(Such an r must exist by Observation 3.8 ii)). Thus by Ramsey's Theorem there must be an infinite $X \subset \omega$, which is homogeneous for f. That is, there is $s \leqslant n$ such that if i, j and k are in X and $i < j < k$ then

$$(*) \qquad M \models \psi_s(\bar{a}_k, \bar{c}_{ij}).$$

Now choose $i_0 < i_1 < i_2 < i_3 < i_4$ all in X. Thus by $(*)$, we have

$$M \models \psi_s(\bar{a}_{i_4}, \bar{c}_{i_0 i_1}) \wedge \psi_s(\bar{a}_{i_2}, \bar{c}_{i_0 i_1}) \wedge \psi_s(\bar{a}_{i_4}, \bar{c}_{i_2 i_3}).$$

But $\psi_s \in \Phi$. Thus it easily follows (from Definition 3.1 ii)) that $M \models \psi_s(\bar{a}_{i_2}, \bar{c}_{i_2 i_3})$. But this contradicts Observation 3.8 i). This contradiction proves that M is relatively homogeneous. As in the proof of Proposition 2.9 it follows that there is some type q of T which is omitted in M. Let N be prime over a realisation of q. Then M, N and the prime and countable saturated models of T give us our four models. So Proposition 3.3 is proved.

I should like to thank the C.N.R.S. for their financial support during a visit to Université Paris VII when some of the above results were obtained.

References

[1] W. Baur, *Elimination of quantifiers for modules*, Israel J. Math. 25 (1976), pp. 64–70.
[2] S. Garavaglia, *Decomposition of totally transcendental modules*, J. Symb. Logic 45 (1980), pp. 155–164.
[3] D. Lascar, *Définissabilité de types en théorie des modèles*, Thèse de Doctorat d'Etat, Université Paris VII, 1975.
[4] — and B. Poizat, *An introduction to forking*, J. Symb. Logic 44 (1979), pp. 77–88.
[5] A. Pillay, *Theories with exactly three countable models and theories with algebraic prime models*, J. Symb. Logic 45 (1980), pp. 302–310.
[6] — and M. Prest, *Forking and pushouts in modules*, preprint 1981.
[7] M. Prest, *Pure-injectives and T-injective hulls of modules*, preprint 1981.
[8] S. Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, North-Holland, Amsterdam 1978.
[9] M. Ziegler, *Model theory of modules*, preprint 1981.

DEPARTMENT OF MATHEMATICS
BEDFORD COLLEGE
Regent's Park
London NW1 4NS

Received 15 June 1981

Fixed point theorems and almost continuity

by

Vladimir N. Akis* (Davis, Ca.)

Abstract. In 1959, John Stallings asked the following question which he attributed to K. Borsuk: Suppose K is a non-separating planar continuum contained in the interior of a disk D. Is there an almost continuous function $r\colon D \to D$ such that $r(D) = K$ and $r|K = \mathrm{id}$? We answer this question negatively. We also show that if $X_0 \supset X_1 \supset \ldots \supset X_n \supset X_{n+1} \supset \ldots$ is a sequence of ARs, with retractions $f_n\colon X_{n-1} \to X_n$, such that $x \in X_{n-1} \sim X_n$ implies $f_n(x) \in \bigcap X_i$, then $\bigcap X_i$ has the fixed point property.

1. Introduction. Throughout this paper X, Y and Z will denote topological spaces. A *map* is a continuous function. When $f\colon X \to Y$ may not be continuous, we refer to it simply as the function f. An *absolute retract* (AR) is a retract of the Hilbert cube. A space X has the *fixed point property*, if for each map $f\colon X \to X$ there exists $x \in X$ such that $f(x) = x$. The *graph* of a function $f\colon X \to Y$ is the subset of $X \times Y$ consisting of the points $(x, f(x))$; this set will be symbolized $\Gamma(f)$.

J. Stallings [11] defined a class of functions, which he named almost continuous, for the purpose of studying the fixed point property.

DEFINITION 1 [11, p. 252]. A function $f\colon X \to Y$ is *almost continuous* if for each open subset $\mathscr{U}$ of $X \times Y$ such that $\Gamma(f) \subset \mathscr{U}$, there exists a map $g\colon X \to Y$ such that $\Gamma(g) \subset \mathscr{U}$.

THEOREM 1 [11, p. 252]. *A Hausdorff space X has the fixed point property if and only if every almost continuous function $f\colon X \to X$ leaves a point fixed.*

THEOREM 2 [11, p. 260]. *If $f\colon X \to Y$ is almost continuous and $g\colon Y \to Z$ is a map, then $gf\colon X \to Z$ is almost continuous.*

DEFINITION 2. If $Y \subset X$ and $r\colon X \to X$ is an almost continuous function such that $r(X) = Y$ and $r(x) = x$ for all $x \in Y$, then r is called a *quasi retraction* and Y is called a *quasi retract* of X [1].

* The author gratefully acknowledges numerous conversations about topics in this paper with Professor C. L. Hagopian.

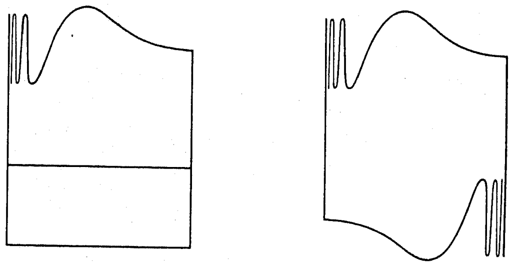
[1] In the Literature quasi retractions have been called almost continuous retractions. We have avoided the term "almost continuous retraction" because it has also been used for almost continuous $r\colon X \to Y$ such that $r(x) = x$ for all $x \in Y$.

THEOREM 3. *If X is a Hausdorff space with the fixed point property, $Y \subset X$ and each map $g: Y \to Y$ has a continuous extension $G: X \to X$, and if Y is a quasi retract of X, then Y has the fixed point property.*

Proof. Let $g: Y \to Y$ be a map and let $G: X \to X$ be its extension. Let $r: X \to X$ be a quasi retraction associated with Y. By Theorem 2, $Gr: X \to X$ is almost continuous. Hence by Theorem 1 there is $x \in X$ such that $Gr(x) = x$. But $r(x) \in Y$. So $Gr(x) = gr(x) \in Y$, therefore $x \in Y$. Thus $r(x) = x$. Hence $x = g(x)$.

COROLLARY. *If X is an AR and Y a closed quasi retract of X, then Y has the fixed point property.*

B. Garrett pointed out the assumption in Theorem 3, that each map $g: Y \to Y$ has a continuous extension $G: X \to X$, is essential by defining the following example. Let S be the $\sin 1/x$ circle and D a disk such that $S \cap D$ is an arc (see Fig. 1a). Let $X = S \cup D$. Let Y be the double $\sin 1/x$ circle, represented in Figure 1b. Even though Y is a quasi retract of X and X has the fixed point property, Y does not have the fixed point property.



(a) (b)

Fig. 1

Stallings' strategy was to prove that a certain continuum Y has the fixed point property, by exhibiting an AR, X, containing Y as a quasi retract. In particular he asked the following question, which he attributes to Borsuk: Let C be an acyclic planar continuum contained in the interior of a disk D. Is Y a quasi retract of D? [11, p. 263]. In this paper we answer this question negatively, by showing that the disk with a spiral about its boundary is not a quasi retract of any AR. This object is represented in Figure 2.

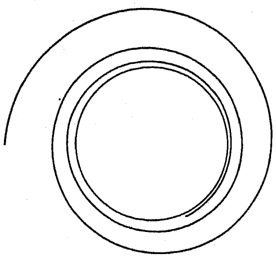


Fig. 2.

2. Almost continuous approximation. An almost continuous function is a function approximated by maps in the sense of Definition 1. A natural question to ask is: How is this approximation associated to the familiar pointwise and uniform convergence of functions? The following definitions and theorems answer this question.

Given spaces X, Y, we let $C(X, Y)$ denote the space of all maps from X into Y, with the compact open topology. We let $A(X, Y)$ denote the set of all almost continuous functions from X into Y. If S is a subset of X by $\mathrm{Cl}\, S$ we denote the closure of S in X.

DEFINITION 3. A sequence $\{f_n\}$ of functions of X into Y *almost continuously approximates* a function $f: X \to Y$ if for every sequence $\{x_n\} \subset X$, either there exists n such that $f_n(x_n) = f(x_n)$ or there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ and $x \in X$ such that $x_{n_i} \to x$ and $f_{n_i}(x_{n_i}) \to f(x)$.

The next theorem shows that if $\{f_n\}$ almost continuously approximates f, then $\{f_n\}$ approximates f in the same sense that maps approximate an almost continuous function.

THEOREM 4. *The sequence $\{f_n\}$ almost continuously approximates f if and only if for each open $\mathcal{U} \subset X \times Y$, $\Gamma(f) \subset \mathcal{U}$ implies that for some n, $\Gamma(f_n) \subset \mathcal{U}$.*

Proof. For the "only if" part let $B = X \times Y \sim \mathcal{U}$. If for all n, $\Gamma(f_n) \not\subset \mathcal{U}$, then for each n there exists x_n such that $(x_n, f_n(x_n)) \in B$. But $\{f_n\}$ almost continuously approximates f and since B is closed we infer that $\Gamma(f) \cap B \neq \emptyset$. A contradiction.

For the "if" part assume $\{x_n\} \subset X$. Let $\mathcal{U} = X \times Y \sim \mathrm{Cl}\,\{(x_n, f_n(x_n))\}$. Since for all n, $\Gamma(f_n) \not\subset \mathcal{U}$, we conclude that $\Gamma(f) \not\subset \mathcal{U}$ so $\Gamma(f) \cap \mathrm{Cl}\,\{(x_n, f_n(x_n))\} \neq \emptyset$. Hence $\{f_n\}$ almost continuously approximates f.

THEOREM 5. *If $\{f_n\} \subset A(X, Y)$ and $\{f_n\}$ almost continuously approximates f, then $f \in A(X, Y)$.*

Proof. For any open set $\mathcal{U}$ such that $\Gamma(f) \subset \mathcal{U} \subset X \times Y$, there is some n such that $\Gamma(f_n) \subset \mathcal{U}$, but since f is almost continuous there is a map $g: X \to Y$ such that $\Gamma(g) \subset \mathcal{U}$.

Theorems 4 and 5 apply to arbitrary topological spaces X and Y. We proceed to show in the case that X and Y are compact and metrizable, $f: X \to Y$ is almost continuous if and only if for some sequence $\{f_n\} \subset C(X, Y)$, $\{f_n\}$ almost continuously approximates f. More specifically we show that if $\{f_n\}$ is a countable dense subset of $C(X, Y)$, then for any almost continuous function $f: X \to Y$, $\{f_n\}$ almost continuously approximates f. Hence by Theorem 4, for every open $\mathcal{U} \subset X \times Y$ if $\Gamma(f) \subset \mathcal{U}$ then for some n, $\Gamma(f_n) \subset \mathcal{U}$.

DEFINITION 4. A sequence $\{f_n\}$ in $C(X, Y)$ *converges continuously* to an $f \in C(X, Y)$ if $f_n(x_n) \to f(x)$ for each $x \in X$ and sequence $x_n \to x$.

It turns out that if X and Y are compact metric spaces, continuous convergence is equivalent to uniform convergence in $C(X, Y)$ [3, p. 268].

THEOREM 6. *Assume X and Y are compact metric spaces. Then $f \in A(X, Y)$ if and only if there exists a sequence $\{f_n\} \subset C(X, Y)$ such that $\{f_n\}$ almost continuously approximates f.*

Proof. The "if" part follows from Theorem 4 and the fact that $C(X, Y) \subset A(X, Y)$.

For the "only if" part suppose $f \in A(X, Y)$. Let $\{f_n\}$ be a countable dense subset of $C(X, Y)$. We claim $\{f_n\}$ almost continuously approximates f. Given any sequence $\{x_n\} \subset X$, let $B = \mathrm{Cl}\{(x_n, f_n(x_n))\}$. If $g \in C(X, Y)$ then there exists a subsequence $\{f_{n_i}\}$ which converges uniformly, and hence continuously to g. We may assume $\{x_{n_i}\}$ is convergent (if not we could take a convergent subsequence of $\{x_{n_i}\}$), say $x_{n_i} \to x$. Then $f_{n_i}(x_{n_i}) \to g(x)$, so $(x, g(x)) \in B$. We have shown that $g \in C(X, Y)$ implies that $\Gamma(g) \cap B \neq \emptyset$ and since $f \in A(X, Y)$ we have that $\Gamma(f) \cap B \neq \emptyset$. From the definition of B it follows that $\{f_n\}$ almost continuously approximates f.

From the proof of Theorem 6, we extract the following:

COROLLARY. *If X and Y are compact metric spaces, then for any $f \in A(X, Y)$, and any countable dense subset $\{f_n\}$ of $C(X, Y)$, $\{f_n\}$ almost continuously approximates f.*

From this corollary we conclude that $\{f_n\}$ almost continuously approximates f does not imply that $\{f_n\}$ converges pointwise to f. Therefore in the spirit of Definition 4, we introduce the idea of almost continuous convergence.

DEFINITION 5. A sequence $\{f_n\}$ in $A(X, Y)$ *converges almost continuously* to f, if $\{f_n\}$ converges pointwise to f and $\{f_n\}$ almost continuously approximates f.

Clearly if some sequence $\{f_n\} \subset C(X, Y)$ converges almost continuously to f, then $f \in A(X, Y)$, the converse however is not true. K. Kellum [6], has defined a function $f \in A(I, Y)$, where I is an arc and Y any 2nd countable space, such that the graph of f is dense in $I \times Y$. From [8, p. 394, Thm. 1] it follows that if X and Y are compact metric spaces and if $\{f_n\} \subset C(X, Y)$ converges pointwise to f, then the graph of f is nowhere dense. In this paper however all the examples of almost continuous functions are of the type described in Definition 5.

3. The main results. We now proceed to show that the disk with a spiral about its boundary is not a quasi retract of an AR.

DEFINITION 6. For any space X, the *cone TX over X*, is the quotient space $(X \times I)/R$, where R is the equivalence relation $(x, t) \sim (y, s)$ if and only if $t = s = 1$ or $x = y$ and $t = s$ for all $x, y \in X$ and $s, t \in I$.

By $\langle x, t\rangle$ we denote the equivalence class of $(x, t) \in X \times I$. Given a function $f\colon X \to Y$ we define a function $Tf\colon TX \to TY$ by the rule $Tf\langle x, t\rangle = \langle f(x), t\rangle$.

LEMMA [3, p. 127]. *If $f \in C(X, Y)$ then $Tf \in C(TX, TY)$.*

LEMMA. *Suppose X and Y are compact metric spaces. If $f \in A(X, Y)$ then $Tf \in A(TX, TY)$.*

Proof. Since $f \in A(X, Y)$ there exists a sequence $\{f_n\} \subset C(X, Y)$ such that $\{f_n\}$ almost continuously approximate f. By the above lemma, $\{Tf_n\} \subset C(TX, TY)$. We claim that $\{Tf_n\}$ almost continuously approximates Tf. Let for $n = 1, 2, 3, \ldots$, $\langle x_n, t_n\rangle \in TX$. Either there is an n such that $f_n(x_n) = f(x_n)$, hence $\langle f_n(x_n), t_n\rangle = \langle f(x_n), t_n\rangle$, or for some subsequence $\{x_{n_i}\}$ and $x \in X$, $x_{n_i} \to x$ and $f_{n_i}(x_{n_i}) \to f(x)$. In the last case assume with no loss of generality that $\{t_{n_i}\}$ is convergent, say $t_{n_i} \to t$. Hence $\langle x_{n_i}, t_{n_i}\rangle \to \langle x, t\rangle$ and $\langle f_{n_i}(x_{n_i}), t_{n_i}\rangle \to \langle f(x), t\rangle$. We have shown that either there is an n such that $Tf_n\langle x_n, t_n\rangle = Tf\langle x_n, t_n\rangle$ or for some subsequence $\{\langle x_{n_i}, t_{n_i}\rangle\}$, $\langle x_{n_i}, t_{n_i}\rangle \to \langle x, t\rangle$ and $Tf_{n_i}\langle x_{n_i}, t_{n_i}\rangle \to Tf\langle x, t\rangle$. Thus $\{Tf_n\}$ almost continuously approximates Tf.

THEOREM 7. *Suppose X is a compact metric space, and that Y is a quasi retract of X, then TY is a quasi retract of TX.*

Proof. If $r\colon X \to X$ is a quasi retraction associated with Y, then $Tr\colon TX \to TX$ is a quasi retraction associated with TY.

COROLLARY. *If X is an* AR *and Y is a quasi retract of X then TY has the fixed point property.*

R. Knill has shown that the cone over the disk with a spiral about its boundary does not have the fixed point property [7], [2]. We conclude that the disk with a spiral about its boundary is not a quasi retract of an AR.

At this point we would like to make the following parenthetical remark. If we alter Definition 2 to require $r \in A(X, Y)$, instead of $r \in A(X, X)$, we obtain a class of functions different from the quasi retracts, which we call *almost continuous retracts*. For a contrast of these two types of retracts we refer the reader to B. Garrett's article "Almost continuous retracts" [4]. It is easy to see that $A(X, Y) \subset A(X, X)$, hence an almost continuous retract of X is a quasi retract of X, but the converse is not true. K. Kellum studied almost continuous retracts and has the following theorem: Given a 2nd countable space Y, there exists a Peano continuum P such that $A(P, Y) \neq \emptyset$ if and only if Y is almost Peano. That Y is almost Peano means that for each finite collection of nonempty open subsets of Y there is a Peano continuum in Y which intersects each of them [6]. Using this result we see that the double sin $1/x$ curve, represented by Figure 3, is not the almost continuous retract of a disk containing it. But as we will show later, this space is a quasi retract of a disk.

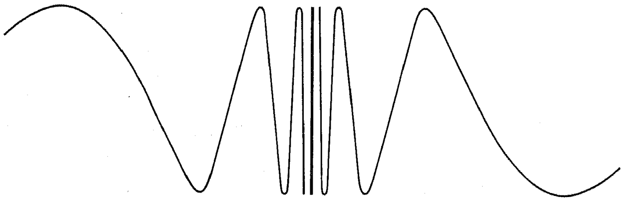


Fig.3

Returning our attention to the disk with a spiral about its boundary, we see that since it is not almost Peano, Kellum's theorem would assert that it is not an almost continuous retract of a disk. This is consistent with our result which says that the disk with a spiral about its boundary is not even a quasi retract of a disk.

We now continue with the study of some other properties of quasi retracts. First we point out that there is a theorem involving the suspension of a space, analogous to Theorem 7.

DEFINITION 7. For any space X, the *suspension* SX *of* X, is the quotient space $(X\times I)/R$, where R is the equivalence relation $(x, t)\sim(y, s)$ if and only if $t=s=1$ or $t=s=0$ or $x=y$ and $t=s$, for $x, y\in X$ and $s, t\in I$.

THEOREM 8. *Suppose X is a compact metric space, and that Y is a quasi retract of X, then SY is a quasi retract of SX.*

The proof of Theorem 8 is similar to the proof of Theorem 7.

DEFINITION 8. A closed subset Y of the Hilbert cube H, is an *absolute quasi retract* (AQR) if Y is a quasi retract of H.

THEOREM 9. *The following are equivalent.*

1) *Y is an* AQR.

2) *Y is a closed subset of the Hilbert cube H, and Y is a quasi retract of any* AR *containing it.*

3) *Y is a closed quasi retract of an* AR.

Proof. We first show that 1) implies 2). Let X be an AR containing Y. Let $f\in A(H, H)$ be a quasi retraction associated with Y. By [11; p. 260, Prop. 2] $f|X\in A(X, H)$. Let r: $H\to X$ be a continuous retraction. By Theorem 2 $rf|X\in A(X, X)$. Thus Y is a quasi retract of X with $rf|X$ its associated quasi retraction. It is clear that 2) implies 3). It remains to show that 3) implies 1). If X is an AR and Y a closed quasi retract of X, let r: $H\to X$ be a continuous retraction and $f\in A(X, X)$ be a quasi retraction associated with Y. Let $\{f_n\}\subset C(X, X)$ such that $\{f_n\}$ almost continuously approximates f. Extend each f_n to $F_n\in C(H, H)$. We claim $\{F_n r\}$ almost continuously approximates fr. Let $\{x_n\}\subset H$, then $\{r(x_n)\}\subset X$. Hence either there exists n such that $f_n r(x_n)=fr(x_n)$, in which case $F_n r(x_n)=fr(x_n)$, or there exists a subsequence $\{x_{n_i}\}$ such that $r(x_{n_i})\to y$ and $f_{n_i} r(x_{n_i})\to f(y)$. We may assume with no loss of generality that $x_{n_i}\to x$. Hence $r(x_{n_i})\to r(x)=y$. Thus $f_{n_i} r(x_{n_i})=F_{n_i} r(x_{n_i})\to fr(x)$. Therefore $fr\in A(H, H)$, is the quasi retraction associated with Y.

We will show that the product of an AR with an AQR is an AQR, for this we will use the following:

THEOREM 10. *Let X, Y and Z be compact metric spaces. If $f\in A(X, Y)$ and $g\in C(X, Z)$ then $F(x)=\big(f(x), g(x)\big)$ is in $A(X, Y\times Z)$.*

Proof. Let $\{f_n\}\subset C(X, Y)$ such that $\{f_n\}$ almost continuously approximates f. For each n let $F_n(x)=\big(f_n(x), g(x)\big)$, $F_n\in C(X, Y\times Z)$. It is easy to show that $\{F_n\}$ almost continuously approximates F.

COROLLARY. *Let X, Y, Z and W be compact metric spaces. If $f\in A(X, Y)$ and $g\in C(Z, W)$, then $F\in A(X\times Z, Y\times W)$, where $F(x, z)=\big(f(x), g(z)\big)$.*

Proof. Let $p_1\in C(X\times Z, X)$, $p_2\in C(X\times Z, Z)$ be the projection maps. Then $fp_1\in A(X\times Z, Y)$ by [11, p. 261] and $gp_2\in C(X\times Z, W)$. Since $F(x, z)=\big(fp_1(x, z), gp_2(x, z)\big)$, by Theorem 10, $F\in A(X\times Z, Y\times W)$.

THEOREM 11. *If X is an* AR *and if Y is an* AQR, *then $X\times Y$ is an* AQR.

Proof. Let H be the Hilbert cube, r: $H\to X$ be a continuous retraction, and f: $H\to H$ be a quasi retraction associated with Y. Let $R(a, b)=\big(r(a), f(b)\big)$. By the previous corollary, $R\in A(H\times H, H\times H)$. Hence $X\times Y$ is a quasi retract of $H\times H$ and thus it is an AQR.

COROLLARY. *If X is an* AR *and Y an* AQR, *then $X\times Y$ has the fixed point property.*

R. Knill [7] has shown that if Y is the can-with-a-skirt in Figure 8 of [2], then Y has the fixed point property, but $I\times Y$ does not have the fixed point property. From the above corollary, we conclude that Y is not an AQR.

One might hope to get a stronger version of Theorem 10, by allowing $g\in A(X, Z)$. The following example shows this cannot be done.

Let $B=\{z\colon |z|<1\}$ be the unit disk in the plane.

Let $I=[-1, 1]$ and define functions f: $I\times I\to I$ and g: $I\times I\to I$ as follows:

$$f(z)=\begin{cases}\cos\dfrac{2\pi}{1-|z|} & \text{if } |z|<1,\\ p_1(z) & \text{if } |z|\geqslant 1,\end{cases}\qquad g(z)=\begin{cases}\sin\dfrac{2\pi}{1-|z|} & \text{if } |z|<1,\\ p_2(z) & \text{if } |z|\geqslant 1.\end{cases}$$

We claim that $f, g\in A(I\times I, I)$. For $n=2, 3, 4, \ldots$ the restriction of the function $\cos x$ to the interval $[n\pi, (n+1)\pi]$ is a homeomorphism. Hence its inverse $\cos_n^{-1}x$ is a homeomorphism of I onto $[n\pi, (n+1)\pi]$. Similarly, define $\sin_n^{-1}$: $I\to[n\pi, (n+1)\pi]$. Let for $n=2, 3, 4, \ldots$

$$f_n = \begin{cases} \cos\dfrac{2\pi}{1-|z|} & \text{if } |z| \leqslant 1-\dfrac{2\pi}{\cos_n^{-1} p_1(z)}, \\ p_1(z) & \text{if } |z| > 1-\dfrac{2\pi}{\cos_n^{-1} p_1(z)}, \end{cases}$$

$$g_n = \begin{cases} \sin\dfrac{2\pi}{1-|z|} & \text{if } |z| \leqslant 1-\dfrac{2\pi}{\sin_n^{-1} p_2(z)}, \\ p_2(z) & \text{if } |z| > 1-\dfrac{2\pi}{\sin_n^{-1} p_2(z)}, \end{cases}$$

$f_n, g_n \in C(I\times I, I)$ because if $|z| = 1-\dfrac{2\pi}{\cos_n^{-1} p_1(z)}$ then $p_1(z) = \cos\dfrac{2\pi}{1-|z|}$, and if $|z| = 1-\dfrac{2\pi}{\sin_n^{-1} p_2(z)}$, then $p_2(z) = \sin\dfrac{2\pi}{1-|z|}$. We claim that $\{f_n\}$ almost continuously approximates f and that $\{g_n\}$ almost continuously approximates g. To check this, let $\{z_n\} \subset I\times I$. If for each $n \geqslant 2$, $f_n(z_n) \neq f(z_n)$, then $|z_n| < 1$ and $f_n(z_n) = p_1(z_n)$. Let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ converging to some $z\in I\times I$. Because $f_{n_i}(z_{n_i}) = p_1(z_{n_i})$, we must have that $|z| = 1$, hence $p_1(z) = f(z)$. By the continuity of p_1 we conclude that $f_{n_i}(z_{n_i}) \to f(z)$. We have shown that $\{f_n\}$ almost continuously approximates f.

Similarly one can show that $\{g_n\}$ almost continuously approximates g. Hence, $f, g\in A(I\times I, I)$. Now let $F(z) = (f(z), g(z))$. If $F\in A(I\times I, I\times I)$ then $I\times I \sim B$ would be a quasi retract of $I\times I$. This is a contradiction because $I\times I\sim B$ does not have the fixed point property. Therefore $F\notin A(I\times I, I\times I)$ even though $f, g\in A(I\times I, I)$.

Theorems 7, 8 and 11 tell us how to construct new quasi retracts from old. The next theorem is of a different nature, giving a sufficient condition for a space to be a quasi retract.

Theorem 12. *If X_0 is compact, and if $X_0 \supset X_1 \supset X_2 \supset \ldots$ is a sequence of subspaces of X_0, with retractions f_n: $X_{n-1} \to X_n$, such that $x\in X_{n-1} \sim X_n$ implies $f_n(x) \in \bigcap X_i$, then $\bigcap X_i$ is a quasi retract of X_0.*

Proof. We define a function f: $X_0 \to X_0$ as follows: $f(x) = x$ if $x\in \bigcap X_i$ and $f(x) = f_n(x)$ if $x\in X_{n-1} \sim X_n$. Let $g_n = f_n f_{n-1} \ldots f_1$ for $n = 1, 2, 3, \ldots$; $g_n \in C(X_0, X_0)$. We claim that $\{g_n\}$ converges almost continuously to f, and hence $f\in A(X_0, X_0)$, thus f is a quasi retraction. It is clear that $\{g_n\}$ converges pointwise to f. We proceed to show that $\{g_n\}$ almost continuously approximates f.

Let $A_n = \{x\in X_0: f(x) \neq g_n(x)\}$, we show that $A_n = X_n \sim \bigcap X_i$. If $x\in X \sim X_n$ then for some $m \leqslant n$, $x\in X_{m-1} \sim X_m$. Hence $f_m(x) = f(x)$ and $f_1(x) = f_2(x) = \ldots = f_{m-1}(x) = x$. Therefore $f(x) = g_m(x) \in \bigcap X_i$. Hence $g_n(x) = f_n f_{n-1} \ldots g_m(x) = g_m(x)$. Thus $f(x) = g_n(x)$, and we conclude that $A_n \subset X_n$. If $x\in \bigcap X_i$ then $f(x) = x = g_n(x)$. Therefore $A_n \subset X_n \sim \bigcap X_i$. If $x\in X_n \sim \bigcap X_i$, then for some $k > n$, $x\in X_{k-1} \sim X_k$. Hence $f(x) = f_k(x) \in \bigcap X_i$, thus $x \neq f(x)$. But since $x\in X_n$, for any $m \leqslant n$, $f_m(x) = x$, thus $g_n(x) = x$. Therefore $f(x) \neq g_n(x)$, and we conclude that $X_n \sim \bigcap X_i \subset A_n$.

Now we are ready to verify the conditions of Definition 3. Let $\{x_n\} \subset X_0$. If for all n, $f_n(x_n) \neq f(x_n)$ then for each n, $x_n \in A_n$. Let $x_{n_i} \to x$. Now since $A_n \subset X_n$, $f_{n_i}(x_{n_i}) = x_{n_i}$. So $f_{n_i}(x_{n_i}) \to x$. Also $x\in \bigcap \mathrm{Cl}\, A_n \subset \bigcap X_i$. Hence $f(x) = x$.

Corollary. *For $n = 1, 2, 3, \ldots$ let X_n be as in Theorem 12, and also assume that X_0 is an AR, then $\bigcap X_i$ has the fixed point property.*

Theorem 12 may be used to show that for the spaces X and Y represented in Figure 1, Y is a quasi retract of X. Also one can use Theorem 12 to show that the double $\sin 1/x$ curve of Figure 3 is a quasi retract of a disk containing it. The cone over a Cantor set, Knaster's U-continuum [9, p. 205] and Ingram's T-like non-chainable continuum [5], represented in Figure 4, are also examples of quasi retracts of a disk containing them, by virtue of Theorem 12.

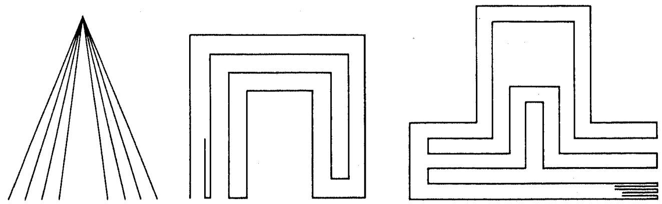
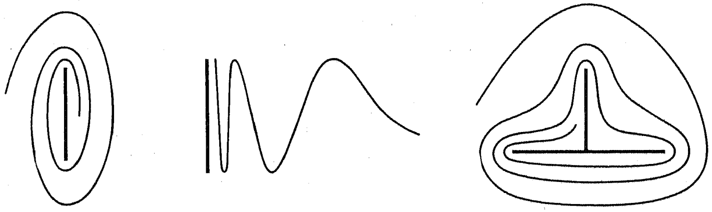


Fig.4

Let D be a topological disk. Dig into D, a canal, by removing from D, the interior of a topological disk which intersects the boundary of D at an arc (this arc is also removed). In the resulting continuum, dig a canal as described above, starting the canal from the boundary of D. Continue this process inductively, always starting a canal from the boundary of D. The continuum thus obtained will be a quasi retract of the disk D by virtue of Theorem 12. Hence it will have the fixed point property. Each of the continua of Figure 4 can be constructed in this manner. As indicated by Theorem 7, the disk with a spiral (Figure 2) is not a quasi retract of a disk therefore it is not of this type. Note that the canal of this continuum is not obtained by removing from a disk the interior of a topological disk. However, according to the Bell-Siekluckі Theorem [1], [10], the continuum of Figure 2 has the fixed point property. This theorem states that if a non-separating planar continuum does not have the fixed point property, then it has an indecomposable subcontinuum in its boundary.

Let X be an arc with a spiral as shown in Figure 5a. From our result that the disk with a spiral of Figure 2 is not a quasi retract of a disk, one may also think that X is not a quasi retract of a disk. This is not so. The continuum X can be embedded as the $\sin 1/x$ curve as shown in Figure 5b. By Theorem 12, the continuum in Figure 5b is a quasi retract of a disk.



(a) (b) (c)

Fig.5

It is also easy to show that being a quasi retract of a disk is a topological property, i.e., it does not depend on the embedding. Hence the continuum in Figure 5a is a quasi retract of the disk. It is not known if the triod with a spiral shown in Figure 5c is a quasi retract of a disk.

References

[1] H. Bell, *On fixed point properties of plane continua*, Trans Amer. Math. Soc. 128 (1967), pp. 539–548.

[2] R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly 76 (1969), pp. 119–132.

[3] J. Dugundji, *Topology*, Allyn and Bacon, Inc. 1978.

[4] B. D. Garrett, *Almost continuous retracts*, in: *General Topology and Modern Analysis* (L. C. McAuley and M. M. Rao, eds.), Academic Press, New York 1981, pp. 229–238.

[5] W. T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. 77 (1972), pp. 99–107.

[6] K. R. Kellum, *Almost continuous images of Peano continua*, Top. and Appl. 11 (1980), pp. 293–296.

[7] R. J. Knill, *Cones, products and fixed points*, Fund. Math. 60 (1967), pp. 35–46.

[8] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1968.

[9] – *Topology*, Vol. II, New York–London–Warszawa 1968.

[10] K. Sieklucki, *On a class of plane acyclic continua with the fixed point property*, Fund. Math. 63 (1968), pp. 257–278.

[11] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47 (1959), pp. 249–263.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Davis, CA. 95616

Received 7 September 1981

Baire category in spaces of probability measures, II

by

J. B. Brown (Auburn, Ala.) and **G. V. Cox** (Redlands, Ca.)

Abstract. Completeness relationships for a space X, and its space of probability measures $P(X)$ are compared. All implications between X and $P(X)$ and between compactness, local compactness, topological completeness, pseudo completeness, Baire completeness, and strong Baire completeness are resolved. The continuum hypothesis has been assumed when needed.

1. Introduction. In [B], completeness relationships between a separable metric space (X, d) and the space of probability measures on X endowed with the separable metric of weak convergence, $(P(X), \varrho)$ were investigated. It was shown that X PC $\to P(X)$ PC $\to P(X)$ BC $\to X$ BC and none of the implications are reversible. Here, as in [B], TC means topologically complete, PC means pseudo complete (i.e., contains a dense TC subspace), and BC means Baire complete (i.e., is a Baire space). We also denote strongly Baire complete by SBC. A space is SBC if every closed subspace is BC.

Based upon results of Prohorov [P] and Luther [L], we know that X compact $\leftrightarrow P(X)$ compact $\leftrightarrow P(X)$ locally compact, and also that X TC $\leftrightarrow P(X)$ TC. The purpose of this paper is to resolve the following diagram.