may also think that $X$ is not a quasi retract of a disk. This is not so. The continuum $X$ can be embedded as the $\sin 1/x$ curve as shown in Figure 5b. By Theorem 12, the continuum in Figure 5b is a quasi retract of a disk.

![Fig. 5](image)

It is also easy to show that being a quasi retract of a disk is a topological property, i.e., it does not depend on the embedding. Hence the continuum in Figure 5a is a quasi retract of the disk. It is not known if the triod with a spiral shown in Figure 5c is a quasi retract of a disk.

References


Baire category in spaces of probability measures, II

by

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Abstract. Completeness relationships for a space $X$, and its space of probability measures $P(X)$ are compared. All implications between $X$ and $P(X)$ and between compactness, local compactness, topological completeness, pseudo completeness, Baire completeness, and strong Baire completeness are removed. The continuum hypothesis has been assumed when needed.

1. Introduction. In [B], completeness relationships between a separable metric space $(X, d)$ and the space of probability measures on $X$ endowed with the separable metric of weak convergence, $(P(X), \varrho)$ were investigated. It was shown that $X$ compact $\iff P(X)$ compact $\iff P(X)$ locally compact, and also that $X$ TC $\iff P(X)$ TC. The purpose of this paper is to resolve the following diagram.

![Diagram](image)
That \( P(X) \text{SBC} \rightarrow X \text{SBC} \) follows from the fact that if \( F \) is closed in \( X \), 
\( \{ \mu \in P(X) : \mu(x) = 1 \text{ for some } x \in F \} \) is homeomorphic to \( F \) and is closed in \( P(X) \). As it is well known that

\[ X \text{TC} \rightarrow X \text{BC} \]

the only remaining items to be shown are (Theorem 4) \( X \text{SBC} \rightarrow P(X) \text{BC} \), and

(5) \( P(X) \text{SBC} \rightarrow P(X) \text{PC} \).

As a tool for obtaining these theorems, we prove two results related to the theory of totally imperfect spaces (cf. [K]); namely we provide a characterization of property \( C^* \) and have shown which of the \textbf{“Lusin-type”} universal null spaces can be SBC and which cannot.

2. Results concerning totally imperfect spaces. For the proof of Theorem 4, we will require a space which is SBC and a \( \beta \) space (one of universal measure zero or equivalently, one which supports atomic measures only). Therefore it was desirable to determine just which of the spaces in the hierarchy of \textbf{“Lusin-type”} totally imperfect spaces discussed in Section 40 of

[K] and elsewhere in the literature (e.g., \( L_1 \) is defined in [B]).

\[
\text{countable} \rightarrow L \rightarrow \tau \rightarrow L_1 \rightarrow P \text{CON}_+ \rightarrow \gamma \rightarrow \beta \rightarrow \text{totally imperfect},
\]

can have SBC property. A space \( X \) (assumed to be embedded in some space \( Y \)) has property CON (relative to \( Y \)) if there exists a countable subset \( M \) of \( Y \) about which it is concentrated, i.e., such that if \( Q \) is an open subset of \( Y \) which contains \( M \), then \( X \setminus Q \) is countable. A space has property \( P \) if it is concentrated about a countable subset of itself. The statement that \( X \) has property \( C^* \) means that if \( \{ U(x, n) : x \in X, \ n = 0, 1, 2, \ldots \} \) is a family of open subsets of \( X \) such that \( x \in U(x, n) \) for each \( x \) and \( n \), then there exists a sequence \( x_0, x_1, x_2, \ldots \) such that \( X = \bigcup_{n=0}^\infty U(x_n, n) \). A space has property \( C \) if it is true that for every sequence \( f_0, f_1, f_2, \ldots \) of positive valued continuous functions with domain \( X \), there exists a sequence \( x_0, x_1, x_2, \ldots \) of elements of \( X \) such that \( X = \bigcup_{n=0}^\infty N(x_n, f_n) \), where \( N(x, f) \) denotes the \( f \)-neighborhood of \( x \). In order to facilitate the construction given in the proof of Theorem 3, we first give the following \( C \)-like characterization of property \( C^* \).

\textbf{Theorem 1.} A space \( X \) has property \( C^* \) if and only if it is true that for every sequence \( f_0, f_1, f_2, \ldots \) of positive valued continuous functions with domain \( X \), there exists a sequence \( x_0, x_1, x_2, \ldots \) of elements of \( X \) such that \( X = \bigcup_{n=0}^\infty N(x_n, f_n) \), where \( N(x, f) \) denotes the \( f \)-neighborhood of \( x \). In order to facilitate the construction given in the proof of Theorem 3, we first give the following \( C \)-like characterization of property \( C^* \).

\[ X, \therefore \text{there exists a sequence } x_0, x_1, x_2, \ldots \text{ of elements of } X \text{ such that } X = \bigcup_{n=0}^\infty N(x_n, f_n(x_n)). \]

Proof. It is obvious that property \( C^* \) implies the latter property, so suppose that \( X \) satisfies the latter property. Let \( \{ U(x, n) : x \in X, \ n = 0, 1, 2, \ldots \} \) be a family of open sets such that \( x \in U(x, n) \) for each \( x \) and \( n \). For each \( n \), let \( g_n \) be the function defined by \( g_n(x) = \sup \{ |x| : x \in U(x, n) \} \). For each \( x \in X \) such that \( N(x, e) \) is a subset of \( U(y, n) \), for each \( x \in X \). (If this sup fails to exist for infinitely many \( n \), the conclusion follows, so we may assume that the sup exists in every case.) The functions \( g_0, g_1, g_2, \ldots \) are continuous and positive valued on \( X \). Let \( f_n = g_n/2 \) for each \( n \), and let \( x_0, x_1, x_2, \ldots \) be a sequence such that \( X = \bigcup_{n=0}^\infty N(x_n, f_n(x_n)) \). For each \( n \), let \( y_n \) be an element of \( X \) such that \( N(x_n, f_n(x_n)) \) lies in \( U(y_n, n) \). Then \( X = \bigcup_{n=0}^\infty U(y_n, n) \).

\textbf{Theorem 2.} There exists no uncountable space which is SBC and has property \( C^* \).

Proof. Let \( X \) be an uncountable subset of the space \( Y \) and suppose that \( X \) is concentrated about the subset \( M = \{ m_0, m_1, m_2, \ldots \} \) of \( Y \). Let \( Q_0 \) be an open subset of \( Y \) containing \( m_0 \) such that \( X \setminus Q_0 \) is uncountable. Let \( K_0 \) be the range of a nonrepeating sequence \( k_0, k_1, k_2, \ldots \) of elements of and condensation points of \( X \setminus Q_0 \). Let \( k_1, k_2, k_3, \ldots \) converges to \( k_0 \) and \( Q_1 \) be an open subset of \( Y \) containing \( m_0 \) and no elements of \( K_0 \). Let \( k_0 \) be the range of a nonrepeating sequence \( k_0, k_1, k_2, \ldots \) of elements of and condensation points of \( X \setminus (Q_0 \cup Q_1) \) converging to that element of \( K_0 \). Pick these sequences in such a way that the union, \( K_1 \), of \( K_0 \) and the ranges of all these sequences is closed in \( Y \). Choose in this manner for each positive integer \( n \) then \( K = K_1 \cup K_2 \cup K_3 \cup \ldots \) is closed relative to \( X \), countable because it is a subset of \( X \setminus \bigcup_{n=0}^\infty Q_n \), and perfect (every point of \( K \) is a limit point of \( K \)). Thus \( K \) is first category relative to itself, and \( X \) is not SBC.

\textbf{Theorem 3.} The continuum hypothesis implies the existence of a subspace \( X \) of the reals such that \( X \) is SBC and has property \( C^* \).

Proof. \( X \) will be constructed as the union of the sets \( X_h \) defined by the following transfinite process. Let \( \{ q_h \}_{h=0}^\infty \) be a well ordered sequence such that for each \( q_h \), \( q_h \) is a sequence \( q_0, q_1, q_2, \ldots \) of non-negative, lower semi-continuous functions with domain \( [0, 1] \). For each \( h \), let \( H_h = O_h^c \cap O_h^c \cap O_h^c \). Assume that each sequence of non-negative, lower semi-continuous functions with domain \( [0, 1] \) which are positive valued on dense open subsets of \([0, 1] \) appears in the sequence \( \{ q_h \}_{h=0}^\infty \). For \( X_h \) be a
listing of the closed perfect subsets of \([0, 1]\) such that each such set appears in the list uncountably many times. For each closed perfect set \(J_\alpha\), list the first category (relative to \(F\)) \(F_\alpha\) subsets of \(F\). For each \(\alpha < \omega_1\), let \(J_\alpha\) be the set obtained as follows: Denote \(F_\alpha\) by \(F_\alpha\). \(F\) appears for the \(\gamma\)th time at ordinal \(\alpha(0 < \gamma < \omega_1)\). Let \(J_\alpha = \bigcup \{K_\beta : J_\alpha\text{ is still }F_\alpha\text{ and first category relative to }F\}\). Therefore, we have that if \(F\) is closed and perfect in \([0, 1]\) and \(K\) is \(F_\alpha\) and first category relative to \(F\), there exists \(x\) such that \(F = F_\alpha\) and \(x < J_\alpha\).

We now begin the process of constructing \(X\).

Level 0: Let \(\alpha(0) = 0\) and \(G_0 = H_0\). Let \(X_0 \subset G_0\) be the range of a sequence \(x_0^0, x_0^1, x_0^2, \ldots\) which includes a dense subset of \([0, 1]\) and if possible at least one element of \(G \cap F_\alpha\). Then let \(Q_0 = \bigcup_{i=0}^\infty N(x_i^\gamma, q^\gamma(x_i^\gamma))\).

Level \(\beta\): Let \(\alpha(\beta)\) be the first ordinal greater than all \(\alpha(\gamma)\) with \(\gamma < \beta\) such that \(\bigcup_{\gamma < \beta} X_\gamma \subset H_\alpha\). Let \(G_\beta = \bigcap_{\gamma < \beta}(G_\gamma \cap Q_\gamma) \cap H_\alpha\). Let \(X_\beta \subset G_\beta\) be the range of a sequence \(x_\beta^0, x_\beta^1, x_\beta^2, \ldots\) which includes all elements of \(\bigcup_{\gamma < \beta} X_\gamma\) and if possible includes at least one element of \(G_\beta \cap F_\beta\). Let \(Q_\beta = \bigcup_{i=0}^\infty N(x_i^\gamma, q^\gamma(x_i^\gamma))\).

Let \(X = \bigcup_{\beta < \omega_1} X_\beta\).

We first show that \(X\) has property \(C^*\). Let \(0_0, g_1, g_2, \ldots\) be a sequence of continuous positive valued functions with domain \(X\). Each \(g_i\) can be extended to a non-negative lower semi-continuous function \(q_i\) defined on \([0, 1]\) which is positive on a dense open subset \(O_i\) of \([0, 1]\). Let \(H = \bigcap_{i=0}^\infty O_i\). Let \(H = \bigcap_{i=0}^\infty O_i\). Since the entire set \(X \cap H\), there will exist a \(\beta\) such that \(\alpha(\beta) = \gamma\). Now, \(Q_\beta = \bigcup_{i=0}^\infty N(x_i^\gamma, q^\gamma(x_i^\gamma))\) contains all of \(X\). Thus we have that \(X = \bigcup_{\beta < \omega_1} X_\beta\) and \(X\) has property \(C^*\).

We now show that \(X\) is SBC. Suppose otherwise, and that \(f \in X\) is a closed relative to \(X\) and perfect subset of \(X\) which is first category relative to itself. Let \(\mu_0, \mu_1, \mu_2, \ldots\) be dense in \(f\). Let \(F = \bigcap_{i=0}^\infty (f)\) and let \(H\) be an \(F\) first category relative to \(F\) subset of \(F\) such that \(f \subset H\). Choose \(\beta\) such that \(F = F_\beta\) \(f \subset J_\beta\), and \(\{y_0, y_1, y_2, \ldots\} \subset X_\beta\). \(G_\beta\) contains all of \(X\), so \(\{y_0, y_1, y_2, \ldots\} \subset G_\beta\). Therefore, \(G_\beta \cap F_\beta\) is a dense \(G_\beta\) set relative to \(F_\beta\), but \(J_\beta\) is first category relative to \(F_\beta\). Therefore, it would have been possible for \(X_\beta\) to include an element of \((G_\beta \cap F_\beta) \cap J_\beta\) at level \(\beta\) in the construction. Thus, \(X\) must contain an element of \(\bigcap_{\gamma < \beta} H_\gamma\). This is a contradiction and completes the proof of Theorem 3.

3. Completeness properties. In [B, Theorem 4] it is argued that if every element of \(P(X)\) has an atom (i.e., \(X\) is a \(\beta\) space), then \(P(X)\) is not BC. Thus, from Theorem 3, we have

Theorem 4. The continuum hypothesis implies the existence of a subspace \(X\) of the reals such that \(X\) is SBC but \(P(X)\) is not BC.

Theorem 5. The continuum hypothesis implies the existence of a subspace \(X\) of the reals such that \(P(X)\) is SBC but \(P(X)\) is not BC.

Proof. Index all dense \(G_\beta\) sets in \([0, 1]\) and all closed subsets of \(P[0, 1]\) as \(\{G_{\alpha^\gamma} : \gamma < \beta\}\), and \(\{M_{\alpha^\gamma} : \gamma < \beta\}\), respectively. For each \(M_{\alpha^\gamma}\), index all first category (rel. \(M_{\alpha^\gamma}\)) \(F_{\gamma^\beta}\) subsets of \(M_{\alpha^\gamma}\) as \(\{M_{\alpha^\gamma} \cap F_{\gamma^\beta} : \gamma < \beta\}\). Let \(g\) be a bijection from \(\omega_1\) onto \(\omega^2_1\). We now construct \(X\).

Level 0: Represent \(g(0)\) as \((x_0, \beta)\). Select a measure \(\mu_0 \in M_{\alpha^\gamma} \cap F_{\gamma^\beta}\) and let \(F_0\) be a first category yet dense in \([0, 1]\) \(F_\alpha\) set which supports \(\mu_0\). Now let \(\eta_\alpha (F_0)\) be a measure such that \(\eta_\alpha (F_0) = 0\). This is possible since most measures assign \(F_0\) measure 0. Finally, let \(H_0\) be a first category \(F_0\) set in \([0, 1]\) that supports \(\eta_\alpha\) but such that \(F_0 \cap H_0 = \emptyset\).

Level \(\gamma\): Represent \(g(\gamma)\) as \((x_\gamma, \beta)\). If possible, select \(\mu_\gamma \in M_{\alpha^\gamma} \cap M_{\alpha^\gamma}\) such that \(\mu_\gamma (\bigcup_{\gamma < \beta} H_\gamma) = 0\), and let \(F_\gamma\) be a first category yet dense \(F_{\gamma^\beta}\) subset of \([0, 1]\) that supports \(\mu_\gamma\) and such that \(F_\gamma \cap (\bigcup_{\gamma < \beta} H_\gamma) = 0\). If the selection of \(\mu_\gamma\) is not possible, let \(F_\gamma = \emptyset\). Now let \(\eta_\gamma \in G_\gamma\) be such that \(\eta_\gamma (\bigcup_{\gamma < \beta} F_\gamma) = 0\), and choose \(H_\gamma\) to be a first category \(F_\gamma\) set in \([0, 1]\) that supports \(\eta_\gamma\) and such that \(H_\gamma \cap (\bigcup_{\gamma < \beta} F_\gamma) = 0\).

Set \(X = \bigcup_{\beta < \omega_1} F_\beta\). Now \(P(X)\) is not PC for it contains no \(G_\beta\); i.e., it contains no dense TC subspace.

However, \(P(X)\) is SBC, for suppose that \(M \subset P(X)\) is closed. Let \(M_{\gamma}\) be the \(P[0, 1]\)-closure of \(M\). It suffices to show that \(M\) is not first category in itself, so suppose that it is. Then \(M \subset M_{\alpha} \cap F_\gamma\) for some \(\alpha\). Let \(\gamma = g^{-1}(x_\alpha, \beta)\). At level \(\gamma\), if it was possible to select \(\mu_\gamma\), then we have a contradiction because \(\mu_\gamma \in M_{\alpha^\gamma} \cap M_{\alpha^\gamma}\) and \(\mu_\gamma \in P(X)\), hence \(\mu_\gamma \in M_{\alpha^\gamma} \cap M_{\alpha^\gamma}\). To finish the proof then, we need only to show that it was possible to select \(\mu_\gamma\). Notice that a dense subset (namely \(M\)) of \(M_{\alpha^\gamma}\) assigns \(H = \bigcup_{\beta < \omega_1} H_\beta\) measure 0. Furthermore, \(\{\mu \in M_{\gamma} : (\mu(0, 1) \cap H) = 1\}\) is \(G_\gamma\), so most of \(M_{\alpha^\gamma}\) assigns \(H\) measure 0, and hence there are measures in \(M_{\alpha^\gamma} \cap M_{\alpha^\gamma}\) which could be selected as \(\mu_\gamma\).
On Michael's problem concerning the Lindelöf property in the Cartesian products

by

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Abstract. In this paper we present a negative solution of Michael's conjecture which says that if $Y \times X$ is Lindelöf, for every hereditarily Lindelöf space $Y$, then $Y \times X^n$ is Lindelöf, for every hereditarily Lindelöf space $Y$.

Introduction. It is known that if $Y$ is a hereditarily Lindelöf space and $X$ a metric separable space then $Y \times X$ and also $Y \times X^n$ are Lindelöf. Z. Frolik proved (see [F]) that if $Y$ is a hereditarily Lindelöf and $X$ is a Lindelöf and complete in the sense of Čech space then $Y \times X$ and also $Y \times X^n$ are Lindelöf. R. Telgarski showed (see [T]) that if $Y$ is a hereditarily Lindelöf space and $X$ a Lindelöf and scattered space then $Y \times X$ is Lindelöf. I have improved the result of Telgarski [Al], by showing that $Y \times X^n$ is Lindelöf. I think that these results were the motivation of Michael's conjecture which says that if the product $Y \times X$ is Lindelöf for every hereditarily Lindelöf space $Y$ then $Y \times X^n$ is Lindelöf for every hereditarily Lindelöf space $Y$. In this paper we proved that the answer to the Michael's conjecture is a negative one.

Examples.

Example 1. There exists $Z$ such that, for every natural number $n$ and for every hereditarily Lindelöf space $Y$, the product $Y \times Z^n$ is Lindelöf but $Z^n$ is not.

Example 2. There exist a separable metric space $M$ and a space $X$ such that, for every Lindelöf space $Y$ and every natural number $n$, the products $Y \times X^n$ and $X^n$ are Lindelöf but $M \times X^n$ is not.

It is easy to see that in order to obtain Example 1 it is enough to put $Z = M \times X$, where $M$ and $X$ are from Example 2.

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