Ramified analysis and the minimal β-models of higher order arithmetics *

by

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Abstract. We discuss the minimal β-models of A_n in terms of ramified analysis. A β-interpretation of A_n in A_n is obtained, which leads from a β-model to the minimal β-model of A_n. We propose an alternative notion of constructibility for A_n. We generalize the notion of ramified analysis and we prove (by Jensen's methods) a generalization of the theorem on "correspondence in levels" between the hierarchy of constructible sets L and the ramified analysis RA. All proofs and lemmas are formulated for $n = 3$. However, they work for $n \geqslant 3$.

Chapter I

Introduction. The present investigation was inspired by the following question, formulated by K. Apt and W. Marek ([1], p. 226): "Can the smallest β-model of A_n ($n > 2$) be characterized "from below" similarly to the "ramified analytical" characterization in the case of A_2?" The answer we give is positive.

Higher order arithmetics, and especially the second order arithmetic A_2, have been throughly investigated since the fifties: particularly in Warsaw by Professor Mostowski and his colleagues and students. It was Mostowski's idea to classify models of A_2 with respect to the notion of well-foundedness: β-models (i.e., models with respect to which the notion of well-ordering is absolute) were carefully investigated.

The class of all β-models of A_2 has a nice property: there exists a smallest β-model of A_2. This model was investigated by R. O. Gandy [4] and others. Gandy proved that this is exactly what we call ramified analysis. The proof of this fact was obtained by recursion-theoretic methods by using the properties of the notion of hyperjump; unfortunately, it does not generalize to the higher order cases. (This is not very surprising: let us recall here some very important differences between A_2 and A_3, such as for example different positions of the notion of well-ordering and also of constructibility. See [1].) A strong connection between constructibility and

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ramified analysis was first established by Boolos [2], who proved that $RA_\alpha = P(\omega) \cap L_\alpha$. We shall generalize R. Jensen's proof of this result. Let us recall also one important result obtained by Zbierski [10]: it shows the equivalence (in the sense of interpretability) of the arithmetic A_n and the set theory ZFC$^-$ + "$P^{n-2}(\omega)$ exists". Zbierski discusses the formula "A is a well-founded tree" (Tree (A)) and shows that for all axioms φ of ZFC$^-$ + + "$P^{n-2}(\omega)$ exists" we have $A_n \vdash \varphi^{\mathrm{Trees}}$. The model theoretic counterpart of this theorem establishes the correspondence between β-models of arithmetics and standard models of set theory.

THEOREM (Zbierski [10]). 1. *If M is a standard transitive model of* ZFC$^-$ + "$P^{n-2}(\omega)$ *exists", then $Ar^n(M)$ is a β-model of A_n.*

2. *If N is a β-model of A_n, then the trees of N with the relation of isomorphism of trees and the relation of being a maximal proper subtree form a model which is isomorphic to a standard transitive model M of* ZFC$^-$ + "$P^{n-2}(\omega)$ *exists", so that $N = Ar^n(M)$.*

We shall give below a very natural generalization of ramified analysis for $n \geqslant 3$ and we shall show its properties. The main feature of our proofs is the absence of any recursion-theoretic notions and methods. This constitutes the main difference between our results and those mentioned above ([1], [2], [4]). A possible generalization of recursion-theoretic methods of Gandy, Boolos, Putnam and others would require an essential development of the recursion-theoretic technique to higher type objects. A general question arises in which scale would it be possible?

To solve the problem of Apt and Marek we introduce generalized ramified analysis in the following way:

DEFINITION 1.1.

$RA_\xi^{(n)} \underset{\mathrm{df}}{=} \langle \omega, RA_\xi^{(n),1}, \ldots, RA_\xi^{(n),n-1}, \in, +, \cdot, <, 0, 1\rangle$,

$RA^{(n)} \underset{\mathrm{df}}{=} \langle \omega, RA^{(n),1}, \ldots, RA^{(n),n-1}, \in, +, \cdot, <, 0, 1\rangle$,

where

$RA_0^{(n),i} \underset{\mathrm{df}}{=} P^i(\omega) \cap HF$,

$RA_{\alpha+1}^{(n),i} \underset{\mathrm{df}}{=} D^i(RA_\alpha^{(n)}) \underset{\mathrm{df}}{=}$ subsets of $RA_\alpha^{(n),i-1}$ definable over $RA_\alpha^{(n)}$ by $L(A_n)$-formulas with parameters from $RA_\alpha^{(n)}$ (here $RA_\alpha^{(n),0} \underset{\mathrm{df}}{=} \omega$),

$RA_\lambda^{(n),i} \underset{\mathrm{df}}{=} \bigcup_{\xi<\lambda} RA_\xi^{(n),i}$ for $\lambda \in \mathrm{Lim}$, $RA^{(n),i} \underset{\mathrm{df}}{=} \bigcup_{\xi} RA_\xi^{(n),i}$.

We call this structure ramified analysis because for $n = 2$ it is just the usual ramified analysis. In the Chapter III we prove the Main Theorem claiming the existence of the β-interpretation of A_n in A_n which defines in every β-model the ramified analysis $RA^{(n)}$.

NOTATIONS. 1. Variables. We shall use letters m, n, i, k, t to denote natural numbers; φ, ψ will denote formulas or their Gödel numbers, capitals will usually denote the highest order objects (classes) and small letters the lower order objects (sets or numbers).

2. Pairing functions. In 1st-order Peano arithmetic we can define a function J being a "one-one" mapping defined on pairs of natural numbers onto all natural numbers. It is possible to write a formula "$\langle x, y\rangle^{\cdot} = z$" defining the pair $\langle x, y\rangle^{\cdot}$ of two objects x and y (of orders i and j respectively) as an object z of the order $\max(i, j)$. One can do this by using the function J. Also one can code n-tuples of objects of order smaller than or equal to i as an object of order i.

3. Coding. We shall use this term in two different meanings.

a. $X\eta Y \underset{\mathrm{df}}{\leftrightarrow} (Ea)(b)(b \in X \leftrightarrow \langle a, b\rangle^{\cdot} \in Y) \underset{\mathrm{df}}{\leftrightarrow} (Ea)(X = (Y)^{(a)})$,
where $X = (Y)^{(a)} \underset{\mathrm{df}}{\leftrightarrow} (b)(b \in X \leftrightarrow \langle a, b\rangle^{\cdot} \in Y)$. We then say that Y is a *code of the family* $\{X\colon X\eta Y\}$. We define similarly $a\eta X$ for a and X of different order. $\mathrm{Dom}(Y) \underset{\mathrm{df}}{=} \{a\colon (Eb)(\langle a, b\rangle^{\cdot} \in Y)\}$. We can imagine that the family coded by Y is numerated by the elements of $\mathrm{Dom}(Y)$.

b. We also say that X *codes the relation* A iff $\langle a, b\rangle^{\cdot} \in X \leftrightarrow \langle a, b\rangle \in A$.
In particular, the class X *codes a tree* iff

– X codes a partial ordering with no loops,

– there exists the unique maximal element in the partial ordering coded by X,

– every linear subordering of the ordering coded by X is finite.

When X codes a tree, we write $\mathrm{Tree}(X)$ or $X \in \mathrm{Trees}$. It is easy to write the formula of $L(A_n)$ defining trees.

Chapter II

Definability of ***RA***. The purpose of this chapter is to formalize the notion of ramified analysis in the language of arithmetic, which yields a formula which is absolute with respect to β-models. This will be accomplished in Theorem 1. The method of proof of this theorem was presented by Mostowski [7] for the β-models of KM and can be directly translated to our case. This permits us to omit all details and to restrict ourselves to some general remarks concerning the proof.

THEOREM 1 (formulated here for $n = 3$). *Let $M = \langle F_1, F_2, \in, \ldots\rangle$ be a β-model of A_3, η being the height of M. Then, for $i = 1, 2$*

1. *there exist formulas* $\mathrm{ra}_i(\cdot,\cdot)$ *and* $\mathrm{ra}_i(\cdot)$ *such that*

$$a \in RA_\alpha^{(3),i} \leftrightarrow \langle \omega, F_1, F_2, \in, \ldots\rangle \models \mathrm{ra}_i(T, a)$$

and

$$a \in RA^{(3),i} \leftrightarrow \langle \omega, F_1, F_2, \in, \ldots\rangle \models \mathrm{ra}_i(a),$$

where T is a code of a well-ordering of order-type α and $T \in F_2$,

2. $RA_{\eta+1}^{(3),i} \subset F_i$,

3. $RA_{\eta+1}^{(3),2}$ ($RA_{\eta+1}^{(n),n-1}$ *in the general formulation*) *does not contain any well-ordering of type η.*

Our aim is to write down in an appropriate way the formulas $\mathrm{ra}_i(\cdot,\cdot)$ and $\mathrm{ra}_i(\cdot)$, which are the arithmetical reconstruction of the set-theoretical definition of ramified analysis. The difficulties which we have to overcome are typical for the problems of formalization of mathematics in arithmetic. One of them is the nonexistence of objects which are collections of elements of different ranks. It is also impossible to discuss explicitly collections of classes, even as small as pairs. (We can not apply here Kuratowski's notion of pair.) We overcome this difficulty by emploing pairs $\langle x, y\rangle^{\cdot}$ and codes. In particular, we can write down formula $\mathrm{Bord}(X)$, which says that the class X is a well-ordering by using the same formula as in the set theory but with $\langle\cdot,\cdot\rangle^{\cdot}$ instead of Kuratowski's pairing function.

Another important problem is connected with inductive definitions. The main difficulty here is due to the absence of ordinal numbers, and so we are reduced to work with well-orderings. The problem is that, is general, there is no distinguished well-ordering of a given order type. We have to prove additionally that a construction where a well-ordering of a given type is to be used does not depend on the particular choice of the well-ordering. In our proof we define ramified analysis by using sequence-constructors whose domains are well-orderings. The totality of these sequence-constructors determines objects of ramified hierarchy. It is important to show that any two sequence-constructors based on isomorphic domains (well-orderings) produce the same objects.

To be able to reconstruct the definition of RA in arithmetic we first have to reconstruct the notion of satisfaction. We write a formula $\mathrm{Sat}(X, Y, \varphi, \vec{n}, \vec{a}, \vec{b})$, where X, Y code some families of objects and $\vec{n}, \vec{a}, \vec{b}$ code some finite sequences of objects, with the following property:

$$M \models \mathrm{Sat}(X, Y, \varphi, \vec{n}, \vec{a}, \vec{b})$$
$$\leftrightarrow \langle\omega, \{a\colon a\eta X\}, \{A\colon A\eta Y\}, \in, \ldots\rangle \models \varphi[\vec{n}, X^{(\vec{a})}, Y^{(\vec{b})}]$$

for any ω-model M. This formula has to represent the inductive definition of satisfaction. For example, one of its inductive conditions is: $\mathrm{Sat}(X, Y, x^{(1)}\in x^{(2)}, a, b) \leftrightarrow X^{(a)}\in Y^{(b)}$. $\mathrm{Sat}(\ldots)$ is Π^2_1 in all ω-models of A_3 (generally $\mathrm{Sat}(\ldots)$ is Π^{n-1}_1 in all ω-models of A_n). Using the formula $\mathrm{Sat}(\ldots)$, we can define the class $D^i(X, Y)$ coding the family of classes which are definable over the structure coded by X, Y, i.e., $(D^i(X, Y))^M$ is a code (in any ω-model M) of the family $\mathrm{Def}^i(\langle\omega, \{a\colon a\eta X\}, \{A\colon A\eta Y\}, \in, \ldots\rangle)$. Now we are able to write the formulas $\mathrm{ra}_i(T, a)$ which, informally speaking, express what follows: "a is an object of rank i which is produced by a suitable sequence-constructor whose domain is the well-ordering T". $\mathrm{ra}_i(a)$ is the formula $(ET)\,(\mathrm{Bord}(T)\,\&\,\mathrm{ra}_i(T, a))$. If T is a "true" well-ordering of order type α, then $M\models \mathrm{ra}_i(T, a)$ iff $a\in RA^{(3),i}_\alpha$ for any β-model M. This fact may be proved by induction with respect to α. As a by-product of his reasoning Mostowski got the existence of a definable well-ordering of RA ([7]). This is also true in our case.

Chapter III

Correspondence theorem. We shall now discuss the main technical lemma of our work. Investigating the usual ramified analysis, Boolos [2] discovered the following interesting fact (we use the term "correspondence theorem" in referring to it): $RA_\alpha = P(\omega)\cap L_{\omega+\alpha}$ for $\alpha<\beta_0$. For some α this correspondence of levels in ramified and constructible hierarchies was known before from Kleene's result about hyperarithmetical sets: $\Delta^1_1 = RA_{\omega_1}$. Therefore, $RA_{\omega_1} = P(\omega)\cap L_{\omega_1}$ and, by relativization, $RA_{\omega^x_1} = P(\omega)\cap L_{\omega^x_1}$ (let us remark that any admissible ordinal is of the form ω^x_1 for some $x\in P(\omega)$). The recursive-theoretical proof given by Boolos seems not to generalize to the higher order cases. Anyway, another proof found by Jensen (unpublished [6]) can be adapted to our case. We shall briefly outline this proof in the present chapter.

The following theorem (formulated here for $RA^{(3)}$) is our main technical lemma.

THEOREM 2 (on correspondence in levels).

$$(*)\qquad RA^{(3),i}_\alpha = P^i(\omega)\cap L_{\omega+\alpha}\quad \textit{for}\quad \alpha<\eta_3 \textit{ and } i=1,2.$$

(η_n *denotes the first n-gap, see* [1], [8].)

We shall prove this theorem by induction: that is why some informations about the "fine structure of constructibility" will be used (Lemmas 3.4 and 3.5).

Notice. The proof we shall outline below works in $\mathrm{ZFC}^- +$"$P^{n-2}(\omega)$ exists" (here for $n=3$). This observation plays the fundamental role in the considerations of the next chapter.

Proof. It can easily be observed that only the nonlimit step in the induction is nontrivial. Let us assume $(*)_\alpha$. $(*)_{\alpha+1}$ results from the following equations for $i=1, 2$: $P^i(\omega)\cap L_\alpha = \mathrm{Def}^i(\mathrm{Ar}^3(L_\alpha))$. The inclusions $\supset$ are easy because of the definability of $\mathrm{Ar}^3(L_\alpha)$ in L_α. (Let us recall that $\mathrm{Ar}^3(L_\alpha) \underset{\mathrm{df}}{=} \langle\omega, P(\omega)\cap L_\alpha, PP(\omega)\cap L_\alpha, \in, \ldots\rangle$ and $\mathrm{Ar}^3(L_\alpha)$ is called the 3th order arithmetic of L_α). To prove the inclusions $\subset$ we shall attempt to code the whole L as a definable subset of $\mathrm{Ar}^3(L_\alpha)$. Then we shall explore the fact that the isomorphism between natural numbers and theirs codes in $\mathrm{Ar}^3(L_\alpha)$ is definable in $\mathrm{Ar}^3(L_\alpha)$. It is convenient to consider separately two cases: $\alpha\in\mathrm{Lim}$ and $\alpha\notin\mathrm{Lim}$.

Let $\alpha=\gamma+1$.

DEFINITION 3.1. 1. Let $E\in P^2(\omega)$. We say that E is a *copy for* L_α iff

$$(EH)\big(H\colon \mathrm{Fld}(E)\overset{1-1}{\leftrightarrow} L_\alpha\ \&\ (a)(b)(\langle a, b\rangle^{\cdot}\in E \leftrightarrow H(a)\in H(b))\big).$$

2. Let $F\in P^2(\omega)$. We say that F is a *copy for* $<_\alpha$ iff

$$(EH)\big(H\colon \mathrm{Fld}(F)\overset{1-1}{\leftrightarrow} L_\alpha\ \&\ (a)(b)(\langle a, b\rangle^{\cdot}\in F \leftrightarrow H(a)<_\alpha H(b))\big).$$

($<_\alpha$ is the canonical constructible well-ordering of L_α; F: $X \overset{1-1}{\leftrightarrow} Y$ means that F is a bijection from X into Y).

The following lemma is due to Boolos:

LEMMA 3.1 (G. Boolos [2], see also W. Marek, M. Srebrny [8]). *If γ is not a 3-gap, then there is a copy for L_γ in $P^2(\omega) \cap L_{\gamma+1}$. The same holds for a copy of $<_\gamma$.*

Having in $RA^{(3)}_{\gamma+1}$ the copies for L_γ and $<_\gamma$, we can easily define over $RA^{(3)}_{\gamma+1}$ a copy of L_α. This permits us to define elements of $P^i(\omega) \cap L_\alpha$ over $RA^{(3)}_\alpha$.

The limit case is more complicated.

Let $\alpha \in \mathrm{Lim}$.

Unfortunately it is not clear how to define a copy for L_α over $\mathrm{Ar}^{(3)}(L_\alpha)$ in this case. Instead, we define uniformly fragments of such a copy, i.e. copies for L_β, $\beta < \alpha$. This may be done with the aid of ramified language (see [9]). Intuitively: terms of the ramified language RL_S (ramified language of rank S, where S is a code for a well-ordering $\beta < \alpha$) are names for constructible sets from L_β. True atomic β-formulas give a copy of L_β. The point is we have to be able to pick in $\mathrm{Ar}^{(3)}(L_\alpha)$ the collection of true well-orderings of length β for all $\beta < \alpha$. To do this we take the set $W_\alpha \underset{\mathrm{df}}{=} \{R$: R codes a well-ordering & $R \in P^2(\omega) \cap L_\alpha$ & $(E\beta)_\alpha (Ef)_{L_\alpha} f$: $\mathrm{Fld}(R) \overset{1-1}{\leftrightarrow} \beta\}$ and we show the following two lemmas.

LEMMA 3.2. *The supremum of order types of elements of W_α equals α* (i.e., $\sup_{S \in W_\alpha} (\bar{S}) = \alpha$).

LEMMA 3.3. *W_α is definable over* $\mathrm{Ar}^3(L_\alpha)$.

To prove Lemma 3.2 we shall use two facts established by Jensen (see Devlin [3], p. 95).

LEMMA 3.4 (Jensen). *If $\beta \in \mathrm{Lim}$ and β is not p.r.-closed (i.e., is not closed with respect to primitive recursive functions), then there is a cofinality function g definable over L_β and such that g: $\mathrm{cf}(\beta) \overset{1-1}{\to} \beta$ cofinally.*

LEMMA 3.5 (Jensen). *If β is p.r.-closed and $\beta < \eta_n$, then there exists a function f in* $\mathrm{Def}(L_\beta)$ *such that f: $P^{n-2}(\omega) \cap L_\beta \overset{\text{onto}}{\to} L_\beta$.*

Proof of Lemma 3.2. We show by induction that for every $\beta \leqslant \alpha$ there exists an $R \in W_\alpha$ such that $\bar{R} \geqslant \beta$. For $\beta \in \mathrm{Lim}$ we analyse separately two cases: 1. β is p.r.-closed, 2. it is not. Here we use Jensen's two lemmas presented above. The case where $\beta \notin \mathrm{Lim}$ is trivial.

To prove Lemma 3.3, the fine point of our proof, we need several facts about the ramified language RL_S, where $S \in P^2(\omega) \cap L_\alpha$. This is a language whose terms can serve as names for constructible sets (see [9]). It is important to define this language in terms of arithmetic. What we do is to use a Gödel style inductive definition with the aid of the pairing function $\langle \cdot, \cdot \rangle^{\cdot}$. For example $\langle 5, \varphi, \psi \rangle^{\cdot}$ will be a formula (intuitively φ & ψ), if φ and ψ are. Then we define in the usual way all fundamental syntactic notions connected with RL_S, in particular T_S — the class of terms, Fl_S — the class of formulas and Fl^0_S — the class of variable-free formulas. We define satisfaction $\models_S$ for RL_S formulas and realization $\|\cdot\|_S$ for RL_S terms. The following property holds:

$$L_\alpha = \{\|t\|_S\colon\ t \in T_S\} = \|\hat{x}^a (x = x)\| \quad \text{for } S \text{ such that } \bar{S} = \alpha,$$

a being the last element in R and $\hat{x}^a$ being the abstraction operator in RL_S.

We shall also use the following lemma. Its straightforward but tedious proof will be omitted.

LEMMA 3.6. *Let $\alpha \in \mathrm{Lim}$. 1. The following relations are definable over* $\mathrm{Ar}^3(L_\alpha)$: "$t \in T_S$", "$\varphi \in \mathrm{Fl}_S$", "$\varphi \in \mathrm{Fl}^0_S$".

2. *There is a formula Θ such that, for each $D, S \in P^2(\omega) \cap L_\alpha$*

$$D = \{\varphi\colon\ \varphi \in \mathrm{Fl}^0_S \ \&\ \models_S \varphi\} \leftrightarrow \mathrm{Ar}^3(L_\alpha) \models \Theta(D, S).$$

3. *If $S, R \in P^2(\omega) \cap L_\alpha$, then $\{\varphi\colon\ \varphi \in \mathrm{Fl}^0_S \ \&\ \models_S \varphi\} \in P^2(\omega) \cap L_\alpha$.*

4. *Any order homomorphism (isomorphism) H of S and R belonging to $P^2(\omega) \cap L_\alpha$ may be extended in a natural way to the homomorphism (isomorphism) $\bar{H}$ of $T_S \cup \mathrm{Fl}_S$ and $T_R \cup \mathrm{Fl}_R$, which also belongs to $P^2(\omega) \cap L_\alpha$.*

To finish the proof of the definability of W_α over $\mathrm{Ar}^3(L_\alpha)$ we analyse two cases. The first of them is: there is no $R \in P^2(\omega) \cap L_\alpha$, R being a linear ordering of type $\alpha + \tau$ and such that $R^{(\nu)} \in W_\alpha$, for $\gamma < \alpha$, where τ is an arbitrary order type, empty or not. (If S is a code of a well-ordering, then $S^{(\nu)}$ is its initial segment of type ν). Our Lemma 3.3 then follows from the fact that $R \in W_\alpha \leftrightarrow \mathrm{Ar}^3(L_\alpha) \models \mathrm{Bord}(R)$ (because $\mathrm{Ar}^3(L_\alpha)$ has the β-property). In the opposite case, i.e., if a well-ordering R with the above property exists, it is enough to show that the set $\{R^{(\nu)}\colon\ \nu < \alpha\}$ is definable over $\mathrm{Ar}^3(L_\alpha)$ (because $W_\alpha = \{S\colon (EH)_{P^2(\omega) \cap L_\alpha} (E\nu)_\alpha (H\colon S \overset{1-1}{\leftrightarrow} R^{(\nu)})\}$). To do this we need two facts (true if $\alpha \in \mathrm{Lim}$).

Fact 1. $(\beta)_\alpha (ED)_{P^2(\omega) \cap L_\alpha} \mathrm{Ar}^3(L_\alpha) \models \Theta(D, R^{(\beta)})$.

Fact 2. For each $a \in \mathrm{Fld}(R)$ such that $R \upharpoonright a$ has the order type $\alpha + \tau$ and each $D \in P^2(\omega) \cap L_\alpha$ we have $\mathrm{Ar}^3(L_\alpha) \models \neg\Theta(D, R \upharpoonright a)$ (Θ is the formula from Lemma 3.6).

This completes the proof of Lemma 3.3.

LEMMA 3.7. *Let φ be a formula of $L(\mathrm{ZF})$. Then there exists a formula $\bar{\varphi}$ of $L(A_3)$ such that*

$$L_\alpha \models \varphi[z_1, \ldots, z_k] \leftrightarrow \mathrm{Ar}^3(L_\alpha) \models \bar{\varphi}[R_1, \ldots, R_k, t_1, \ldots, t_k]$$

where $R_i \in W_\alpha$, $t_i \in T_{R_i}$, $z_i = \|t_i\|_{R_i}$.

The proof of this lemma is by induction. Let φ be a bounded formula from $L(\mathrm{ZF})$ and $R_i \in W_\alpha$, $t_i \in T_{R_i}$, $z_i = \|t_i\|_{R_i}$.

$L_\alpha \models \varphi[z_1, \ldots, z_k] \leftrightarrow (E_\beta)_\alpha L_\beta \models \varphi[z_1, \ldots, z_k] \leftrightarrow (ES)_{W_\alpha}(ES_1) \ldots (ES_k)(EH_1) \ldots (EH_k)(Et'_1) \ldots (Et'_k)$ ("S_i are initial segments of S isomorphic to R_i via isomorphisms H_i" & (H) ("H is a common extension of isomorphisms H_i to terms and formulas of RL_{S_i} such that $H(t_i) = t'_i$" $\rightarrow \models_S H(\varphi)[t'_1, \ldots, t'_k]$)).

To verify those equalities we use the well-known properties of $\models_S$ and $\|\cdot\|_S$. From the definability of syntactical notions of RL and from the definability of W_α it follows that the last sentence can be relativized to $\mathrm{Ar}^3(L_\alpha)$.

The inductive step: we use the following equivalence.

$$L_\alpha \models (Ex)\varphi(x)[z_1, \ldots, z_k]$$

$$\leftrightarrow \mathrm{Ar}^3(L_\alpha) \models (ER)(Et)(\text{"}R \in W_\alpha\text{" \& "}t \in T_R\text{" \& } \bar{\varphi}(R, t))[R_1, \ldots, R_k, t_1, \ldots, t_k].$$

COROLLARY. *There is a formula* $\in$ *such that, for* $n \in \omega$,

$$n \in a \leftrightarrow \mathrm{Ar}^3(L_\alpha) \models \in[R_\omega, R, \underline{n}, t_a],$$

where R_ω *is a code of a well-ordering of type* ω, $R \in W_\alpha$, $t_a \in T_R$, $\|t_a\|_R = a$ *and* $\underline{n}$ *is a term of the ramified language which denotes* n.

Now we can easily obtain the inclusions we need to complete the proof of Theorem 2. For example: let $a \in P^2(\omega) \cap L_\alpha$. Then

$$a = \{b \in L_\alpha \colon L_\alpha \models \varphi[b, \ldots]\} \quad \text{for a certain } \varphi.$$

We show that

$$b \in a \leftrightarrow \mathrm{Ar}^3(L_\alpha) \models (ER)(Et)(\text{"}R \in W_\alpha\text{" \& "}t \in T_R\text{" \& } \bar{\varphi}(R, t, \ldots) \,\&$$
$$\& \,(n)(n \in b \leftrightarrow \in(R_\omega, R, n, t))).$$

It follows that $a \in \mathrm{Def}^2(\mathrm{Ar}^3(L_\alpha))$.

Chapter IV

The Main Theorem: $\mathrm{ra}(\cdot)$ is a standard β-interpretation of A_3 in A_3.

We are ready now to prove the main theorem about the ramified analysis $\mathrm{ra}(\cdot)$ (formulated below for A_3).

THEOREM 3 (ra-theorem). 1. *Formulas* $\mathrm{ra}_1(\cdot), \mathrm{ra}_2(\cdot)$ *from Theorem* 2 (*Chapter* II) *give a standard*(1) *interpretation of* A_3 *in* A_3, *i.e.*, $A_3 \vdash \varphi^{\mathrm{ra}(\cdot)}$ *for all axioms* φ *for* A_3. *For an arbitrary* β-*model* $M \models A_3$ *the equalities* $RA^{(3),i} = (\mathrm{ra}_i(\cdot))^M$ *hold.*

2. $M_0 \underset{\mathrm{df}}{=} \langle \omega, RA^{(3),1}, RA^{(3),2}, \in, +, \cdot, <, 0, 1\rangle$ *is the smallest* β-*model for* A_3.

3. *The following inequalities hold:* $\beta_0 < \xi_3^1 < \xi_3^2 = \eta_3$ ($\beta_0 < \xi_n^1 < \ldots \ldots < \xi_n^{n-2} < \xi_n^{n-1} = \eta_n$ *in the general case*).

4. $M_0 \models V = \mathrm{ra}$.

5. ra *is a* β-*interpretation, i.e.*,

$$A_3 \vdash (S)\big(\mathrm{ra}_2(S) \rightarrow ((\mathrm{Bord}(S))^{\mathrm{ra}} \leftrightarrow \mathrm{Bord}(S))\big).$$

(β_0 *is the first* 2-*gap*, η_n *is the first* n-*gap*, $\xi_n^i \underset{\mathrm{df}}{=} (\mu\xi)[RA_\xi^{(n),i} = RA_{\xi+1}^{(n),i}]$ *and* $V = \mathrm{ra}$ *abbreviates* (*for* A_n) *the conjunction* $\bigwedge_{i=1}^{n-1} (x^{(i)})(\mathrm{ra}_i(x^{(i)}))$.)

Notice. The formula $V = \mathrm{ra}$ may serve as a new form of the axiom of constructibility.

Proof. The existence of the smallest β-model of A_n is well known and follows from the Characterization Theorem for Gaps (Srebrny [8]). From this theorem it follows that the smallest β-model for A_n equals $\mathrm{Ar}^n(L_{\eta_n})$ (where η_n is the first n-gap) and its height equals η_n. Now from the Correspondence Theorem (Chapter III) we get 2.

3 follows easily from 2.

4 follows easily from 1 and 2.

We shall now sketch the proof of 1. As we have noticed, both the theorem of Srebrny and the Correspondence Theorem can be carried in the theory $\mathrm{ZFC}^- +$"$P(\omega)$ exists". From this remark it follows (in $\mathrm{ZFC}^- +$"$P(\omega)$ exists") that $RA^{(3)}$ is the smallest β-model of A_3 ($RA^{(3)}$ exists as a definable class in this set theory).

More precisely:

LEMMA 4.1. $\mathrm{ZFC}^- +$"$P(\omega)$ *exists*" $\vdash \varphi^{RA^{(3)}}$, *where* φ *is an axiom of* A_3 *and* $\varphi^{RA^{(3)}}$ *its relativization to the* $L(\mathrm{ZF})$-*formula defining a class* $RA^{(3)}$ (*arithmetical operations and constants being replaced by their set-theoretical definitions*).

Using Zbierski's theorem we obtain:

LEMMA 4.2. $A_3 \vdash (\varphi^{RA^{(3)}})^{\mathrm{Trees,Eps,Eq}}$ *for all axioms* φ *of* A_3.

(If ψ is a ZF-formula, then by $\psi^{\mathrm{Trees,Eps,Eq}}$ we denote its translation into $L(A_3)$ obtained by relativization of variables to the collection of trees and by replacing $\in$ by Eps and $=$ by Eq. Now it is clear that $(\varphi^{RA^{(3)}})^{\mathrm{Trees,Eps,Eq}}$ is $L(A_3)$-formula.)

For each formula $\psi \in L(A_3)$ the following equivalence holds:

$$(\psi^{RA^{(3)}})^{\mathrm{Trees,Eps,Eq}} \leftrightarrow \psi^{(RA^{(3)})^{\mathrm{Trees,Eps,Eq}}, \mathrm{Eps,Eq}, +^{\mathrm{Trees}}, \cdot^{\mathrm{Trees}}, <^{\mathrm{Trees}}, 0^{\mathrm{Trees}}, 1^{\mathrm{Trees}}}.$$

This equivalence is easily provable by induction with respect to ψ. The right-hand side of this formula will be abbreviated to $\psi^{(RA^{(3)})^{\mathrm{Trees}}}$ and the left-hand side to $(\psi^{RA^{(3)}})^{\mathrm{Trees}}$. We then have the following lemma:

LEMMA 4.3. $A_3 \vdash \varphi^{(RA^{(3)})^{\mathrm{Trees}}}$ *for all axioms* φ *for* A_3.

(1) "standard" means here that $\in, +, \cdot, \ldots$ are interpreted by themselves.

Formulas $(RA^{(3),i})^{\mathrm{Trees}}$ with $+^{\mathrm{Trees}}$, $\cdot^{\mathrm{Trees}}$, etc., give a nonstandard interpretation of A_3 in A_3. Now we shall find out how to pass from this interpretation to a standard one.

Let M be a β-model of A_3. From Zbierski's theorem it follows that trees from M form a model of set theory uniquely isomorphic to a standard transitive one. The collapsing isomorphism is called "realization". It follows that a fragment of this realization is definable in A_3. This definable fragment is just the restriction of the isomorphism in question to trees coding arithmetical objects. Informally speaking, we define in A_3 functions g and f such that:

1. g = (tree coding arithmetical object $\mapsto$ its realization).
2. f = (object $a \mapsto$ tree coding object a).
3. g is a surjection and f is defined everywhere.
4. $g \circ f$ is an identity.

We shall limit ourselves to giving only the definition of the function g.

Let t_n be a natural number which codes (via the pairing function J) the canonical tree for n. Such coding can be done uniformly. Moreover, we may assume that the relations "$m \in \mathrm{Fld}(t_n)$" and "$m <_n r$" as well as the function $(n \mapsto t_n)$ are recursive ($m <_n r$ means that m is less than r in the sense of the tree coded by t_n).

We define g by means of formulas g_0, g_1, g_2.

$$g_0(T, n) \underset{\mathrm{df}}{\leftrightarrow} T \in \mathrm{Trees} \ \& \ (EX)(\text{"}X \text{ is bijection of } \mathrm{Fld}(T) \text{ and } \mathrm{Fld}(t_n)\text{"} \ \&$$
$$\& \ (m)(r)(c)(d)(\langle c, m\rangle^{\bullet} \in X \ \& \ \langle d, r\rangle^{\bullet} \in X \to (m <_n r \leftrightarrow \langle c, d\rangle^{\bullet} \in T)))$$

$g_0(T, n)$ says that T is isomorphic to the canonical tree for N.

$$g_{k+1}(T, A) \underset{\mathrm{df}}{\leftrightarrow} T \in \mathrm{Trees} \ \& \ (T')(T' \widetilde{\mathrm{Eps}}\, T \to (Ea) g_k(T', a)) \ \&$$
$$\& \ (a)(a \in A \leftrightarrow (ET')(T' \widetilde{\mathrm{Eps}}\, T \ \& \ g_k(T', a)))$$

$g_{k+1}(T, A)$ says that A is a set of objects which are coded by the maximal proper subtrees of T. (Let us recall that

$$X \,\mathrm{Eps}\, Y \underset{\mathrm{df}}{\leftrightarrow} (Ea)_{\mathrm{Amax}(Y)}(X \,\mathrm{Eq}\, (Y)_a)$$

and

$$X \,\widetilde{\mathrm{Eps}}\, Y \underset{\mathrm{df}}{\leftrightarrow} (Ea)_{\mathrm{Amax}(Y)}(X = (Y)_a),$$

see also [1], p. 204.)

LEMMA 4.4 (homomorphism lemma).

1. $A_3 \vdash (T_1)_{RA^{\mathrm{Trees}}}(T_2)_{RA^{\mathrm{Trees}}}(T_1 \,\mathrm{Eq}\, T_2 \leftrightarrow g(T_1) = g(T_2))$,
2. $A_3 \vdash (T)_{RA^{\mathrm{Trees}}}(\bigvee\!\!\bigvee_{i=0}^{2} (Ex^{(i)}) g_i(T, x^{(i)}))$,
3. $A_3 \vdash (T_1)_{RA^{\mathrm{Trees}}}(T_2)_{RA^{\mathrm{Trees}}}(T_1 \,\mathrm{Eps}\, T_2 \leftrightarrow g(T_1) \in g(T_2))$,
4. $A_3 \vdash g_0''(RA^{\mathrm{Trees}}) = \omega$,
5. $A_3 \vdash (T_1)(T_2)(T_3)(\bigwedge\!\!\bigwedge_{i=1}^{3} T_i \mathrm{Eps}\, \omega^{\mathrm{Trees}}$
$\to ((T_1 +^{\mathrm{Trees}} T_2) \mathrm{Eq}\, T_3 \leftrightarrow g(T_1) + g(T_2) = g(T_3)))$,
6. $A_3 \vdash (T_1)(T_2)(T_3)(\bigwedge\!\!\bigwedge_{i=1}^{3} T_i \mathrm{Eps}\, \omega^{\mathrm{Trees}}$
$\to ((T_1 \cdot^{\mathrm{Trees}} T_2) \mathrm{Eq}\, T_3 \leftrightarrow g(T_1) \cdot g(T_2) = g(T_3)))$,
7. $A_3 \vdash (T_1)(T_2)(\bigwedge\!\!\bigwedge_{i=1}^{2} T_i \mathrm{Eps}\, \omega^{\mathrm{Trees}} \to (T_1 \,\mathrm{Eps}\, T_2 \leftrightarrow g(T_1) < g(T_2)))$,
8. $g_0(0^{\mathrm{Trees}}) = 0$, $g_0(1^{\mathrm{Trees}}) = 1$.

The homomorphism lemma shows that g is a definable homomorphism of a nonstandard model $\langle RA^{\mathrm{Trees}}, \mathrm{Eps}, \mathrm{Eq}, +^{\mathrm{Trees}}, \ldots \rangle$ onto $\langle g''RA^{\mathrm{Trees}}, \in, =, +, \ldots \rangle$, i.e., onto a standard one: taking the superposition of the interpretation of Zbierski with the function g, we obtain the required standard interpretation of A_3 in A_3. We have proved that $A_3 \vdash \varphi^{g''RA^{\mathrm{Trees}}}$ for all axioms φ of A_3.

All that remains to be proved is Lemma 4.5.

LEMMA 4.5. $A_3 \vdash \bigwedge\!\!\bigwedge_{i=1}^{2} (g_i''(RA^{(3),i})^{\mathrm{Trees}} = \mathrm{ra}^i)$.

The proof is by induction. First we introduce the relation $A(T)$ (for any given T), which is defined as follows:

$$\langle a, b\rangle^{\bullet} \in A(T) \underset{\mathrm{df}}{\leftrightarrow} (On(T))^{\mathrm{Trees}} \ \& \ (ET_1)(ET_2)(T_1 \widetilde{\mathrm{Eps}}\, T \ \& \ T_2 \widetilde{\mathrm{Eps}}\, T \ \&$$
$$\& \ a = \max(T_1) \ \& \ b = \max(T_2) \ \& \ T_1 \,\mathrm{Eps}\, T_2).$$

If T is a tree which codes an ordinal number, then $A(T)$ is a well-ordering of its almost maximal elements. We have A: $On^{\mathrm{Trees}} \to \mathrm{Bord}$. Conversely: for every X such that $\mathrm{Bord}(X)$ there is a tree T such that $A(T)$ is isomorphic to X. This may be shown by induction (in A_3); namely, we show by induction that

$$(a)_{\mathrm{Fld}(X)}(ET)_{\mathrm{Trees}}(On^{\mathrm{Trees}}(T) \ \& \ A(T) \cong X \upharpoonright a).$$

Employing $A(\cdot)$ we can reformulate the lemma to get a form more convenient for induction (in A_3):

$$(X)_{\mathrm{Bord}}(T)_{\mathrm{Trees}}(On^{\mathrm{Trees}}(T) \ \& \ A(T) \cong X$$
$$\to \bigwedge\!\!\bigwedge_{i=1}^{2} (a)\,[(ES)(S \in RA_T^{\mathrm{Trees}} \ \& \ g_i(S, a)) \leftrightarrow \mathrm{ra}_i(X, a)]).$$

We leave out the tedious details of this induction.

The remaining part of 1 is contained in Theorem 1 in Chapter II. To complete the proof of Theorem 3 it remains to prove 5.

We use essentially the same methods as in the proof of 1. Let $\mathrm{Ar}^{(3)}$ be a formula defining the "full arithmetic" $\mathrm{Ar}^{(3)}(V)$ and let $\dot{\mathrm{B}}\mathrm{ord}(T)$ be the arithmetical definition of the class of well-orderings, i.e., $\dot{\mathrm{B}}\mathrm{ord}(T) \leftrightarrow \mathrm{Bord}^{\mathrm{Ar}^{(3)}}(T)$. Let us recall that the β-property of RA was established in $\mathrm{ZFC}^- +$"$P(\omega)$ exists", i.e.,

$$\mathrm{ZFC}^- + \text{"}P(\omega) \text{ exists"} \vdash (T)\big(T \in RA \to (\dot{\mathrm{B}}\mathrm{ord}^{RA}(T) \leftrightarrow \dot{\mathrm{B}}\mathrm{ord}(T))\big).$$

Now we can employ Zbierski's theorem as in 1.

$$A_3 \vdash (T)_{RA^{\mathrm{Trees}}}\big((\dot{\mathrm{B}}\mathrm{ord}^{RA})^{\mathrm{Trees,Eps,Eq}}(T) \leftrightarrow \dot{\mathrm{B}}\mathrm{ord}^{\mathrm{Trees,Eps,Eq}}(T)\big),$$

$$A_3 \vdash (T)_{RA^{\mathrm{Trees}}}\big(\dot{\mathrm{B}}\mathrm{ord}^{g''(RA^{\mathrm{Trees}}),\in,=}(g(T)) \leftrightarrow \dot{\mathrm{B}}\mathrm{ord}^{g''(\mathrm{Trees}),\in,=}(g(T))\big),$$

$$A_3 \vdash (S)\big(S \in g''(RA^{\mathrm{Trees}}) \to (\dot{\mathrm{B}}\mathrm{ord}^{g''(RA^{\mathrm{Trees}}),\in,=}(S) \leftrightarrow \dot{\mathrm{B}}\mathrm{ord}(S))\big),$$

$$A_3 \vdash (S)_{\mathrm{ra}}\big(\dot{\mathrm{B}}\mathrm{ord}^{\mathrm{ra}}(S) \leftrightarrow \dot{\mathrm{B}}\mathrm{ord}(S)\big).$$

This means that $A_3 \vdash$ "ra has β-property".

Chapter V

Other nice properties of ra. **Reflection.** In Chapter IV it was proved that ra is an inner model of A_3. Zbierski's theorem shows that it is reasonable to regard A_3 as a set theory. Following this line we can investigate some characteristic properties of the set-theoretical universe, such as for example the property of reflection with respect to a given hierarchy of objects. We shall establish that reflection with respect to ramified analytical hierarchy holds.

DEFINITION 5.1.

1. "ra becomes stabilized on X" $\underset{\mathrm{df}}{\leftrightarrow} \dot{\mathrm{B}}\mathrm{ord}(X) \;\&\; \big(\mathrm{ra}(X,\cdot) = \mathrm{ra}(X+1,\cdot)\big) \;\&$
$\& \; (Y)\big(Y \gtrsim X \leftrightarrow (\mathrm{ra}(Y,\cdot) \neq \mathrm{ra}(Y+1,\cdot))\big).$

2. "ra does not stabilize" $\underset{\mathrm{df}}{\leftrightarrow} (X)_{\dot{\mathrm{B}}\mathrm{ord}}(\neg$"ra becomes stabilized on X").

($Y \gtrsim X$ means that Y is isomorphic to an initial segment of X).

Now we are ready to formulate and to prove the reflection principle.

THEOREM 4 (Principle of reflection).

1. $A_3 \vdash \big[$"ra$(\cdot)$ becomes stabilized on X" $\to (Y)_{\dot{\mathrm{B}}\mathrm{ord}}(\varphi)\big[Y \gtrsim X$
$\to (EZ)_{\dot{\mathrm{B}}\mathrm{ord}}\big(Y \gtrsim Z \gtrsim X \;\&\; (p)\big(\mathrm{ra}(Z, p)$
$\to (\mathrm{ra}(X,\cdot) \models \varphi[p] \leftrightarrow \mathrm{ra}(Z,\cdot) \models \varphi[p])\big)\big)\big]\big]$,

2. $A_3 \vdash \big[$"ra$(\cdot)$ does not stabilize" $\to (Y)_{\dot{\mathrm{B}}\mathrm{ord}}(EZ)_{\dot{\mathrm{B}}\mathrm{ord}}\big(Y \gtrsim Z \;\&$
$\& \; (p)\big(\mathrm{ra}(Z, p) \to (\varphi^{\mathrm{ra}(\cdot)}(p) \leftrightarrow \varphi(p))\big)\big)\big]$ (*for all formulas* φ).

Proof. Let us now introduce two abbreviations (in L(ZF)):

a. "RA becomes stabilized on α_0" $\underset{\mathrm{df}}{\leftrightarrow} (RA_{\alpha_0+1} = RA_{\alpha_0} \;\&$
$\& \; (\beta)_{\alpha_0}(RA_\beta \neq RA_{\beta+1}))$.

b. "RA does not stabilize" $\underset{\mathrm{df}}{\leftrightarrow} (\alpha)(\neg$"$RA$ becomes stabilized on α").

It is well known (W. Marek, M. Srebrny [8]) that L_{η_3} is pointwise

definable and satisfies the principle of reflection (η_3 is the first 3-gap). From this it follows that

$$\mathrm{ZFC}^- + \text{"}P(\omega) \text{ exists"} \vdash \text{"}RA \text{ satisfies the principle of reflection"},$$

i.e.,

$$\mathrm{ZFC}^- + \text{"}P(\omega) \text{ exists"} \vdash (\alpha)\big(\text{"}RA \text{ becomes stabilized on } \alpha\text{"}$$
$$\to (\beta)_\alpha(\varphi)_{\mathrm{Form}}(E\gamma)_\alpha\big(\gamma > \beta \;\&\; (p)_{RA_\gamma}(RA_{\alpha_0} \models \varphi[p] \leftrightarrow RA_\gamma \models \varphi[p])\big)\big)$$

and

$$\mathrm{ZFC}^- + \text{"}P(\omega) \text{ exists"} \vdash \big(\text{"}RA \text{ does not stabilize"}$$
$$\to (\beta)(E\gamma)\big(\beta < \gamma \;\&\; (\beta)_{RA_\gamma}(\varphi^{RA_\gamma}(p) \leftrightarrow \varphi^{RA}(p))\big)\big)$$

for all formulas φ.

We can apply, as usual, Zbierski's theorem to the following formula:

(*) $\quad A_3 \vdash (T)_{\mathrm{Trees}}\big((\text{"}RA \text{ becomes stabilized on } T\text{"})^{\mathrm{Trees}}$
$\to (T)(T'\,\mathrm{Eps}\,T \to (ET'')\big[T'\,\mathrm{Eps}\,T'' \;\&\; T''\,\mathrm{Eps}\,T \;\&$
$\& \; (\varphi)_{\mathrm{Form}^{\mathrm{Trees}}}(p)_{RA^{\mathrm{Trees}}}(RA_T \models \varphi[p] \leftrightarrow RA_{T''} \models \varphi[p])^{\mathrm{Trees}}\big]\big)$.

The formula ("RA becomes stabilized on T")$^{\mathrm{Trees}}$, i.e., the formula

$$\text{"}RA_T^{\mathrm{Trees}} = RA_{T+1}^{\mathrm{Trees}} \;\&\; (T')_{On^{\mathrm{Trees}}}(T'\,\mathrm{Eps}\,T \to RA_{T'+1}^{\mathrm{Trees}} \neq RA_{T'}^{\mathrm{Trees}})\text{"}$$

is equivalent in A_3 to the formula

$$g''RA_{g(T)}^{\mathrm{Trees}} = g''RA_{g(T+1)} \;\&\; (T')(T'\,\mathrm{Eps}\,T \to g''RA_{g(T'+1)}^{\mathrm{Trees}} = g''RA_{g(T')}^{\mathrm{Trees}})\text{"}$$

and by virtue of Lemma 4.5 is A_3-equivalent to "ra$(\cdot)$ becomes stabilized", namely to the formula

$$\mathrm{ra}(A(T+1),\cdot) = \mathrm{ra}(A(T),\cdot) \;\&\; (X)_{\mathrm{Bord}}\big[X \gtrsim A(T) \to (\mathrm{ra}(X,\cdot) \neq \mathrm{ra}(X+1,\cdot))\big]$$

(here $g''A$ is the image of A by g). So in A_3 the formula "ra$(\cdot)$ becomes stabilized on $A(T)$" is equivalent to the formula ("RA becomes stabilized on T")$^{\mathrm{Trees}}$.

We make use of a lemma stating that

$$A_3 \vdash \big[\mathrm{SAT}^{\mathrm{Trees}}(N_1, N_2, f(\varphi), f(\vec{m}), f(Y_1^{(\vec{p}_1)}), f(Y_2^{(\vec{p}_2)}))$$
$$\leftrightarrow \mathrm{Sat}(Y_1, Y_2, \varphi, m, p_1, p_2)\big],$$

where Sat is arithmetical but SAT is a set-theoretical formula of satisfaction, N_i is such that

$$A_3 \vdash \big[X\eta N_i \leftrightarrow (ES)(S\,\mathrm{Eps}\,N_i \;\&\; g_i(S, X))\big] \quad \text{for} \quad i = 1, 2,$$

g_i are functions defined in Chapter IV and $Y_i = \{X\colon X\eta N_i\}$. (Incidentally this fact is useful in the proof of Lemma 4.5 in Chapter IV).

Using this and (*) together with Lemma 4.5, we obtain the principle of reflection.

Another nice property of the inner model ra is established in the following theorem. It corresponds to the absoluteness (with respect to the class of constructible sets) of the notion of constructibility.

THEOREM 5. $A_3 \vdash \mathrm{ra}^{\mathrm{ra}} = \mathrm{ra}$.

Before beginning the proof we introduce some auxiliary notions.

DEFINITION 5.2. We say that $(\psi^1, \ldots, \psi^{n-1})$ is a *transitive system of formulas of* $L(A_n)$ if and only if the only free variable in ψ^i is of order i and

1. $A_n \vdash [\psi^i(a^{(i)}) \mathbin{\&} b^{(i-1)} \in a^{(i)} \to \psi^{i-1}(b^{(i-1)})]$ for $i = 2, 3, \ldots, n-1$,
2. $A_n \vdash [\psi^i(a, b) \to (\psi^i(a) \mathbin{\&} \psi^i(b))]$.

EXAMPLE. $(\mathrm{ra}_1(\ ), \mathrm{ra}_2(\ ), \ldots, \mathrm{ra}_{n-1}(\ ))$ is a transitive system of formulas of $L(A_n)$.

DEFINITION 5.3. TC, T, T_0, T_n are A_n-definable operations such that

$$T \underset{\mathrm{df}}{=} (x \mapsto \{a\colon a \in x\} \cup \{\langle a, b\rangle^{\cdot}\colon (Ec)_x(\langle a, b\rangle^{\cdot} = c)\})$$

and

$$T_0(X) \underset{\mathrm{df}}{=} T(X),$$
$$T_{n+1}(X) \underset{\mathrm{df}}{=} T_n(X) \cup T'' T_n(X),$$
$$\mathrm{TC}(X) \underset{\mathrm{df}}{=} \bigcup_{n \in \omega} T_n(X).$$

($\mathrm{TC}(X)$ is a kind of transitive closure of X.)

The following easy lemma is a counterpart to the lemma on absoluteness of Δ_1^{ZF}-formulas with respect to standard transitive models of ZF.

LEMMA 5.1. 1. *Formulas of the form* $(EX^{(n-1)})(\varphi(X^{(n-1)}, \ldots,))^{\mathrm{TC}(X^{(n-1)})}$ *are upward absolute with respect to transitive systems of formulas.*

2. *Formulas of the form* $(X^{(n-1)})(\varphi(X^{(n-1)}, \ldots))^{\mathrm{TC}(X^{(n-1)})}$ *are downward absolute with respect to transitive systems of formulas.*

By a careful analysis of Π_1^{n-1} and Σ_1^{n-1} definitions of ramified analysis we obtain:

LEMMA 5.2. *There exist formulas P_i and Q_i such that*:

$$A_n \vdash [\mathrm{ra}_i(\cdot, \cdot\cdot) \leftrightarrow (EX^{(n-1)}) P_i(X^{(n-1)}, \cdot, \cdot\cdot)^{\mathrm{TC}(X^{(n-1)})}]$$

and

$$A_n \vdash [\mathrm{ra}_i(\cdot, \cdot\cdot) \leftrightarrow (X^{(n-1)}) Q_i(X^{(n-1)}, \cdot, \cdot\cdot)^{\mathrm{TC}(X^{(n-1)})}].$$

Theorem 5 follows easily from the above lemmas and from the fact that $(\mathrm{ra}_1(\cdot), \mathrm{ra}_2(\cdot))$ is a transitive system of formulas of $L(A_3)$.

Notice. For $n = 2$ the theorem immediately follows from the absoluteness of $(\Delta_1^1)^{A_2}$-formulas with respect to $\mathrm{ra}(\cdot)$.

I dedicate this paper to my Friend R. Z. Kufner.

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