[4] C. E. Burgess, *Homogeneous Continua*, Summer Inst. on Set Theor. Topology, Madison, Wisconsin, 1955, pp. 75–78.
[5] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. 81 (1965), pp. 38–55.
[6] P. Fletcher, *Note on quasi-uniform spaces and representable spaces*, Coll. Math. 23 (1971), pp. 263–265.
[7] L. R. Ford, Jr., *Homeomorphism groups and coset spaces*, Trans. Amer. Math. Soc. 77 (1974), pp. 490–497.
[8] K. Kuperberg, W. Kuperberg, and W. R. R. Transue, *On the 2-homogeneity of Cartesian products*, Fund. Math. 110 (1980), pp. 131–134.
[9] G. S. Ungar, *Countable dense homogeneity and n-homogeneity*, Fund. Math. 99 (1978), pp. 155–160.
[10] — *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. 212 (1975), pp. 393–400.
[11] — *Products and n-homogeneity*, preprint.

Received 5 May 1980

A generalized version of the singular cardinals problem

by

Arthur W. Apter (Newark, N. J.)

Abstract. We show that it is consistent, relative to the existence of an unbounded class of cardinals each of which possesses a certain degree of supercompactness, for every limit cardinal to be a strong limit cardinal and for the ωth successor of any cardinal to violate GCH.

The behaviour of the power sets of singular cardinals has long been of interest to set theorists. Shortly after Cohen invented forcing, Easton in his thesis [2] showed that, roughly speaking, the power sets of regular cardinals could be anything desired within the technical restrictions of $2^{\varkappa_1} \leqslant 2^{\varkappa_2}$ if $\varkappa_1 \leqslant \varkappa_2$ and $\operatorname{cof}(2^{\varkappa}) > \varkappa$. No such results, however, were known for singular cardinals for quite a while. Indeed, the famous singular cardinals problem asks whether it is consistent to have $2^{\aleph_n} = \aleph_{n+1}$ for all natural numbers n and yet also have $2^{\aleph_\omega} = \aleph_{\omega+2}$, or more generally, whether or not it is consistent for a singular cardinal to be the least cardinal that violates GCH.

Much light has been shed on the singular cardinals problem within the last few years. It is of course now known by the work of Silver [13] that if a singular cardinal of uncountable cofinality violates GCH, then there is a stationary set of cardinals less than it which also violates GCH. This settles the generalized version of the singular cardinals problem. It is also known, by the work of Jensen [1], that if a singular cardinal (by necessity of cofinality ω) is the first cardinal to violate GCH, then there is an inner model with a measurable cardinal.

Magidor was the first person who obtained positive results in the direction of the singular cardinals problem: Starting with models in which $\varkappa$ possessed a certain degree of supercompactness and violated GCH, he was able to force and obtain a model in which $\aleph_\omega$ is a strong limit cardinal and yet violates GCH [8]. Then, starting with an enormously powerful hypothesis, namely the existence of a supercompact cardinal with a huge cardinal above it, Magidor was able to get a model in which $2^{\aleph_n} = \aleph_{n+1}$ for every natural number n and yet $2^{\aleph_\omega} = \aleph_{\omega+2}$ [9], i.e., Magidor was able to solve the singular cardinals problem. Of course, by the above mentioned result of Jensen, some large cardinal hypothesis is necessary.

Given Magidor's results, one wonders just how far this pattern for singular cardinals can extend. For example, is it consistent for every cardinal $\varkappa$ of cofinality ω to satisfy $2^\varkappa = \varkappa^{++}$ and yet have every other cardinal λ satisfy $2^\lambda = \lambda^+$? Or, in a weaker form, is it possible to have every cardinal of cofinality ω a strong limit cardinal which violates GCH?

We have obtained a partial answer to the second of the above two questions. We have shown that, relative to certain hypotheses, it is consistent for every limit cardinal to be a strong limit cardinal and for the ωth successor of any cardinal to violate GCH. Specifically, we have the following theorem:

THEOREM 1. *Let* $V \models$ "ZFC $+ 2^\varkappa = \varkappa^+$ *for* $\varkappa$ *a singular cardinal* $+$ $+ \langle \delta_\alpha : \alpha \in \text{Ordinals}\rangle$ *is an unbounded sequence of cardinals such that*:

1. *Each* δ_α *is* δ_α^+ *supercompact.*
2. $2^{\delta_\alpha} = \delta_\alpha^{++}$ *and* $2^{\delta_\alpha^+} = \delta_\alpha^{+++}$".

Then there is a model, $\bar{V}$, *for the theory* "ZFC $+$ *Every limit cardinal is a strong limit cardinal* $+$ *The* ω*-th successor of any cardinal violates* GCH".

Note that in $\bar{V}$, for λ the ωth successor of a cardinal, we have $2^\lambda = \lambda^{++}$. If we want the power sets of such λ to be larger, e.g., if we want $2^\lambda = \lambda^{+n}$ for n a natural number, then we assume that each δ_α is $\delta_\alpha^{+(n-1)}$ supercompact, $2^{\delta_\alpha} = \delta_\alpha^{+n}$, and $2^{\delta_\alpha^{+(n-1)}} = \delta_\alpha^{+n}$.

Note also that the hypotheses of Theorem 1 are much weaker than the existence of a supercompact cardinal. For example, using Silver's backwards Easton techniques ([10] and [12]), the existence of a cardinal λ which is a regular limit of cardinals δ each of which are δ^{++} supercompact gives a model for the hypothesis of Theorem 1.

The proof of Theorem 1 will use a generalized version of the partial ordering introduced by Magidor in [8] to obtain a model in which $\aleph_\omega$ is a strong limit cardinal which violates GCH. First, though, we shall briefly digress to give our notation and recall certain useful facts.

1. Preliminaries. Throughout, we work in ZFC. Our notation is reasonably standard. Lower case Greek letters $\alpha, \beta, \gamma, \ldots$ will be used to denote ordinals, with the letters $\varkappa, \lambda, \delta$ generally being reserved for cardinals. For ultrafilters and measures, we use the letters U and μ.

For $\varkappa$ a cardinal, $\varkappa^{+\alpha}$ will denote the αth least cardinal $> \varkappa$. The cofinality of $\varkappa$, $\text{cof}(\varkappa)$, is the least possible cardinality of an unbounded subset of $\varkappa$.

Given a set x, we shall let 2^x denote the power set of x. $\bar{x}$ will be the cardinality of x, and $\bar{\bar{x}}$ is the order type of x. Further, for f a function, $f''x$ is the range of f on x, and $f \restriction x$ is f restricted to x.

For α an ordinal, $R(\alpha)$ is taken to be the collection of all sets of rank $< \alpha$. For $\varkappa$ and λ cardinals, $\varkappa < \lambda$, $P_\varkappa(\lambda) = [\lambda]^{<\varkappa} = \bigcup_{\alpha<\varkappa} \{f\colon f$ is a strictly increasing function from α to $\lambda\}$.

The symbol $\Vdash$ will mean, as usual, "weakly forces", and $\|$ will mean "decides". By convention, we will say that for forcing conditions p and q, $p \leqslant q$ means that q is stronger than p.

We will make use of Solovay's Product Lemma for product forcing, so we recall its statement here. Let P and R be partial orderings defined in V. Then if $G_1 \subseteq P$, $G_2 \subseteq R$, the following are equivalent:

1. $G_1 \times G_2$ is V-generic over $P \times R$.
2. G_1 is V-generic over P and G_2 is $V[G_1]$-generic over R.
3. G_2 is V-generic over R and G_1 is $V[G_2]$-generic over P.

We assume that the reader is quite familiar with the notions of measurable cardinal and supercompact cardinal. For definitions and facts about these cardinals, consult Solovay–Reinhardt–Kanamori [14].

We recall Magidor's notion of forcing [8] which makes $\aleph_\omega$ a strong limit cardinal that violates GCH, as well as some key lemmas about this notion of forcing.

First of all, for $p, q \in P_\varkappa(\lambda)$, we say that $p \subsetneqq q$ iff $p \subseteq q$ and $\bar{p} < \overline{q \cap \varkappa}$. Next, we recall the following general lemma about normal ultrafilters over $P_\varkappa(\lambda)$.

LEMMA 2. *Let* U *be a normal ultrafilter over* $P_\varkappa(\lambda)$. *Then*:

1. $\{p\colon \overline{p \cap \varkappa}$ *is strongly inaccessible*$\} \in U$.
2. *For* $\alpha \leqslant \lambda$ *a cardinal* (*a regular cardinal*) $\{p\colon \overline{p \cap \alpha}$ *is a cardinal* (*a regular cardinal*)$\} \in U$.
3. *For* $\gamma < \varkappa$, $\alpha, \beta \leqslant \lambda$, $\alpha^{+\gamma} = \beta$, $\{p\colon (\overline{p \cap \alpha})^{+\gamma} = \overline{p \cap \beta}\} \in U$.
4. *For* $\alpha, \beta \leqslant \lambda$ *and either* $2^\alpha = \beta$, $2^\alpha < \beta$, *or* $2^\alpha > \beta$, $\{p\colon 2^{\overline{p\cap\alpha}} = \overline{p \cap \beta}\} \in U$, $\{p\colon 2^{\overline{p\cap\alpha}} < \overline{p \cap \beta}\} \in U$, *or* $\{p\colon 2^{\overline{p\cap\alpha}} > \overline{p \cap \beta}\} \in U$.

We leave the proof of this lemma to the reader.

The standard notion of forcing, due to Lévy, for collapsing a strongly inaccessible cardinal β to the successor of a regular cardinal α will also be useful, and so we briefly recall its definition and some of its properties. Let $\text{Col}(\alpha, \beta) = \{f\colon f$ is a function from $\alpha \times \beta$ into β such that $\overline{\overline{\text{dmn}}}(f) < \alpha$ and such that $\langle \gamma, \delta\rangle \in \text{dmn}(f) \Rightarrow f(\gamma) < \delta\}$, and let $\text{Col}(\alpha, \beta)$ be ordered by inclusion. Then any set of compatible conditions of length $< \alpha$ has an upper bound.

Now, let $\varkappa$ be $\varkappa^{+(k-1)}$ supercompact, and let $2^\varkappa = \varkappa^{+k}$. If U is any normal ultrafilter over $P_\varkappa(\varkappa^{+(k-1)})$, then by Lemma 2, the set

$$D = \{p \in P_\varkappa(\varkappa^{+(k-1)})\colon \overline{p \cap \varkappa} \text{ is an inaccessible cardinal and } (\overline{p \cap \varkappa^{+i}})^+ = (\overline{p \cap \varkappa^{+(i+1)}}) \text{ for } 0 \leqslant i \leqslant k-2\} \in U.$$

Magidor's forcing notion P is now defined as the set of all sequences π of the form $\langle p_1, \ldots, p_n, f_0, \ldots, f_n, A, G\rangle$ where:

1. For $1 \leqslant i \leqslant n$, $p_i \in D$, and for $1 \leqslant i \leqslant n-1$, $p_i \subsetneq p_{i+1}$.

2. If we let $\delta_i = \overline{p_i \cap \varkappa}$, then $f_0 \in \mathrm{Col}(\omega_1, \delta_1)$, for $1 \leqslant i \leqslant n-1$, $f_i \in \mathrm{Col}(\delta_i^{+k}, \delta_{i+1})$, and $f_n \in \mathrm{Col}(\delta_n^{+k}, \varkappa)$.

3. $A \subseteq D$, $A \in U$, and for every $q \in A$, $p_n \subsetneq q$ and $f_n \in \mathrm{Col}(\delta_n^{+k}, \overline{q \cap \varkappa})$.

4. G is a function defined on A such that for $q \in A$, $G(q) \in \mathrm{Col}\big((\overline{\varkappa \cap q})^{+k}, \varkappa\big)$, and if $p \in A$, $q \subsetneq p$, then $G(q) \in \mathrm{Col}\big((\overline{\varkappa \cap q}^{+k}, \overline{\varkappa \cap p})\big)$.

If π and π' are elements of P,

$$\pi = \langle p_1, \ldots, p_n, f_0, \ldots, f_n, A, G\rangle \quad \text{and} \quad \pi' = \langle q_1, \ldots, q_l, g_0, \ldots, g_l, B, H\rangle,$$

then we say that $\pi \leqslant \pi'$ iff:

1. $n \leqslant l$ and $q_i = p_i$ for $1 \leqslant i \leqslant n$.
2. $f_i \subseteq g_i$ for $0 \leqslant i \leqslant n$.
3. $q_i \in A$, $G(q_i) \subseteq g_i$ for $n < i \leqslant l$.
4. $B \subseteq A$.
5. For every $p \in B$, $G(p) \subseteq H(p)$.

We also recall the notions of j-direct extension and j-length preserving extension. Let π and π' be as in the above, $\pi \leqslant \pi'$. Then for $0 \leqslant j \leqslant n$, π' is said to be a *j-direct extension of* π if:

1. $f_i = g_i$ for $j \leqslant i \leqslant n$.
2. $G(q_i) = g_i$ for $n < i \leqslant l$.
3. $B = \{p\colon p \in A \text{ and } q_l \subsetneq p\}$.
4. For $p \in B$, $G(p) = H(p)$.

Note that if π' is a j direct extension of π, then π' is uniquely determined by $\langle g_0, \ldots, g_{j-1}\rangle$ and $\langle q_{n+1}, \ldots, q_l\rangle$. When $j = 0$, we omit $\langle g_0, \ldots, g_{j-1}\rangle$.

The dual of the above notion is the notion of j-length preserving extension which we now define. If π and π' are again as above, $\pi \leqslant \pi'$, then π' is called a *j-length preserving extension of* π if:

1. $n = l$.
2. For $0 \leqslant i < j$, $f_i = g_i$.

Note that if $\pi \leqslant \pi'$, then there is a unique π'' such that:

1. $\pi \leqslant \pi'' \leqslant \pi'$.
2. π'' is a j-direct extension of π.
3. π' is a j-length preserving extension of π''.

This unique π'' is called the *j interpolant of π and π'*, written as $j\,\mathrm{int}(\pi, \pi')$. Note that the following 2 statements are true:

1. If $\pi \leqslant \pi' \leqslant \pi''$, then $j\,\mathrm{int}(\pi, \pi'') \leqslant j\,\mathrm{int}(\pi', \pi'')$.
2. If $\pi'' = j\,\mathrm{int}(\pi, \pi')$, then $j\,\mathrm{int}(\pi', \pi'') = \pi''$.

LEMMA 3 (Magidor [8]). *Let $\langle \pi_\alpha\colon \alpha < \lambda\rangle$ be an increasing sequence such that for $\alpha < \beta < \lambda$, π_β is a j-length preserving extension of π_α, where $\lambda \leqslant \delta_j^{+(k-1)}$ if $j > 0$, and $\lambda \leqslant \omega$ otherwise. Then there is one condition π which is a j-length preserving extension of every member of this sequence.*

Proof. For $\alpha < \lambda$, let

$$\pi_\alpha = \langle p_1, \ldots, p_n, f_0, \ldots, f_{j-1}, f_j^\alpha, \ldots, f_n^\alpha, A^\alpha, G^\alpha\rangle.$$

For $j \leqslant i \leqslant n$, by the closure properties of $\mathrm{Col}(\delta_i^{+k}, \delta_{i+1})$, we let $g_i = \bigcup_{\alpha<\lambda} f_i^\alpha$, and for $p \in \bigcap_{\alpha<\lambda} A^\alpha = B$, $H(p) = \bigcup_{\alpha<\lambda} G^\alpha(p)$. By the fact that U is $\varkappa$ additive, we thus have that $\langle p_1, \ldots, p_n, f_0, \ldots, f_{j-1}, g_j, \ldots, g_n, B, H\rangle$ is the desired condition. ■

LEMMA 4 (Magidor [8]). *Let π be a condition of length n, let $j \leqslant n$, and let φ be a statement in the forcing language appropriate for P. Then there is a j length preserving extension of π, π', such that if $\pi' \leqslant \pi''$ and $\pi'' \| \varphi$, then $j\,\mathrm{int}(\pi', \pi'') \| \varphi$.*

Lemma 4 is the main technical lemma that Magidor uses to show that forcing with P makes $\aleph_\omega$ a strong limit cardinal that violates GCH. We will not prove this lemma here, but instead refer the reader to [8] for a proof of this fact.

2. The Main Theorem. We turn now to the proof of Theorem 1. Before starting the proof, we feel that a few intuitive remarks are in order. After some reflection, it becomes clear that in order to prove Theorem 1, some kind of partial ordering will be needed which makes $\aleph_\omega$ a strong limit cardinal which violates GCH, makes $\aleph_{\omega+\omega}$ a strong limit cardinal which violates GCH without harming the facts about $\aleph_\omega$, and so forth. Hence, a natural partial ordering to consider is some sort of iteration or product of Magidor's original partial ordering. This is exactly the sort of partial ordering which shall be employed in the proof of Theorem 1.

Proof of Theorem 1. Let V be as is the hypotheses for Theorem 1, and let $A = \{\delta\colon \delta \text{ is } \delta^+ \text{ supercompact and } 2^\delta = \delta^{++}\}$. For each $\delta \in A$, let U^δ be a normal ultrafilter on $P_\delta(\delta^+)$(1), and let $D^\delta = \{p \in P_\delta(\delta^+)\colon \overline{p \cap \delta} \text{ is an inaccessible cardinal } > \bigcup\{\alpha \in A\colon \alpha < \delta\} \text{ and } (\overline{p \cap \delta})^+ = (\overline{p \cap \delta^+})\}$. We may assume that $\bigcup\{\alpha \in A\colon \alpha < \delta\} < \delta$, for otherwise, if $\bigcup\{\alpha \in A\colon \alpha < \delta\} = \delta$ for some δ, then for δ_0 the least such δ, $R(\delta_0)$ is a model for the hypotheses of Theorem 1 in which for all $\delta \in A$, $\bigcup\{\alpha \in A\colon \alpha < \delta\} < \delta$. Thus, using Lemma 2 and the additivity of the measure U^δ, $D^\delta \in U^\delta$.

We are now in a position to define the forcing conditions P which will be used to prove Theorem 1. Keeping in mind our earlier intuitive remarks,

(1) We assume throughout this paper that we have at our disposal a well-ordering W of V. That this is consistent is a well known fact; see, for example, Felgner [3] or Gitik [4].

we define P as follows: P consists of all sequences of the form $\langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, f_0^\alpha, \ldots, f_{l_\alpha}^\alpha, A^\alpha, G^\alpha\rangle_{\alpha\in A}$ where:

1. $l_\alpha \in \omega$.
2. For $1 \leqslant i \leqslant l_\alpha$ $p_i^\alpha \in D^\alpha$, and for $1 \leqslant i \leqslant l_\alpha - 1$ $p_i^\alpha \subsetneqq p_{i+1}^\alpha$.
3. Let $\delta_i^\alpha = \overline{\overline{p_i^\alpha \cap \alpha}}$. Then $f_0^{\alpha_0} \in \mathrm{Col}(\omega_1, \delta_1^{\alpha_0})$, where α_0 is the least element of A. For $\alpha \in A$, $f_0^\alpha \in \mathrm{Col}(\bigcup\{\beta\in A\colon \beta < \alpha\}^{+++}, \delta_1^\alpha)$. And, for $\alpha \in A$, $1 \leqslant i \leqslant l_\alpha - 1$, $f_i^\alpha \in \mathrm{Col}(\delta_i^{\alpha++}, \delta_{i+1}^\alpha)$, and $f_{l_\alpha}^\alpha \in \mathrm{Col}(\delta_{l_\alpha}^{\alpha++}, \alpha)$.
4. For $\alpha\in A$, $A^\alpha \subseteq D^\alpha$, $A^\alpha \in U^\alpha$.
5. For every $q \in A^\alpha$, $p_{l_\alpha}^\alpha \subsetneqq q$, and $f_{l_\alpha}^\alpha \in \mathrm{Col}(\delta_{l_\alpha}^{\alpha++}, \overline{\overline{q\cap\alpha}})$.
6. For every $\alpha\in A$, G^α is a function defined on A^α such that for $q \in A^\alpha$, $G^\alpha(q) \in \mathrm{Col}((\overline{\overline{\alpha\cap q}})^{++}, \alpha)$, and if $p \in A^\alpha$, $q \subsetneqq p$, then $G^\alpha(q) \in \mathrm{Col}(\overline{\overline{q\cap\alpha}}^{++}, \overline{\overline{p\cap\alpha}})$.
7. Length $(\langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, f_0^\alpha, \ldots, f_{l_\alpha}^\alpha, A^\alpha, G^\alpha\rangle) = 1$ except finitely often, where the length is defined as l_α.

Note that the above definition parallels the intuition mentioned earlier, as the least element of A is changed into $\aleph_\omega$, and each succeeding $\beta\in A$ is changed into $[\bigcup\{\alpha\in A\colon \alpha<\beta\}]^{+\omega}$.

It is appropriate to introduce here the notation which we shall use when talking about P. Given a condition $\pi\in P$, we let π_α, for $\alpha\in A$, be the αth coordinate of π. $\pi\restriction\alpha = \langle\pi_\beta\colon \beta<\alpha \,\&\, \beta\in A\rangle$, and P_α is the collection of all $\pi\restriction\alpha$. (Note that P_α is a set.) An element of the partial ordering P^α will be one of the form $\langle\pi_\beta\colon \beta\in A-\alpha\rangle$. Occasionally it will be useful to write a condition as $\langle\pi, \psi\rangle$, where $\pi\in P_\alpha$ and $\psi\in P^\alpha$.

We can now define the ordering on P which will just be the componentwise ordering. Specifically, if $\pi, \pi'\in P$, $\pi = \langle\pi_\alpha\rangle_{\alpha\in A}$, $\pi' = \langle\pi'_\alpha\rangle_{\alpha\in A}$, say that $\pi\leqslant\pi'$ iff $\forall\alpha\in A\,[\pi_\alpha\leqslant\pi'_\alpha]$; the ordering $\leqslant$ on π_α and π'_α is the obvious analogue of the ordering $\leqslant$ of [8].

There is a distance function $|\ |$ associated with P which is similar to the distance function of [7]. Given conditions $\pi, \pi'\in P$, $\pi\leqslant\pi'$, we define $|\pi_\alpha - \pi'_\alpha|$ as length (π'_α) − length (π_α). The definition of P ensures that $|\pi_\alpha-\pi'_\alpha|$ is non-zero for only finitely many $\alpha\in A$; hence, as in [7], the non-zero values of $|\pi-\pi'|$ for any $\pi'\geqslant\pi$ are well-ordered in the anti-lexicographical ordering. The definition of P also ensures that for each $\alpha\in A$, $\bar{\bar{P}}_\alpha < \alpha$.

We can now state and prove the two main technical lemmas which we shall need. The intuition behind them is as follows: The partial ordering P which has just been defined is in some sense like Magidor's notion of iterated Prikry forcing [7] in that we wish to change the cofinalities of a class of cardinals to ω "at the same time" (while simultaneously collapsing them). Hence, we would like, given a condition $\pi\in P$ and a formula ψ in the forcing language associated with P, to be able to extend π to a condition π' which decides φ and which is such that $\forall\alpha\in A\,[|\pi_\alpha-\pi'_\alpha| = 0]$, i.e., for all α, π'_α is a 0-length preserving extension of π_α (we can, of course, do the analogous thing

with iterated Prikry forcing; this is just Lemma 2.1 of [7]). We can then infer that some bounded piece of the generic object is enough to determine the cardinal structure through a certain point. Lemma 5, our next lemma, is a technical device which is used in Lemma 6 to show that any condition π can be extended to a condition π' as above.

LEMMA 5. *Let* $\pi = \langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, f_0^\alpha, \ldots, f_{l_\alpha}^\alpha, A^\alpha, G^\alpha\rangle_{\alpha\in A}$, *and let* φ *be a formula in the forcing language appropriate for* P. *Then there is a condition* $\pi'\in P$, $\pi' = \langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, g_0^\alpha, \ldots, g_{l_\alpha}^\alpha, B^\alpha, H^\alpha\rangle_{\alpha\in A}$ *so that:*

1. $|\pi_\alpha - \pi'_\alpha| = 0$ *for all* α.
2. *If* $\pi'\leqslant\pi''$, $\pi''\|\varphi$, *then* $\langle\langle\pi''_\alpha\rangle_{\alpha<\beta}, \langle 0\,\mathrm{int}(\pi'_\alpha, \pi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle\|\varphi$, *where* β *is the last coordinate such that* $|\pi'_\beta-\pi''_\beta|\neq 0$.

Proof. We shall define each coordinate of π' inductively. So, suppose that $\langle\pi'_\alpha\rangle_{\alpha<\beta}$ have been defined. We shall show how to define π'_β.

The proof of Lemma 5 is very similar to the proof of Theorem 2.6 of [8]. In particular, as in [8], Lemma 5 is broken into a number of sublemmas. The first sublemma is the analogue of Lemma 2.8 of [8].

SUBLEMMA a. *Let* χ *be the condition* $\langle\langle\pi'_\alpha\rangle_{\alpha<\beta}, \pi_\beta, \langle\pi_\alpha\rangle_{\alpha>\beta}\rangle$. *Let* k *be a fixed natural number. Then for each* $\alpha\geqslant\beta$, *there is a condition* ψ'_α *so that:*

1. $\psi'_\alpha\geqslant\pi_\alpha$ *and* $|\pi_\alpha-\psi'_\alpha| = 0$.
2. *Let* ψ' *be the condition* $\langle\langle\pi'_\alpha\rangle_{\alpha<\beta}, \langle\psi'_\alpha\rangle_{\alpha\geqslant\beta}\rangle$. *If* $\psi''\geqslant\psi'$, $\psi''\restriction\beta = \psi'\restriction\beta$, $|\psi'_\beta-\psi''_\beta| = k$, $|\psi'_\alpha-\psi''_\alpha| = 0$ *for* $\alpha>\beta$, $\psi''\|\varphi$, *then* $\langle\psi'_\alpha\restriction\beta, \langle 0\,\mathrm{int}(\psi'_\alpha, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle\|\varphi$.

Proof. The proof of this sublemma is quite similar to the proof of Lemma 2.8 of [8]. Specifically, the proof is by induction on k. Let $k = 0$, and distinguish 2 cases.

Case I. There is a condition ψ, $\psi\restriction\beta = \langle\pi'_\alpha\rangle_{\alpha<\beta}$, $|\pi_\beta-\psi_\beta| = 0$, $|\pi_\alpha-\psi_\alpha| = 0$ for $\alpha>\beta$ such that $\psi\|\varphi$ and $\chi\leqslant\psi$. Let ψ' be the condition $\langle\langle\pi'_\alpha\rangle_{\alpha<\beta}, \langle\psi_\alpha\rangle_{\alpha\geqslant\beta}\rangle$. ψ' is then our desired condition:

Let $\psi'\leqslant\psi''$, $\psi''\|\varphi$, $\psi''\restriction\beta = \langle\pi'_\alpha\rangle_{\alpha<\beta}$, $|\psi'_\alpha-\psi''_\alpha| = 0$ for $\alpha\geqslant\beta$. Then $\langle\langle\pi'_\alpha\rangle_{\alpha<\beta}, \langle 0\,\mathrm{int}(\psi'_\alpha, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle\|\varphi$, since it is equal to ψ'.

Case II. Case I fails. Then let $\psi' = \chi$. The sublemma is then vacuously true, since there is no condition with the required properties.

We now come to the induction step. We assume that the sublemma is true for k, and then show that it is true for $k+1$.

We are given the condition $\chi = \langle\langle\pi'_\alpha\rangle_{\alpha<\beta}, \pi_\beta, \langle\pi_\alpha\rangle_{\alpha\geqslant\beta}\rangle$, where $\pi_\beta = \langle p_1^\beta, \ldots, p_{l_\beta}^\beta, f_0^\beta, \ldots, f_{l_\beta}^\beta, A^\beta, G^\beta\rangle$. Since the partial ordering $\subsetneqq$ on A^β is well-founded ($p^\beta\subsetneqq q^\beta \Rightarrow \overline{\overline{\beta\cap p^\beta}} < \overline{\overline{\beta\cap q^\beta}}$), it can be extended to a well-ordering $\preccurlyeq$ of A^β. By induction on $\preccurlyeq$, we are going to define a sequence of conditions $\langle u_p\colon p\in A^\beta\rangle$ such that:

1. u_q is of the form

$$\langle\langle\pi'_\alpha\rangle_{\alpha<\beta}, \langle p_1^\beta, \ldots, p_{l_\beta}^\beta, q, f_0^{\beta,q}, \ldots, f_{l_\beta}^{\beta,q}, f^{\beta,q}, B^{\beta,q}, H^{\beta,q}\rangle, \langle v_\alpha^q\rangle_{\alpha>\beta}\rangle.$$

2. $\chi \leqslant u_q$.
3. $|\pi_\alpha - v_\alpha^q| = 0$ for $\alpha > \beta$.
4. If $q_0, q, q_1 \in A^\beta$, $q_0 \subsetneqq q_1$, $q \subsetneqq q_1$ and $q_1 \in B^{\beta,q} \cap B^{\beta,q_0}$, then $H^{\beta,q}(q_1)$ and $H^{\beta,q_0}(q_1)$ are compatible. Actually, if $q_0 \subsetneqq q$, then $H^{\beta,q_0}(q_1) \subseteq H^{\beta,q}(q_1)$.
5. For $\alpha > \beta$, if $p \preccurlyeq q$, then $v_\alpha^p \leqslant v_\alpha^q$.

Assume now that we have already defined u_q for $q \preccurlyeq p$, $q \neq p$ such that (1)–(5) hold. Note that since $\preccurlyeq$ has been picked to extend $\subsetneqq$, we have that u_q is defined for $q \subsetneqq p$.

Define u'_p as the condition

$$\langle \langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle p_1^\beta, \ldots, p_{l_\beta}^\beta, p, f_0^\beta, \ldots, f_{l_\beta}^\beta, g^{\beta,p}, A^{\beta,p}, G^{\beta,p} \rangle, \langle v_\alpha'^p \rangle_{\alpha>\beta} \rangle$$

where:

1. $g^{\beta,p} = \bigcup \{H^{\beta,q}(p): q \subsetneqq p, q \in A^\beta\}$.
2. $A^{\beta,p} = \{r: r \in \bigcap \{B^{\beta,q}: q \subsetneqq r, q \preccurlyeq p\}\} \cap \{r: p \subsetneqq r\}$. By the normality of U^β, $A^{\beta,p} \in U^\beta$ (see [8] which explains the normality properties of U^β which are being used here).
3. $G^{\beta,p}(r) = \bigcup \{H^{\beta,q}(r): q \subsetneqq r\}$ if $r \in A^{\beta,p}$.
4. For $\alpha > \beta$, $q \preccurlyeq p$, let

$$v_\alpha^q = \langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, h_0^{\alpha,q}, \ldots, h_{l_\alpha}^{\alpha,q}, C^{\alpha,q}, I^{\alpha,q} \rangle.$$

$v_\alpha'^p$, which we shall occasionally also write as $\bigcup_{q \preccurlyeq p} v_\alpha^q$, is then defined as

$$\langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, \bigcup_{q\preccurlyeq p} h_0^{\alpha,q}, \ldots, \bigcup_{q\preccurlyeq p} h_{l_\alpha}^{\alpha,q}, \bigcap_{q\preccurlyeq p} C^{\alpha,q}, I^{\alpha,p} \rangle,$$

where for $r \in C^{\alpha,q}$, $I^{\alpha,p}(r) = \bigcup_{q\preccurlyeq p} I^{\alpha,q}(r)$.

We have that u'_p is a condition: $g^{\beta,p} \in \mathrm{Col}((\overline{\beta \cap p})^{++}, \beta)$ since each $H^{\beta,q}(p) \in \mathrm{Col}((\overline{\beta \cap p})^{++}, \beta)$. By induction, $\{H^{\beta,q}(p): q \subsetneqq p, q \in A^\beta\}$ is a compatible set of functions whose cardinality is at most the cardinality of $\{q: q \subsetneqq p\}$ which is $[\bar{p}]^{<p\cap\beta}$. Since $\bar{p} = \overline{(p\cap\beta)}^{+}$ and $\overline{p\cap\beta}$ is inaccessible, $[\bar{p}]^{<p\cap\beta} = \bar{p} = \overline{(p\cap\beta)}^{+}$. Hence, since $\mathrm{Col}((\overline{\beta\cap p})^{++}, \beta)$ is closed under unions of $\overline{(p\cap\beta)}^{+}$ compatible functions, $g^{\beta,p} \in \mathrm{Col}((\overline{p\cap\beta})^{++}, \beta)$.

It is also easily seen that, for $\alpha > \beta$, the definition of $v_\alpha'^p$ makes sense. This is since $\overline{\overline{\{q: q \preccurlyeq p\}}} \leqslant \beta^{++}$, so since unions of Lévy functions are taken in partial orderings which are at least β^{++} closed, each union of compatible functions produces a function in the correct Lévy ordering. Also, the additivity of each ultrafilter ensures that we always get a measure 1 set. Hence, each $v_\alpha'^p$ for $\alpha > \beta$ is a well defined condition.

The other clauses in the definition of a condition can also be verified as holding. In particular, a similar argument to one given earlier shows that $G^{\beta,p}(r) \in \mathrm{Col}((\overline{r\cap\beta})^{++}, \beta)$. Hence, u'_p is a condition.

Now by the induction hypothesis, we apply Sublemma a using as our parameters the condition u'_p at the coordinate β and the natural no. k (note that the induction hypothesis is that the sublemma holds for the natural no. k and *any* condition of the requisite form). We obtain a condition u''_p which satisfies the conditions of the sublemma for k and u'_p. Let $u_p = u''_p$. Conditions (1)–(5) are preserved because of the particular way in which things were defined. This completes the inductive definition of $\langle u_p: p \in A^\beta \rangle$.

We are now almost ready for the definition of the condition ψ' which will witness the sublemma for χ, $k+1$, and β. We first observe that there is a set $B^{\beta,1} \subseteq A^\beta$, $B^{\beta,1} \in U^\beta$ so that for $p \in B^{\beta,1}$ $f_0^{\beta,p}, \ldots, f_{l_\beta}^{\beta,p}$ are constant. For $0 \leqslant i \leqslant l_\beta - 1$, we have $f_i^{\beta,p} \in \mathrm{Col}(\delta_i^{\beta++}, \delta_{i+1}^\beta)$ and

$$\overline{\overline{\mathrm{Col}(\delta_i^{\beta++}, \delta_{i+1}^\beta)}} = [\delta_{i+1}^\beta]^{<(\delta_i^\beta)^{++}} = \delta_{i+1}^\beta < \beta,$$

so by the β completeness of U^β, we can find $B^{\beta,1} \subseteq A^\beta$, $B^{\beta,1} \in U^\beta$ such that for $p \in B^{\beta,1}$, $f_0^{\beta,p}, \ldots, f_{l_\beta-1}^{\beta,p}$ are constant.

For all $p \in A^\beta$, $f_{l_\beta}^{\beta,p} \in \mathrm{Col}(\delta_{l_\beta}^{\beta++}, \overline{\beta\cap p})$. Since $\overline{\beta\cap p}$ is inaccessible, $f_{l_\beta}^{\beta,p} \in \mathrm{Col}(\delta_{l_\beta}^{\beta++}, \alpha_p)$ for some $\alpha_p < \overline{\beta\cap p}$, and hence, $\alpha_p \in \overline{\beta \cap p}$. By the normality of U^β there is a set $B^{\beta,2} \in U^\beta$, $B^{\beta,2} \subseteq B^{\beta,1}$ such that on $B^{\beta,2}$, α_p is some constant $\alpha < \beta$. As above, we invoke the β completeness of U^β to obtain $B^\beta \subseteq B^{\beta,2}$, $B^\beta \in U^\beta$ such that for $p \in B^\beta$, $f_{l_\beta}^p$ is constant. We let $g_0^\beta, \ldots, g_{l_\beta}^\beta$ be the appropriate constant values.

ψ' is defined as

$$\langle \langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle p_1^\beta, \ldots, p_{l_\beta}^\beta, g_0^\beta, \ldots, g_{l_\beta}^\beta, C^\beta, H^\beta \rangle, \langle v_\alpha \rangle_{\alpha>\beta} \rangle$$

where:

1. $C^\beta = B^\beta \cap \{p: p \in \bigcap \{B^{\beta,q}: q \in A^\beta, q \subsetneqq p\}\}$.
2. For $p \in C^\beta$, $H^\beta(p) = f^{\beta,p}$.
3. For $\alpha > \beta$, $v_\alpha = \bigcup_{p\in A^\alpha} v_\alpha^p$.

Note that by our definition of $f^{\beta,p}$, $\bigcup \{H^{\beta,q}(p): q \in A^\beta, q \subsetneqq p\} \subseteq f^{\beta,p}$ and $G^\beta(p) \subseteq f^{\beta,p}$. Note also that, as before, since unions and intersections are taken over small enough sets, the definition of v_α for $\alpha > \beta$ makes sense.

We now claim that ψ' will satisfy the conditions of the sublemma for $k+1$. Clearly, $\chi \leqslant \psi'$, and for $\alpha \geqslant \beta$, ψ'_α is a 0-length preserving extension of π_α.

Now let $\psi' \leqslant \psi''$, $\psi'' \upharpoonright \beta = \langle \pi'_\alpha \rangle_{\alpha<\beta}$, $|\psi'_\beta - \psi''_\beta| = k+1$, $|\psi'_\alpha - \psi''_\alpha| = 0$ for $\alpha > \beta$, $\psi'' \| \varphi$. Consider $u'' = \langle \langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\,\mathrm{int}(\psi'_\alpha, \psi''_\alpha) \rangle_{\alpha \geqslant \beta} \rangle$. u''_β is the 0 direct extension of ψ'_β determined by $p, q_1, \ldots, q_k$ for some $p, q_1, \ldots, q_k \in C^\beta$, and for $\alpha > \beta$, $u''_\alpha = \psi'_\alpha$.

By construction, $u_p \leqslant u'' \leqslant \psi''$. This is since $u_p \upharpoonright \beta = u'' \upharpoonright \beta = \psi'' \upharpoonright \beta = \langle \langle \pi'_\alpha \rangle_{\alpha<\beta} \rangle$. For $\alpha > \beta$, $u_{p,\alpha} \leqslant u''_\alpha$ $(= \psi'_\alpha) \leqslant \psi''_\alpha$ by a simple examination of the construction. For β, we have $u_{p,\beta} \leqslant u''_\beta \leqslant \psi''_\beta$ because $q_1, \ldots, q_k \in C^\beta$, $p \subsetneq q_1 \subsetneq \ldots \subsetneq q_k$ implies $q_1, \ldots, q_k \in B^{\beta,p}$, for $i = 1, \ldots, l_\beta$,

$$H^{\beta,p}(q_i) \subseteq H^\beta(q_i), \quad f_i^{\beta,p} = g_i^\beta, \quad \text{and} \quad H^\beta(p) = f^{\beta,p}.$$

We now use the definition of u_p. $u_p \leqslant \psi''$, $\psi'' \upharpoonright \beta = \langle \pi'_\alpha \rangle_{\alpha<\beta}$, $|u_{p,\beta} - \psi''_\beta| = k$, for $\alpha > \beta$, $|u_{p,\alpha} - \psi''_\alpha| = 0$, and $\psi'' \| \varphi$. Hence, $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}$, $0\mathrm{int}(u_{p,\beta}, \psi''_\beta)$, $\langle 0\mathrm{int}(u_{p,\alpha}, \psi''_\alpha)\rangle_{\alpha>\beta}\rangle \| \varphi$. But $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(u''_\alpha, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle = u''$ extends the above, so $u'' \| \varphi$. ■

We now know that given any condition χ of the form $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}$, $\langle \pi_\alpha \rangle_{\alpha\geqslant\beta}\rangle$, formula φ, and natural no. k, there is an extension ψ' of χ of the form $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle \psi'_\alpha \rangle_{\alpha\geqslant\beta}\rangle$ such that:

1. For all $\alpha \geqslant \beta$, $|\pi_\alpha - \psi'_\alpha| = 0$.
2. If $\psi' \leqslant \psi''$, $\psi'' \upharpoonright \beta = \langle \pi'_\alpha \rangle_{\alpha<\beta}$, $|\psi'_\beta - \psi''_\beta| = k$, $|\psi'_\alpha - \psi''_\alpha| = 0$ for $\alpha > \beta$, $\psi'' \| \varphi$, then $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(\psi'_\alpha, \psi''_\alpha)\rangle_{\alpha>\beta}\rangle \| \varphi$.

By induction, we define the following sequence of conditions:

$u_0 = \langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle \pi_\alpha \rangle_{\alpha\geqslant\beta}\rangle$,

$u_{k+1} = \langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle \pi_{\alpha,k} \rangle_{\alpha\geqslant\beta}\rangle$ is an extension of u_k which satisfies the conditions of Sublemma a for k.

Let $u = \langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle \bigcup_{k\in\omega} \pi_{\alpha,k} \rangle_{\alpha\geqslant\beta}\rangle$. Clearly, by construction, for all i $u_i \leqslant u$.

Claim. *If $u \leqslant \psi''$, $\psi'' \upharpoonright \beta = \langle \pi'_\alpha \rangle_{\alpha<\beta}$, $|u_\alpha - \psi''_\alpha| = 0$ for $\alpha > \beta$, and $\psi'' \| \varphi$, then $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(u_\alpha, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$.*

Proof. We know that $|u'_\beta - \psi''_\beta| = k$ for some natural number k. But we also know, by construction, that $u_{k+1} \leqslant u \leqslant \psi''$. By Sublemma a, we have that

$$\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(u_{k+1,\alpha}, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi.$$

But $\langle\langle \pi_{\alpha'} \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(u_\alpha, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$, since it extends the above. ■

We thus have

Sublemma b. *Let $\chi = \langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle \pi_\alpha \rangle_{\alpha\geqslant\beta}\rangle$ be a given condition, and let φ be a formula in the forcing language appropriate for P. Then there is ψ', $\chi \leqslant \psi'$ such that:*

1. *$\psi' \upharpoonright \beta = \langle \pi'_\alpha \rangle_{\alpha<\beta}$.*
2. *For $\alpha \geqslant \beta$, ψ'_α is a 0 length preserving extension of π_α.*
3. *If $\psi' \leqslant \psi''$, $\psi'' \upharpoonright \beta = \langle \pi'_\alpha \rangle_{\alpha<\beta}$, $|\psi'_\alpha - \psi''_\alpha| = 0$ for $\alpha > \beta$, $\psi'' \| \varphi$, then $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(\psi'_\alpha, \psi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$.*

We can now resume the proof of Lemma 5. As a reminder, we are given a condition $\pi = \langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, f_0^\alpha, \ldots, f_{l_\alpha}^\alpha, A^\alpha, G^\alpha \rangle_{\alpha\in A}$, a formula φ in the forcing language associated with P, and we wish to obtain a condition $\pi' \geqslant \pi$ such that:

1. $\forall \alpha\, [|\pi_\alpha - \pi'_\alpha|] = 0$.
2. If $\pi' \leqslant \pi''$, $\pi'' \| \varphi$, and β is the last coordinate so that $|\pi'_\beta - \pi''_\beta| = 0$, then $\langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(\pi'_\alpha, \pi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$.

So, as mentioned earlier, we proceed inductively as follows: We assume that the condition $\chi = \langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \pi_\beta, \langle \pi_\alpha \rangle_{\alpha\geqslant\beta}\rangle$ has already been defined. We then show how to define π'_β, the βth coordinate of the desired condition π'.

By our GCH assumptions, we know that

$$\bar{T} = \{t\colon t \in P_\beta \,\&\, \langle \pi'_\alpha \rangle_{\alpha<\beta} \leqslant t\} \leqslant \bigcup \{\alpha\colon \alpha < \beta \,\&\, \alpha \in A\}^{++};$$

this is since $\bigcup\{\alpha\colon \alpha < \beta \,\&\, \alpha \in A\}$ is either some $\alpha \in A$, in which case $\bar{P}_\beta \leqslant \alpha^{++}$, or it is some singular cardinal, in which case

$$\bar{P}_\beta \leqslant 2^{\bigcup\{\alpha\colon \alpha<\beta \& \alpha\in A\}} = (\bigcup\{\alpha\colon \alpha < \beta \,\&\, \alpha \in A\})^+.$$

Let $\gamma_0 = (\bigcup\{\alpha\colon \alpha < \beta \,\&\, \alpha \in A\})^{++}$, and let $\langle t_\alpha\colon \alpha < \gamma_0\rangle$ enumerate T. By induction on $\gamma < \gamma_0$, we shall define a sequence $\langle \psi^\gamma\colon \gamma < \gamma_0\rangle$ of elements of P so that for each $\gamma < \gamma_0$, $(*)\, \psi^\gamma_\sigma$ for $\sigma \geqslant \beta$ will always be a 0 length preserving extension of ψ^α_σ for $\alpha < \gamma$.

We proceed as follows: Assume that ψ^α has been defined for $\alpha < \gamma$. To define ψ^γ, we first consider the conditon $s^\gamma = \langle t^\gamma, \langle \bigcup_{\alpha<\gamma} \psi^\alpha_\sigma \rangle_{\sigma\geqslant\beta}\rangle$. Again, since $\gamma < \gamma_0$, by the additivity of the measures involved and the closure properties of the Lévy orderings, we know that s^γ is a condition. (This, incidentally, is the point where we use the fact that

$$f_0^\sigma \in \mathrm{Col}((\bigcup\{\alpha \in A\colon \alpha < \sigma\})^{+++}, \delta_1^\sigma).)$$

Now apply Sublemma b to s^γ and β and obtain a condition ψ^γ such that $s^\gamma \leqslant \psi^\gamma$ and such that the conditions of Sublemma b are met. Since for $\sigma \geqslant \beta$, ψ^γ_σ is a 0 length preserving extension of $\bigcup_{\alpha<\gamma} \psi^\alpha_\sigma$, property $(*)$ is preserved.

To define π'_β and the condition u' which will be used in the next stage of the induction, we let

$$u' = \langle\langle \pi'_\alpha \rangle_{\alpha<\beta}, \langle \bigcup_{\alpha<\gamma_0} \psi^\alpha_\sigma \rangle_{\sigma\geqslant\beta}\rangle.$$

Again, by the additivity properties of the measures and the closure properties of the Lévy orderings, u' is a condition.

We now claim that the condition $\pi' = \langle \pi'_\alpha \rangle_{\alpha\in A}$ so obtained witnesses the truth of Lemma 5, i.e., if $\pi' \leqslant \pi''$, β is the last coordinate where $|\pi'_\beta - \pi''_\beta| > 0$, $\pi'' \| \varphi$, then $\langle\langle \pi''_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(\pi'_\alpha, \pi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$.

To see this, we note, using the same notation as in the inductive definition of π', that $\langle \pi''_\alpha \rangle_{\alpha<\beta} \in T$, so it equals t_γ for some $\gamma < \gamma_0$. Hence, by construction, we have $\psi^\gamma \leqslant \pi' \leqslant \pi''$. By Sublemma b, ψ^γ is such that $\langle\langle \pi''_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(\psi^\gamma_\alpha, \pi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$. But for all $\alpha \geqslant \beta$, $0\mathrm{int}(\psi^\gamma_\alpha, \pi''_\alpha) \leqslant 0\mathrm{int}(\pi'_\alpha, \pi''_\alpha)$, so $\langle\langle \pi''_\alpha \rangle_{\alpha<\beta}, \langle 0\mathrm{int}(\pi'_\alpha, \pi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle \| \varphi$. ■

Armed with the above lemma, we can now show that any condition π may by extended to a larger condition, all of whose coordinates have the same length as π, that decides a formula φ. Specifically, we have:

LEMMA 6. *Let* $\pi\in P$, $\pi = \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, f_0^\alpha, \ldots, f_{i_\alpha}^\alpha, A^\alpha, G^\alpha\rangle_{\alpha\in A}$, *and let* φ *be a formula in the forcing language appropriate for* P. *Then there is* $\chi \geqslant \pi$ *such that:*

1. $\forall\alpha\in A\,[|\pi_\alpha-\chi_\alpha| = 0]$.
2. $\chi\|\varphi$.

Proof. Lemma 6 is the analogue of Lemma 2.1 of [7], and its proof is virtually the same as the proof of this lemma. First, by Lemma 5, let $\pi'\geqslant\pi$, $\pi' = \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, B^\alpha, H^\alpha\rangle$ be such that:

1. $\forall\alpha\in A\,[|\pi_\alpha-\pi'_\alpha|] = 0$.
2. If $\pi'\leqslant\pi''$, $\pi''\|\varphi$, β is the last coordinate such that $|\pi'_\beta-\pi''_\beta| = 0$, then $\langle\langle\pi''\rangle_{\alpha<\beta}, \langle 0\,\mathrm{int}(\pi'_\alpha, \pi''_\alpha)\rangle_{\alpha\geqslant\beta}\rangle\|\varphi$. We will obtain an extension of π' which satisfies the conclusions of Lemma 6.

Let G be V-generic on P. Recall that by the Product Lemma, for any $\beta\in A$, $G_\beta = \{\psi\restriction\beta\colon \psi\in G\}$ is V-generic on P_β, and $G^\beta = \{\langle\psi_\alpha\rangle_{\alpha\geqslant\beta}\colon \psi\in G\}$ is $V[G_\beta]$-generic on P^β.

Now assume that $\pi'\in G$. Define a partition F_β of $[B^\beta]^{<\omega} = \{\langle p_1, \ldots, p_n\rangle\colon p_1, \ldots, p_n\in B^\beta$ and $p_1\subsetneq\ldots\subsetneq p_n\}$ into three pieces as follows:

$$F_\beta(\{q_1\subsetneq\ldots\subsetneq q_n\}) = \begin{cases} 0 & \text{if there is some } \psi\in G_\beta \text{ and } \langle C^\alpha\rangle_{\alpha\geqslant\beta} \text{ such that each } C^\alpha\in U^\alpha \text{ and } \\ & \langle\psi, \langle p_1^\beta, \ldots, p_{i_\beta}^\beta, q_1, \ldots, q_n, g_0^\beta, \ldots, g_{i_\beta}^\beta, H^\beta(q_1), \ldots, H^\beta(q_n), C^\beta, H^\beta\restriction C^\beta\rangle, \\ & \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C^\alpha, H^\alpha\restriction C^\alpha\rangle_{\alpha>\beta}\rangle\Vdash\varphi, \\ 1 & \text{if there is some } \psi\in G_\beta \text{ and } \langle C^\alpha\rangle_{\alpha\geqslant\beta} \text{ such that each } C^\alpha\in U^\alpha \text{ and } \langle\psi, \langle p_1^\beta, \ldots, p_{i_\beta}^\beta, \\ & q_1, \ldots, q_n, g_0^\beta, \ldots, g_{i_\beta}^\beta, H^\beta(q_1), \ldots, H^\beta(q_n), C^\beta, H^\beta\restriction C^\beta\rangle, \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, \\ & g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C^\alpha, H^\alpha\restriction C^\alpha\rangle_{\alpha>\beta}\rangle\Vdash\neg\varphi. \\ 2 & \text{otherwise.}\end{cases}$$

Note that F_β is not actually a partition, but only a term which denotes a partition in $V[G\restriction\beta]$. Also, F_β is well-defined, for if it were not, then we could get $\psi_1, \psi_2\in G\restriction\beta$, $\langle C^\alpha\colon \alpha\geqslant\beta, \alpha\in A\rangle$, $\langle C'^\alpha\colon \alpha\geqslant\beta, \alpha\in A\rangle$ so that:

$$\langle\psi_1, \langle p_1^\beta, \ldots, p_{i_\beta}^\beta, q_1, \ldots, q_n, g_0^\beta, \ldots, g_{i_\beta}^\beta, H^\beta(q_1), \ldots, H^\beta(q_n), C^\beta, H^\beta\restriction C^\beta\rangle, \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C^\alpha, H^\alpha\restriction C^\alpha\rangle_{\alpha>\beta}\rangle\Vdash\varphi$$

and

$$\langle\psi_2, \langle p_1^\beta, \ldots, p_{i_\beta}^\beta, q_1, \ldots, q_n, g_0^\beta, \ldots, g_{i_\beta}^\beta, H^\beta(q_1), \ldots, H^\beta(q_n), C'^\beta, H^\beta\restriction C'^\beta\rangle, \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C'^\alpha, H^\alpha\restriction C'^\alpha\rangle_{\alpha>\beta}\rangle\Vdash\neg\varphi.$$

If we let ψ be a common extension of ψ_1 and ψ_2, then we get that the condition $\langle\psi, \langle p_1^\beta, \ldots, p_{i_\beta}^\beta, q_1, \ldots, q_n, g_0^\beta, \ldots, g_{i_\beta}^\beta, H^\beta(q_1), \ldots, H^\beta(q_n), C^\beta\cap C'^\beta, H^\beta\restriction C^\beta\cap C'^\beta\rangle, \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C^\alpha\cap C'^\alpha, H^\alpha\restriction C^\alpha\cap C'^\alpha\rangle_{\alpha>\beta}\rangle$ forces both φ and $\neg\varphi$.

Now we know by the work of Levy and Solovay on mild Cohen extensions [6] that β is still β^+ supercompact in $V[G\restriction\beta]$ (recall that P_β has been defined so that $\bar{\bar{P}}_\beta<\beta$). Hence, by Menas' theorem [11], let $C^\beta\subseteq B^\beta$ be a measure 1 set which always denotes a homogeneous set. We may assume, by the above mentioned work of Levy and Solovay on mild Cohen extensions, that $C^\beta\in U^\beta$, $C^\beta\in V$. We now claim that $\chi = \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C^\alpha, H^\alpha\restriction C^\alpha\rangle_{\alpha\in A}\|\varphi$.

If the claim is false, then let $\psi\geqslant\chi$, $\psi'\geqslant\chi$ be such that

$$\psi = \langle q_1^\alpha, \ldots, q_{m_\alpha}^\alpha, h_0^\alpha, \ldots, h_{m_\alpha}^\alpha, D^\alpha, I^\alpha\rangle_{\alpha\in A}\Vdash\varphi$$

and

$$\psi' = \langle r_1^\alpha, \ldots, r_{n_\alpha}^\alpha, i_0^\alpha, \ldots, i_{n_\alpha}^\alpha, E^\alpha, J^\alpha\rangle_{\alpha\in A}\Vdash\neg\varphi,$$

and let us assume that $|\chi-\psi|$ is minimal. We claim that $|\chi-\psi| = |\chi-\psi'| = 0$. If not, let $|\chi-\psi|\neq 0$, and let $\beta_1, \ldots, \beta_k = \beta$ be a monotone enumeration of the indices on which $|\chi-\psi|\neq 0$. We will define a condition χ' so that $|\chi-\chi'|<|\chi-\psi|$ and $\chi'\Vdash\varphi$.

Let $t_1\subsetneq\ldots\subsetneq t_n$ enumerate $\psi_\beta-\chi_\beta$. Now $\pi'\leqslant\chi\leqslant\psi$, and $|\pi'-\chi| = 0$. Hence, β is the last coordinate on which $|\pi'-\psi|\neq 0$. But by the choice of π', $\langle\psi\restriction\beta, \langle 0\,\mathrm{int}(\pi'_\alpha, \psi_\alpha)\rangle_{\alpha\geqslant\beta}\rangle\Vdash\varphi$, and for all $\alpha\geqslant\beta$, $0\,\mathrm{int}(\pi'_\alpha, \psi_\alpha)\leqslant 0\,\mathrm{int}(\chi_\alpha, \psi_\alpha)$. Hence, $\langle\psi\restriction\beta, \langle 0\,\mathrm{int}(\chi_\alpha, \psi_\alpha)\rangle_{\alpha\geqslant\beta}\rangle\Vdash\varphi$, so $\psi\restriction\beta\Vdash$ "$F_\beta(\{t_1\subsetneq\ldots\subsetneq t_n\}) = 0$".

Thus, by the homogeneity of C^β for F_β, $\psi\restriction\beta\Vdash$ "For every $\sigma_1\subsetneq\ldots\subsetneq\sigma_n$, $\sigma_1, \ldots, \sigma_n\in C^\beta$, $F_\beta(\{\sigma_1, \ldots, \sigma_n\}) = 0$". Therefore, assuming $\psi\in G$, working in $V[G_\beta]$, for every $\sigma_1, \ldots, \sigma_n\in C^\beta$ we can find $\chi_{\sigma_1,\ldots,\sigma_n}\in G_\beta$ and measure 1 sets $C^{\alpha,\sigma_1,\ldots,\sigma_n}$ for $\alpha\geqslant\beta$ such that

$$\pi_{\sigma_1,\ldots,\sigma_n} = \langle\chi_{\sigma_1,\ldots,\sigma_n}, \langle p_1^\beta, \ldots, p_{i_\beta}^\beta, \sigma_1, \ldots, \sigma_n, g_0^\beta, \ldots, g_{i_\beta}^\beta, H^\beta(\sigma_1), \ldots, H^\beta(\sigma_n), C^{\beta,\sigma_1,\ldots,\sigma_n}, H^\beta\restriction C^{\beta,\sigma_1,\ldots,\sigma_n}\rangle, \langle p_1^\alpha, \ldots, p_{i_\alpha}^\alpha, g_0^\alpha, \ldots, g_{i_\alpha}^\alpha, C^{\alpha,\sigma_1,\ldots,\sigma_n}, H^\alpha\restriction C^{\alpha,\sigma_1,\ldots,\sigma_n}\rangle_{\alpha>\beta}\rangle\Vdash\varphi$$

(the mapping $\sigma_1, \ldots, \sigma_n, \alpha\mapsto C^{\alpha,\sigma_1,\ldots,\sigma_n}$ is in $V[G_\beta]$). Since each $C^{\beta,\sigma_1,\ldots,\sigma_n}\in U^\beta$ and since β is still β^+ supercompact in $V[G_\beta]$, by a theorem of Menas (the analogue of the diagonal intersection property for measurables for measures with the partition property) [11], $T_\beta = \{t\colon t\in C^{\beta,\sigma_1,\ldots,\sigma_n}$ for $\sigma_1\subsetneq\ldots\subsetneq\sigma_n\subsetneq t\}$ will denote a measure 1 set, and let $T^\beta\subseteq T_\beta$ be a measure 1 subset which is present in V. Let $S^\beta = T^\beta\cap C^\beta$. Similarly, for each $\alpha>\beta$, we can work in $V[G\restriction\alpha]$ and define $T_\alpha = \bigcap_{\sigma_1\subsetneq\ldots\subsetneq\sigma_n} C^{\alpha,\sigma_1,\ldots,\sigma_n}$, let $T^\alpha\subseteq T_\alpha$ be a measure 1 set present in V, and define $S^\alpha = T^\alpha \cap C^\alpha$. Then if we define $\chi'' = \langle \psi\upharpoonright\beta, \langle p_1^\alpha, \ldots, p_{l_\alpha}^\alpha, g_0^\alpha, \ldots, g_{l_\alpha}^\alpha, S^\alpha, H^\alpha\upharpoonright S^\alpha\rangle_{\alpha\geqslant\beta}\rangle$, we claim that $\chi''\Vdash\varphi$ (note that the class sequence $\langle S^\alpha\rangle_{\alpha\geqslant\beta}$ is definable in V, so χ'' is indeed a forcing condition).

If $\chi' \nVdash \varphi$, then let $\psi'' \geqslant \chi''$, $\psi'' = \langle s_1^\alpha, \ldots, s_{\theta_\alpha}^\alpha, j_0^\alpha, \ldots, j_{\theta_\alpha}^\alpha, S'^\alpha, K^\alpha\rangle_{\alpha\in A} \Vdash \neg\varphi$. Without loss of generality, we assume that $m = |\chi''_\beta - \psi''_\beta| > n$, and we let $\sigma_1, \ldots, \sigma_m$ enumerate $\psi''_\beta - \chi''_\beta$.

Now we argue in $V[G_\beta]$, assuming that $\psi'' \in G$. Since $\psi''\upharpoonright\beta \geqslant \chi''\upharpoonright\beta = \psi\upharpoonright\beta$, $\chi_{\sigma_1,\ldots,\sigma_n}$ is defined and is a particular member of G_β, and $\langle C^{\alpha,\sigma_1,\ldots,\sigma_n}\rangle_{\alpha\geqslant\beta}$ is a definable class sequence in V, so $\chi_{\sigma_1,\ldots,\sigma_n}$ is compatible with $\psi''\upharpoonright\beta$; assume therefore that $\chi_{\sigma_1,\ldots,\sigma_n} \leqslant \psi''\upharpoonright\beta$. Hence, since

$$\{\sigma_{n+1}, \ldots, \sigma_m\} \subseteq \{t \in S^\beta\colon \sigma_n \subsetneqq t\} \subseteq \{t \in C^{\beta,\sigma_1,\ldots,\sigma_n}\colon \sigma_n \subsetneqq t\},$$

$$\{s_{\theta_\alpha - l_\alpha}^\alpha, \ldots, s_{\theta_\alpha}^\alpha\} \subseteq S^\alpha \subseteq C^{\alpha,\sigma_1,\ldots,\sigma_n} \quad \text{for} \quad \alpha > \beta,$$

$$S'^\alpha \subseteq S^\alpha \quad \text{for} \quad \alpha \geqslant \beta,$$

we get that $\pi_{\sigma_1,\ldots,\sigma_n} \leqslant \psi''$, a contradiction. Thus, $\psi\Vdash\varphi$, and $|\chi - \psi| = 0$. Hence, by Lemma 5,

$$\langle 0\,\mathrm{int}(\chi_\alpha, \psi_\alpha)\rangle_{\alpha\in A} = \chi\Vdash\varphi.$$

Similarly, we can show that $|\chi - \psi'| = 0$ and that $\chi\Vdash\neg\varphi$. This is a contradiction, so Lemma 6 is proven. ∎

Armed with Lemmas 5 and 6, we are in a position to prove Theorem 1. Let G be V-generic on P (G is of course a proper class). As in the previous lemma, we know that for $\beta \in A$, G_β is V-generic over P_β. The full generic extension of V, $V[G]$ (which, by standard class forcing arguments ([2] or [5]) can be defined as $\bigcup_{\beta\in A} V[G_\beta]$) will not be our desired model; rather, a certain submodel of $V[G]$, which we shall call $\bar{V}$, will be the model which witnesses Theorem 1.

Before defining $\bar{V}$, we note that by arguments similar to those in [8], for each $\beta\in A$ $\langle\delta_n^\beta\colon n<\omega\rangle$ is a generic ω sequence through β. And, associated with $\langle\delta_n^\beta\colon n<\omega\rangle$ is a generic sequence $\langle F_n^\beta\colon n<\omega\rangle$, where each F_n^β is a Levy generic collapsing function on $\mathrm{Col}(\delta_n^{\beta^{++}}, \delta_{n+1}^\beta)$ for $n \geqslant 1$; for $n = 0$, F_0^β is either Levy generic on $\mathrm{Col}(\omega_1, \delta_1^{\alpha_0})$, for α_0 the least element of A, or on $\mathrm{Col}\big((\bigcup\{\alpha<\beta\colon \alpha\in A\})^{+++}, \delta_1^\beta\big)$.

Let $V_1^\beta = V\big[\langle\langle\delta_n^\alpha\colon n<\omega\rangle, \langle F_n^\alpha\colon n<\omega\rangle\rangle_{\alpha\in A\cap\beta}\big]$, i.e., let $V_1^\beta \subseteq V[G_\beta]$ be the least model of ZFC that contains, for $\alpha \in A\cap\beta$, the generic sequences $\langle\delta_n^\alpha\colon n<\omega\rangle$ and $\langle F_n^\alpha\colon n<\omega\rangle$. Of necessity, V_1^β will also contain a generic function f defined on $A\cap\beta$ so that for $\alpha<\beta$, $f(\alpha) = \langle\langle\delta_n^\alpha\colon n<\omega\rangle, \langle F_n^\alpha\colon n<\omega\rangle\rangle$. $\bar{V}$ is then defined as $\bigcup_{\beta\in A} V_1^\beta$. To show that $\bar{V}$ is the desired model, we have to show that $\bar{V}\models$ "ZFC + Every limit cardinal is a strong limit cardinal + The ωth successor of any cardinal violates GCH".

LEMMA 7. *In $\bar{V}$, the ω-th successor of any cardinal is a strong limit cardinal that violates* GCH.

Proof. Let $\bar{V}\models$ "δ is a cardinal". Let $\lambda = \bigcup\{\alpha\in A\colon \alpha\leqslant\delta\}$, and let β be the least member of $A > \delta$. We show that $\bar{V}\models$ "$\delta^{+\omega}$ is a strong limit cardinal that violates GCH" by showing that this is true in a certain submodel of $\bar{V}$ and that this behavior is preserved in $\bar{V}$.

Consider the model $V_1 = V[\langle\delta_n^\beta\colon n<\omega\rangle, \langle F_n^\beta\colon n<\omega\rangle]$. Using Magidor's arguments [8], the definition of P, and the Product Lemma, we know that $V_1\models$ "$\lambda^{+\omega}$ is a strong limit cardinal that violates GCH". Further, we know that $V_1\models$ "$\beta = \lambda^{+\omega}$". Hence, since $\lambda\leqslant\delta<\beta$, $V_1\models$ "$\delta^{+\omega}(=\lambda^{+\omega})$ is a strong limit cardinal that violates GCH".

Let β_1 be the least element of $A>\beta$, and let us define the model V^* as $V^* = \bigcup_{\alpha\in A, \alpha\geqslant\beta_1} V\big[\langle\langle\delta_n^\gamma\colon n<\omega\rangle, \langle F_n^\gamma\colon n<\omega\rangle\rangle_{\gamma\geqslant\beta_1,\gamma<\alpha}\big]$. We claim that in V^*, the subsets of $(\beta^{++})^V$ are precisely the same as in V. It is not too hard to see this; one just uses a similar argument as in Prikry forcing. Specifically, let τ be a term, in the language appropriate for P^{β_1} such that for some $\pi_0\in P^{\beta_1}$, $\pi_0\Vdash$ "$\tau\subseteq\beta^{++}$". Using Lemma 6, let, for $\alpha<\beta^{++}$, $\pi_{\alpha+1}\geqslant\pi_\alpha$ be such that

$$|\pi_\alpha - \pi_{\alpha+1}| = 0 \quad \text{and} \quad \pi_{\alpha+1}\| \text{“}\alpha\in\tau\text{”}.$$

For $\sigma<\beta^{++}$ a limit, let $\pi_\sigma = \bigcup_{\alpha<\sigma}\pi_\alpha$; since $\sigma\leqslant\beta^{++}$, π_σ is going to be defined. Finally, let $\pi_{\beta^{++}} = \bigcup_{\alpha<\beta^{++}}\pi_\alpha$. $\pi_{\beta^{++}}$ will completely determine τ, so τ will denote a set that is actually present in V (this is another place where we use $f_0^\alpha\in\mathrm{Col}\big((\bigcup\{\gamma\in A\colon \gamma<\alpha\})^{+++}, \delta_1^\alpha\big)$).

Since there are no new subsets of $(\beta^{++})^V$ present in V^*, we have that $V^*\models$ "U^β is a normal ultrafilter on $P_\beta(\beta^+)\;\&\;2^\beta = \beta^{++}$". Thus, again using the Product Lemma, $\langle\langle\delta_n^\beta\colon n<\omega\rangle, \langle F_n^\beta\colon n<\omega\rangle\rangle$ is a V^*-generic sequence, so $\tilde{V} = V^*\big[\langle\langle\delta_n^\beta\colon n<\omega\rangle, \langle F_n^\beta\colon n<\omega\rangle\rangle\big]\models$ "$\delta^{+\omega}(=\beta)$ is a strong limit cardinal that violates GCH".

Consider now what happens when we force over $\tilde{V}$ with P_β. By our GCH assumptions in V and by definition of P_β, $\bar{P}_\beta\leqslant(\lambda^{++})^V$, so forcing over $\tilde{V}$ with P_β will not change the fact that $(\lambda^{+\omega})^{\tilde{V}} = (\delta^{+\omega})^{\tilde{V}}$ is a strong limit cardinal that violates GCH, i.e., $\tilde{V}[G_\beta]\models$ "$\delta^{+\omega}$ is a strong limit cardinal that violates GCH". Since $\tilde{V}\big[\langle\langle\delta_n^\alpha\colon n<\omega\rangle, \langle F_n^\alpha\colon n<\omega\rangle\rangle_{\alpha<\beta}\big]\subseteq\tilde{V}[G_\beta]$, we have $\tilde{V}\big[\langle\langle\delta_n^\alpha\colon n<\omega\rangle, \langle F_n^\alpha\colon n<\omega\rangle\rangle_{\alpha<\beta}\big]\models$ "$\delta^{+\omega}$ is a strong limit cardinal that violates GCH". However, by the Product Lemma, this model is just $\bar{V}$, so $\bar{V}\models$ "$\delta^{+\omega}$ is a strong limit cardinal that violates GCH". ∎

COROLLARY 8. *In $\bar{V}$, every limit cardinal is a strong limit cardinal.*

Proof. Let λ be a limit cardinal in $\bar{V}$, and let $\delta<\lambda$ be a cardinal. $\bar{V}\models$ "$\delta^{+\omega}\leqslant\lambda$", and by the above lemma, $\bar{V}\models$ "$2^\delta<\delta^{+\omega}$". ∎

To complete the proof of Theorem 1, we need only show that $\bar{V} \models$ ZFC. As is usual when forcing with a proper class, the customary proofs will show that all of the axioms of ZFC, with the exception of Power Set and Replacement, will hold in $\bar{V}$ (see [2] or [5]). The proof of Lemma 7 actually shows that, in $\bar{V}$, the power set of any cardinal is determined by forcing with a set; specifically, using the notation of Lemma 7, the power set of β, for β as in Lemma 7, is determined by forcing with P_{β_1}. Thus, $\bar{V} \models$ Power Set Axiom, and we need only show that the Replacement axioms hold in $\bar{V}$.

The proof of Replacement will hinge on the fact that each $R(\alpha)^{\bar{V}}$ will actually be an $R(\alpha)^{V[\langle\langle \delta_n^\alpha: n<\omega\rangle, \langle F_n^\alpha: n<\omega\rangle\rangle_{\alpha<\beta}]}$ for some suitable P_β. It is not too hard to show this fact; indeed, the proof of this is quite similar to the proof given in the last lemma. Specifically, let α be an arbitrary ordinal, and let $\beta_1 \in A$ be such that there is $\beta \in A$ so that $\alpha < \beta < \beta_1$. β_1 has been chosen so that given a sequence $\langle \pi_\delta: \delta \leqslant \beta\rangle$ of elements of P^{β_1} such that $\delta < \gamma$ implies that π_γ is a 0-length preserving extension of π_δ, there is one condition π which is a 0-length preserving extension of all of them. Thus, we can argue as we did in the previous lemma and see that in V^*, where V^* has the same meaning as it did in Lemma 7, the subsets of β are the same as the ones in V. Since β is thus still strongly inaccessible in V^*, we have that $R(\beta)^{V^*} = R(\beta)^V$. As previously, the sequence $\langle\langle \delta_n^\gamma: n<\omega\rangle, \langle F_n^\gamma: n<\omega\rangle_{\gamma<\beta_1}\rangle$ is generic over V^*, and

$$V^*[\langle\langle \delta_n^\gamma: n<\omega\rangle, \langle F_n^\gamma: n<\omega\rangle_{\gamma<\beta_1}\rangle] = \bar{V},$$

so we have that the new subsets of $R(\alpha)$ present in $\bar{V}$ are not present in V^*, but are present in

$$V^*[\langle\langle \delta_n^\gamma: n<\omega\rangle, \langle F_n^\gamma: n<\omega\rangle\rangle_{\gamma<\beta_1}] = \bar{V}.$$

Thus, the new subsets of $R(\alpha)$ present in $\bar{V}$ are actually those present in

$$V[\langle\langle \delta_n^\gamma: n<\omega\rangle, \langle F_n^\gamma: n<\omega\rangle\rangle_{\gamma<\beta_1}]$$

and so form a set.

The usual proof will show that Aussonderung holds in $\bar{V}$. Hence, to show that $\bar{V} \models$ Replacement, all we need show is that the Bounding Principle (Collection Schema) holds in $\bar{V}$, since it is well known that Aussonderung + Bounding Principle $\vdash$ Replacement. So, suppressing unnecessary parameters, we have to show that if $\bar{V} \models \forall x \exists y\, \varphi(x, y)$, and if a bound u on x is given, then there is some $v \in \bar{V}$ so that $\bar{V} \models \forall x \in u \exists y \in v\, \varphi(x, y)$. To do this, let $p \in P$ be so that $p \Vdash \forall x \exists y\, \varphi(x, y)$.

Suppose a bound u on x is given; without loss of generality, suppose that u is an $R(\alpha_0)$ for some α_0. As we have already observed, each $R(\alpha)^{\bar{V}}$ is in $V[\langle\langle \delta_n^\varrho: n<\omega\rangle, \langle F_n^\varrho: n<\omega\rangle\rangle_{\varrho\leqslant\beta}]$ for some suitable β. Hence, we can let $\langle \tau_\gamma: \gamma<\eta\rangle$ be a set of terms such that each τ_γ always denotes some element of $R(\alpha_0)^{\bar{V}}$ and each element of $R(\alpha_0)^{\bar{V}}$ is always denoted by some τ_γ. As $p \Vdash \forall x \exists y\, \varphi(x, y)$ for each τ_γ $p \Vdash \exists y\, \varphi(\tau_\gamma, y)$; let therefore σ_γ be a term so that $p \Vdash \varphi(\tau_\gamma, \sigma_\gamma)$. As σ_γ will always denote an element of $\bar{V}$, we can let β_γ be so that for each term $\langle\langle \delta_n^\varrho: n<\omega\rangle, \langle F_n^\varrho: n<\omega\rangle\rangle_{\varrho<\beta_\gamma}$ which appears in σ_γ, $\beta_\gamma > \beta$.

Now, define $\beta' = \bigcup_{\gamma<\eta} \beta_\gamma$, and let γ_0 be the least element of $A > \beta'$. By definition of γ_0, the Product Lemma, and our previous remarks, each σ_γ will be interpretable in $V[\langle\langle \delta_n^\varrho: n<\omega\rangle, \langle F_n^\varrho: n<\omega\rangle\rangle_{\varrho<\gamma_0}]$. Hence, if we let b be the collection of the interpretations of the σ_γ, b will be a set in $V[\langle\langle \delta_n^\varrho: n<\omega\rangle, \langle F_n^\varrho: n<\omega\rangle\rangle_{\varrho<\gamma_0}]$ so that $\bar{V} \models \forall x \in R(\alpha_0) \exists y \in b\, [\varphi(x, y)]$. Thus, we will have the Bounding Principle true in $\bar{V}$, and hence have Replacement true in $\bar{V}$. Thus, $\bar{V} \models$ ZFC.

Lemma 7, Corollary 8, and the above complete the proof of Theorem 1.

Added in proof. We would like to remark that although we have not shown them, the usual lemmas about forcing (the Truth Lemma, etc.) remain true even though P is a proper class and each $\pi \in P$ is a proper class. These lemmas are proven in the usual way for P a proper class; see, for example, [2], [4], or [5].

Indeed, it is possible, by a strengthening of the hypotheses, to eliminate the class forcing argument. If we assume, for example, the existence of a measurable cardinal $\varkappa$ so that GCH holds at all singular cardinals below $\varkappa$ and so that $\varkappa$ is the limit of cardinals δ with the property that each δ is δ^+ supercompact and $2^\delta = \delta^{++}$, then Magidor's arguments of [7] can be used to show that $V[G] \models$ "$\varkappa$ is a measurable cardinal"; hence, $V[\langle\langle \delta_n^\gamma: n<\omega\rangle, \langle F_n^\gamma: n<\omega\rangle\rangle_{\gamma<\varkappa}] \models$ "$\varkappa$ is inaccessible". This will immediately imply that $R(\varkappa)$ of this last model satisfies ZFC, so $R(\varkappa)$ will be the desired model.

In conclusion, we remark that Woodin and Foreman have dramatically improved the results contained in this paper. Starting with a cardinal $\varkappa$ which is $2^\varkappa$ supercompact, they use Radin forcing to construct a model in which for each cardinal δ, $2^\delta = \delta^{++}$.

References

[1] A. Dodd and R. Jensen, *The core model*, circulated notes.
[2] W. Easton, *Powers of regular cardinals*, Ann. Math. Logic 1 (1970), pp. 139–178.
[3] U. Felgner, *Comparison of the axioms of local and universal choice*, Fund. Math. 71 (1971), pp. 43–62.
[4] M. Gitik, *All uncountable cardinals can be singular*, to appear.
[5] T. Jech, *Set Theory*, Academic Press 1978.
[6] A. Levy and R. Solovay, *Measurable cardinals and the continuum hypothesis*, Israel J. Math. 5 (1967), pp. 234–248.
[7] M. Magidor, *How large is the first strongly compact cardinal?*, Ann. Math. Logic 10 (1976), pp. 33–57.
[8] — *On the singular cardinals problem I*, Israel J. Math. 28 (1977), pp. 1–31.
[9] — *On the singular cardinals problem II*, Ann. of Math. 106 (1977), pp. 517–547.
[10] T. Menas, *Consistency results concerning supercompactness*, Trans. Amer. Math. Soc. 223 (1976), pp. 61–91.
[11] — *A combinatorial property of $P_\varkappa(\lambda)$*, J. Symb. Logic 41 (1976), pp. 225–233.
[12] J. Silver, *Large cardinals and GCH*, to appear.

[13] J. Silver, *On the singular cardinals problem*, Proc. Int. Congr. Math. Vancouver 1974, Vol. 1, pp. 265–268.
[14] R. Solovay, W. Reinhardt and A. Kanamori, *Strong axioms of infinity and elementary embeddings*, Ann. Math. Logic 13 (1978), pp. 73–116.

DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
Newark, New Jersey 07102

Received 9 February 1981

Relation entre le rang U et le poids

par

Daniel Lascar (Paris)

Abstract. Let T be a superstable theory; we prove, in T^{eq} (see [S]) that a regular type is non-orthogonal to a regular type whose U-rank is of the form ω^α. If $U(p) = \omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \ldots$ $\ldots + \omega^{\alpha_k} n_k$, then its weight is at most $\sum_{i=1}^{k} n_i$.

I. Introduction. Dans tout cet article on supposera que T est une théorie complète et superstable.

La notion de poids (voir les rappels) semble être trés importante lorsqu'on essaie de classifier les modèles d'une théorie superstable, et indispensable lorsqu'on s'intèresse aux modèles dénombrables. Dans [L1] on borne le poids $w(p)$ du type p en fonction du developpement de Cantor de son rang U: si

$$U(p) = \omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \ldots + \omega^{\alpha_k} n_k$$

alors

$$w(p) \leqslant (n_1+1)(n_2+1)\ldots(n_k+1).$$

On va raffiner cette inégalité en montrant que $w(p) \leqslant \sum_{i=1}^{k} n_i$. En fait cette relation est la meilleure possible. On verra aussi quelques relations entre le developpement de Cantor, la régularité et l'orthogonalité. Une autre motivation qui nous a poussé à écrire cet article est qu'on y fait usage du théorème de la base canonique (cf. [S], chapitre III) et qu'à notre connaissance, il n'en existe nulle part, hors du livre de Shelah, d'application ou même de référence.

On supposera que le lecteur possède des connaissances générales sur la stabilité (comme il peut les acquérir dans [LP] ou [S] par exemple). De plus on utilisera le rang U (voir [L1]), la notion de poids (chapitre V de [S], ou [L2]), et les éléments imaginaires (chapitre III de [S]). Le paragraphe suivant est consacré à rappeler les faits essentiels sur ces trois points. Les notations sont celles qui sont habituelles (celles de [L2] par exemple). Par rang, nous voulons toujours dire rang U.