let Q_1 be an open subset of Y containing m_1 and no elements of K_0 such that every element of K_0 is still a condensation point of $X \setminus (M \cup Q_0 \cup Q_1)$. For each element of K_0 pick a sequence of elements of and condensation points of $X \setminus (M \cup Q_0 \cup Q_1)$ converging to that element of K_0, and pick these sequences in such a way that the union, K_1, of K_0 and the ranges of all these sequences is closed in Y. Continue in this manner for every positive integer n. Then $K = \mathrm{Cl}_X(K_0 \cup K_1 \cup K_2 \cup \ldots)$ is closed relative to X, countable because it is a subset of $X \setminus \bigcup_{n=0}^{\infty} Q_n$, and perfect (every point of K is a limit point of K). Thus K is first category relative to itself, and X is not SBC.

THEOREM 3. *The continuum hypothesis implies the existence of a subspace X of the reals such that X is* SBC *and has property C''.*

Proof. X will be constructed as the union of the sets X_β defined by the following transfinite process. Let $\{q^\alpha\}_{\alpha < \omega_1}$ be a well ordered sequence such that for each α, q^α is a sequence $q_0^\alpha, q_1^\alpha, q_2^\alpha, \ldots$ of non-negative, lower semi-continuous functions with domain $[0, 1]$ such that each q_i^α is positive valued on some dense open subset O_i^α of $[0, 1]$. For each α, let $H_\alpha = O_0^\alpha \cap O_1^\alpha \cap O_2^\alpha \cap \ldots$ Assume that each sequence of non-negative, lower semi-continuous functions with domain $[0, 1]$ which are positive valued on dense open subsets of $[0, 1]$ appears in the sequence $\{q^\alpha\}_{\alpha < \omega_1}$. Let $\{F_\alpha\}_{\alpha < \omega_1}$ be a listing of the closed perfect subsets of [0, 1] such that each such set appears in the list uncountably many times. For each closed perfect set F, let $\{K_\alpha^F\}_{\alpha<\omega_1}$, list the first category (relative to F) F_σ subsets of F. For each $\alpha < \omega_1$, let J_α be the set obtained as follows: Denote F_α by F. F appears for the γth time at ordinal $\alpha (0 < \gamma < \omega_1)$. Let $J_\alpha = \bigcup_{\sigma\leqslant\gamma} K_\sigma^F$ (J_α is still F_σ and first category relative to F). Therefore, we have that if F is closed and perfect in [0, 1] and K is F_σ and first category relative to F, there exists α such that $F = F_\alpha$ and $K \subset J_\alpha$.

We now begin the process of constructing X.

Level 0: Let $\alpha(0) = 0$ and $G_0 = H_0$. Let $X_0 \subset G_0$ be the range of a sequence $x_0^0, x_1^0, x_2^0, \ldots$ which includes a dense subset of [0, 1] and if possible at least one element of $(G_0 \cap F_0)\setminus J_0$. Then let $Q_0 = \bigcup_{i=0}^{\infty} N(x_i^0, q_i^{\alpha(0)}(x_i^0))$.

Level β: Let $\alpha(\beta)$ be the first ordinal greater than all $\alpha(\gamma)$ with $\gamma < \beta$ such that $\bigcup_{\gamma<\beta} X_\gamma \subset H_{\alpha(\beta)}$. Let $G_\beta = \bigcap_{\gamma<\beta} (G_\gamma \cap Q_\gamma) \cap H_{\alpha(\beta)}$. Let $X_\beta \subset G_\beta$ be the range of a sequence $x_0^\beta, x_1^\beta, x_2^\beta, \ldots$ which includes all elements of $\bigcup_{\gamma<\beta} X_\gamma$ and if possible includes at least one element of $(G_\beta \cap F_\beta)\setminus J_\beta$. Let $Q_\beta = \bigcup_{i=0}^{\infty} N(x_i^\beta, q_i^{\alpha(\beta)}(x_i^\beta))$.

Let $X = \bigcup_{\beta<\omega_1} X_\beta$.

We first show that X has property C''. Let $g_0, g_1, g_2, \ldots$ be a sequence of continuous positive valued functions with domain X. Each g_i can be extended to a non-negative lower semi-continuous function q_i defined on [0, 1] which is positive on a dense open subset O_i of [0, 1]. Let $H = \bigcap_{i=0}^{\infty} O_i$. Let γ be the first ordinal such that $H_\gamma = H$ and $q^\gamma = q_0, q_1, q_2, \ldots$ Since the entire set $X \subset H_\gamma$, there will exist a β such that $\alpha(\beta) = \gamma$. Now, $Q_\beta = \bigcup_{i=0}^{\infty} N(x_i^\beta, q_i^{\alpha(\beta)}(x_i^\beta))$ contains all of X. Thus we have that $X = \bigcup_{i=0}^{\infty} N(x_i^\beta, g_i(x_i^\beta))$, and X has property C''.

We now show that X is SBC. Suppose otherwise, and that $f \subset X$ is a closed relative to X and perfect subset of X which is first category relative to itself. Let $\{y_0, y_1, y_2, \ldots\}$ be dense in f. Let $F = \mathrm{Cl}_{[0,1]}(f)$ and let H be an F_σ first category relative to F subset of F such that $f \subset H$. Choose β such that $F = F_\beta$, $H \subset J_\beta$, and $\{y_0, y_1, y_2, \ldots\} \subset X_\beta$. G_β contains all of X, so $\{y_0, y_1, y_2, \ldots\} \subset G_\beta$. Therefore, $G_\beta \cap F_\beta$ is a dense G_δ set relative to F_β, but J_β is first category relative to F_β. Therefore, it would have been possible for X_β to include an element of $(G_\beta \cap F_\beta)\setminus J_\beta$ at level β in the construction. Thus, X must contain an element of $f\setminus H$. This is a contradiction and completes the proof of Theorem 3.

3. Completeness properties. In [B, Theorem 4] it is argued that if every element of $P(X)$ has an atom (i.e., X is a β space), then $P(X)$ is not BC. Thus, from Theorem 3, we have

THEOREM 4. *The continuum hypothesis implies the existence of a subspace X of the reals such that X is* SBC *but $P(X)$ is not* BC.

THEOREM 5. *The continuum hypothesis implies the existence of a subspace X of the reals such that $P(X)$ is* SBC *but $P(X)$ is not* PC.

Proof. Index all dense G_δ sets in $P[0, 1]$ and all closed subsets of $P[0, 1]$ as $\{G_\alpha\}_{\alpha<\omega_1}$ and $\{M_\alpha\}_{\alpha<\omega_1}$ respectively. For each M_α, index all first category (rel. M_α) F_σ subsets of M_α as $\{M_{\alpha\beta}\}_{\beta<\omega_1}$. Let g be a bijection from ω_1 onto ω_1^2. We now construct X.

Level 0: Represent $g(0)$ as (α, β). Select a measure $\mu_0 \in M_\alpha \setminus M_{\alpha\beta}$ and let F_0 be a first category yet dense in [0, 1] F_σ set which supports μ_0. Now let $\eta_0 \in G_0$ be a measure such that $\eta_0(F_0) = 0$. This is possible since most measures assign F_0 measure 0. Finally, let H_0 be a first category F_σ set in [0, 1] that supports η_0 but such that $F_0 \cap H_0 = \emptyset$.

Level γ: Represent $g(\gamma)$ as (α, β). If possible, select $\mu_\gamma \in M_\alpha \setminus M_{\alpha\beta}$ such that $\mu_\gamma(\bigcup_{\sigma<\gamma} H_\sigma) = 0$, and let F_γ be a first category yet dense F_σ subset of [0, 1] that supports μ_γ and such that $F_\gamma \cap (\bigcup_{\sigma<\gamma} H_\sigma) = \emptyset$. If the selection of μ_γ is not possible, let $F_\gamma = \emptyset$. Now let $\eta_\gamma \in G_\gamma$ be such that $\eta_\gamma(\bigcup_{\sigma\leqslant\gamma} F_\sigma) = 0$, and choose H_γ to be a first category F_σ set in [0, 1] that supports η_γ and such that $H_\gamma \cap (\bigcup_{\sigma\leqslant\gamma} F_\sigma) = \emptyset$.

Set $X = \bigcup_{\gamma<\omega_1} F_\gamma$. Now $P(X)$ is not PC for it contains no G_γ; i.e., it contains no dense TC subspace.

However, $P(X)$ is SBC, for suppose that $M \subset P(X)$ is closed. Let M_α be the $P[0, 1]$-closure of M. It suffices to show that M is not first category in itself, so suppose that it is. Then $M \subset M_{\alpha\beta}$ for some β. Let $\gamma = g^{-1}(\alpha, \beta)$. At level γ, if it was possible to select μ_γ, then we have a contradiction because $\mu_\gamma \in M_\alpha \setminus M_{\alpha\beta}$ and $\mu_\gamma \in P(X)$, hence $\mu_\gamma \in M \setminus M_{\alpha\beta}$. To finish the proof then, we need only to show that it was possible to select μ_γ. Notice that a dense subset (namely M) of M_α assigns $H = \bigcup_{\sigma<\gamma} H_\sigma$ measure 0. Furthermore, $\{\mu \in M_\alpha\colon \mu([0, 1]\setminus H) = 1\}$ is G_δ, so most of M_α assigns H measure 0, and hence there are measures in $M_\alpha \setminus M_{\alpha\beta}$ which could be selected as μ_γ.

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On Michael's problem concerning the Lindelöf property in the Cartesian products

by

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Abstract. In this paper we present a negative solution of Michael's conjecture which says that if $Y\times X$ is Lindelöf, for every hereditarily Lindelöf space Y, then $Y\times X^{\omega}$ is Lindelöf, for every hereditarily Lindelöf space Y.

Introduction. It is known that if Y is a hereditarily Lindelöf space and X a metric separable space then $Y\times X$ and also $Y\times X^{\omega}$ are Lindelöf. Z. Frolík proved (see [F]) that if Y is a hereditarily Lindelöf and X is a Lindelöf and complete in the sense of Čech space then $Y\times X$ and also $Y\times X^{\omega}$ are Lindelöf. R. Telgarski showed (see [T]) that if Y is a hereditarily Lindelöf space and X a Lindelöf and scattered space then $Y\times X$ is Lindelöf. I have improved the result of Telgarski [Al_1], by showing that $Y\times X^{\omega}$ is Lindelöf. I think that these results were the motivation of Michael's conjecture which says that if the product $Y\times X$ is Lindelöf for every hereditarily Lindelöf space Y then $Y\times X^{\omega}$ is Lindelöf for every hereditarily Lindelöf space Y. In this paper we proved that the answer to the Michael's conjecture is a negative one.

Examples.

Example 1. There exists Z such that, for every natural number n and for every hereditarily Lindelöf space Y, the product $Y\times Z^{n}$ is Lindelöf but Z^{ω} is not.

Example 2. There exist a separable metric space M and a space X such that, for every Lindelöf space Y and every natural number n, the products $Y\times X^{n}$ and X^{ω} are Lindelöf but $M\times X^{\omega}$ is not.

It is easy to see that in order to obtain Example 1 it is enough to put $Z = M\times X$, where M and X are from Example 2.

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