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Accepté par la Rédaction le 12. 10. 1981

Maps of cotriples and a change of rings theorem

by

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Abstract. A well-known theorem on "change of rings" asserts that a ring homomorphism $f\colon R \to S$ induces an inequality $\dim_R(M) \leqslant \dim_S(M) + \dim_R(S)$ for each S-module M. We generalize this result to a categorical setting and obtain an analogous "change of cotriples" inequality induced by a map of cotriples.

Suppose that S is a ring and M is a left S-module; we denote by $\dim_S(M)$ the projective dimension of M as an S-module. A well-known theorem on "change of rings" (see [5, p. 172]) asserts that if $f\colon R \to S$ is a ring homomorphism and M is an S-module, then M and S are R-modules in a natural way, and one has the inequality

$$(*) \qquad \dim_R(M) \leqslant \dim_S(M) + \dim_R(S).$$

In this paper we obtain a similar inequality in a more general categorical setting. If $\boldsymbol{A}$ and $\boldsymbol{X}$ are additive categories with finite limits and $\boldsymbol{G} = (G, \varepsilon, \delta)$ and $\boldsymbol{H} = (H, \varepsilon', \delta')$ are suitable cotriples on $\boldsymbol{A}$ and $\boldsymbol{X}$, respectively, then one can define $\dim_{\boldsymbol{G}}(A)$ and $\dim_{\boldsymbol{H}}(X)$ via resolutions for all objects A of $\boldsymbol{A}$ and X of $\boldsymbol{X}$ (see [4]). We show that in this case a "nice" adjunction (U, F) from $\boldsymbol{A}$ to $\boldsymbol{X}$ and morphisms of cotriples $\tau\colon FH \Rightarrow GF$ and $\mu\colon HU \Rightarrow UG$ induce natural definitions of $\dim_{\boldsymbol{H}}(\boldsymbol{G})$ and $\dim_{\boldsymbol{G}}(\boldsymbol{H})$ and two "change of cotriples" inequalities analogous to $(*)$.

To see the relationship between the "change of rings" theorem and the categorical situation described above, we begin by recalling the following facts. It is well known that if $E\colon \boldsymbol{B} \to \boldsymbol{A}$ is a functor having a left adjoint $F\colon \boldsymbol{A} \to \boldsymbol{B}$, then the adjoint pair (E, F) induces a triple $\boldsymbol{T}$ on $\boldsymbol{A}$ and a cotriple $\boldsymbol{G}$ on $\boldsymbol{B}$ (see [7, p. 134]). It is also well known (see [1, p. 290]) that given a triple $\boldsymbol{T}$ or a cotriple $\boldsymbol{G}$ on a category $\boldsymbol{A}$ one can construct resolutions in $\boldsymbol{A}$ which generalize resolutions of modules over a ring in the following way. If S is a ring, ${}_S\boldsymbol{Mod}$ and $\boldsymbol{Sets}$ the categories of left S-modules and sets, respectively, $E_S\colon {}_S\boldsymbol{Mod} \to \boldsymbol{Sets}$ the "underlying set" functor, and $F_S\colon \boldsymbol{Sets} \to {}_S\boldsymbol{Mod}$ the "free S-module" functor, then F_S is left adjoint to E_S. If $\boldsymbol{G}_S = (G_S, \varepsilon_S, \delta_S)$ denotes the "free" cotriple on ${}_S\boldsymbol{Mod}$ induced by the adjoint pair (E_S, F_S), then $\boldsymbol{G}_S$-resolutions in ${}_S\boldsymbol{Mod}$ are just projective resolutions. Now suppose that R and S are rings, so that we have "free" cotriples $\boldsymbol{G}_R = (G_R, \varepsilon_R, \delta_R)$ on

${}_R\mathbf{Mod}$ and $\mathbf{G}_S = (G_S, \varepsilon_S, \delta_S)$ on ${}_S\mathbf{Mod}$, and suppose that $f\colon R \to S$ is a ring homomorphism. Then f induces a "scalar restriction" functor $U = \operatorname{Hom}_S(S, -)\colon {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$ (which has a left adjoint $F = S \otimes_R -$ and a right adjoint $K = \operatorname{Hom}_R(S, -)$), and for any S-module M we have $\dim_{\mathbf{G}_S}(M) = \dim_S(M)$ and $\dim_{\mathbf{G}_R}(UM) = \dim_R(M)$. (Remark: Functors which have both left and right adjoints have received special attention of late. For example, see [2], proof of Theorem B, which uses Mikkelsen's theorem that a logical morphism between toposes has a left adjoint iff it has a right adjoint; or consider Lawvere's current work on "extra right adjoints" [6].) Furthermore, there are two canonical morphisms of cotriples $\mu\colon \mathbf{G}_R \to \mathbf{G}_S$ and $\tau\colon \mathbf{G}_R \to \mathbf{G}_S$: a natural transformation $\mu\colon G_R U \to U G_S$ (for which $U\varepsilon_S \cdot \mu = \varepsilon_R U$ and $U\delta_S \cdot \mu = \mu G_S \cdot G_R \mu \cdot \delta_R U$), where for each S-module M the map $\mu_M\colon \bigoplus_{x \in M} Rx \to \bigoplus_{x \in M} Sx$ is naturally induced by f, and a natural transformation $\tau\colon FG_R \to G_S F$ (for which $\varepsilon_S F \cdot \tau = F\varepsilon_R$ and $\delta_S F \cdot \tau = G_S \tau \cdot \tau G_R \cdot F\delta_R$), where for each R-module N the map $\tau_N\colon S \otimes_R (\bigoplus_{y \in N} R_y) \to \bigoplus_{x \in S \otimes_R N} S_x$ is naturally induced by f. Thus we are led to ask in the module case: Do U, F, K, τ, and μ provide us with definitions of $\dim_{\mathbf{G}_R}(\mathbf{G}_S)$ and $\dim_{\mathbf{G}_S}(\mathbf{G}_R)$ such that $\dim_{\mathbf{G}_R}(\mathbf{G}_S) = \dim_R(S)$ and $\dim_{\mathbf{G}_S}(\mathbf{G}_R) = \dim_S(R)$, and hence with "change of cotriples" inequalities

(1) $$\dim_{\mathbf{G}_S}(FN) \leqslant \dim_{\mathbf{G}_R}(N) + \dim_{\mathbf{G}_S}(\mathbf{G}_R),$$

and

(2) $$\dim_{\mathbf{G}_R}(UM) \leqslant \dim_{\mathbf{G}_S}(M) + \dim_{\mathbf{G}_R}(\mathbf{G}_S)?$$

We show in this paper that U, F, and τ give us inequality (1) (we leave the (dual) derivation of (2) from K, U, and μ to the reader), and we determine conditions under which a more general result may be obtained. We gratefully acknowledge the helpful suggestions of the referee with regard to the appropriate cotriple maps and generalizations.

Before giving a specific summary of the contents of this paper, we introduce some notation and conventions. We shall assume throughout this paper that $\mathbf{A}$ and $\mathbf{X}$ are additive categories with finite limits, $\mathbf{G} = (G, \varepsilon, \delta)$ and $\mathbf{H} = (H, \varepsilon', \delta')$ are cotriples on $\mathbf{A}$ and $\mathbf{X}$, respectively, $F\colon \mathbf{X} \to \mathbf{A}$ is a functor, and $\tau\colon \mathbf{H} \Rightarrow \mathbf{G}$ is a morphism of cotriples (i.e., a natural transformation $\tau\colon FH \Rightarrow GF$ such that $\varepsilon F \cdot \tau = F\varepsilon'$ and $G\tau \cdot \tau H \cdot F\delta' = \delta F \cdot \tau$). For objects A and B of $\mathbf{A}$, we let $\mathbf{A}(A, B)$ denote the abelian group of morphisms from A to B. If $f\colon A \to B$ is a morphism in $\mathbf{A}$ and C is an object of $\mathbf{A}$, we denote by f_* the morphism $\mathbf{A}(C, f)\colon \mathbf{A}(C, A) \to \mathbf{A}(C, B)$.

In § 1 we dualize some of the results in [4]. We recall the notions of $\mathbf{G}$-projective, $\mathbf{G}$-exactness, and $\mathbf{G}$-resolution. We note that in general one does not have a comparison theorem for $\mathbf{G}$-resolutions, but that if G reflects isomorphisms (that is, $Gf\colon GA \to GB$ is an isomorphism iff $f\colon A \to B$ is an isomorphism), then the usual results about dimensions of modules in a short exact sequence carry over to short $\mathbf{G}$-exact sequences and $\mathbf{G}$-projective dimensions. If G does not reflect isomorphisms, we construct a category $\Sigma^{-1}\mathbf{A}$ and a functor $r\colon \mathbf{A} \to \Sigma^{-1}\mathbf{A}$ (as in [3]), obtain a cotriple $\bar{\mathbf{G}}$ on $\Sigma^{-1}\mathbf{A}$ whose functor reflects isomorphisms, and define $\dim_{\mathbf{G}}(A) = \dim_{\bar{\mathbf{G}}}(rA)$ for every object A of $\mathbf{A}$.

In § 2 we consider the special case in which $\mathbf{A} = \mathbf{X}$ and $F = 1_{\mathbf{A}}$. We define $\dim_{\mathbf{G}}(\mathbf{H}) = \inf\{n \geqslant 0/\tau^{n+1}\colon H^{n+1} \Rightarrow G^{n+1}$ is an isomorphism$\}$, and we show that $\dim_{\mathbf{G}}(A) \leqslant \dim_{\mathbf{H}}(A) + \dim_{\mathbf{G}}(\mathbf{H})$ for every object A of $\mathbf{A}$.

In § 3 we consider the general case. We suppose that $\mathbf{A}$ and $\mathbf{X}$ are exact categories (in the sense of [9]) and that F is a faithful, zero-preserving functor having a right adjoint U, and we define $\dim_{\mathbf{G}}(\mathbf{H}) = \inf\{n \geqslant 0/(\tau U)^n \tau\colon (FHU)^n FH \Rightarrow (GFU)^n GF$ is an isomorphism$\}$. We show that if G and H reflect isomorphisms and if $\mathbf{H}$ satisfies an additional identity relating it to the adjoint pair (U, F), then $\dim_{\mathbf{G}}(FX) \leqslant \dim_{\mathbf{H}}(X) + \dim_{\mathbf{G}}(\mathbf{H})$ for every X in $\mathbf{X}$. If G and H do not reflect isomorphisms, we consider the categories $\Sigma^{-1}\mathbf{A}$ and $\Sigma^{-1}\mathbf{X}$ and cotriples $\bar{\mathbf{G}}$ and $\bar{\mathbf{H}}$, as described in § 1. We obtain a functor $\bar{F}\colon \Sigma^{-1}\mathbf{X} \to \Sigma^{-1}\mathbf{A}$ having a right adjoint $\bar{U}$, and a map of cotriples $\bar{\tau}\colon \bar{\mathbf{H}} \Rightarrow \bar{\mathbf{G}}$. We define $\dim_{\mathbf{G}}(\mathbf{H}) = \dim_{\bar{\mathbf{G}}}(\bar{\mathbf{H}})$, and we obtain as before the inequality $\dim_{\bar{\mathbf{G}}}(\bar{F}X) \leqslant \dim_{\bar{\mathbf{H}}}(X) + \dim_{\bar{\mathbf{G}}}(\bar{\mathbf{H}})$ for every object X of $\Sigma^{-1}\mathbf{X}$.

§ 1. Dimension with respect to a cotriple. In this section we dualize some of the results in [4]. Let $\mathbf{A}$ be an additive category with finite limits and let $\mathbf{G} = (G, \varepsilon, \delta)$ be a cotriple on $\mathbf{A}$. Following [1], we say that an object A of $\mathbf{A}$ is *$\mathbf{G}$-projective* if there exists a morphism $k_A\colon A \to GA$ such that $\varepsilon_A \cdot k_A = 1_A$. A sequence $A' \xrightarrow{p} A \xrightarrow{q} A''$ in $\mathbf{A}$ is *$\mathbf{G}$-exact* if $q \cdot p = 0$ and if $\mathbf{A}(P, A') \xrightarrow{p_*} \mathbf{A}(P, A) \xrightarrow{q_*} \mathbf{A}(P, A'')$ is exact in $\mathbf{Ab}$ for every $\mathbf{G}$-projective P. A *$\mathbf{G}$-resolution* of an object A of $\mathbf{A}$ is a $\mathbf{G}$-exact sequence $\ldots \to P_1 \to P_0 \to A \to 0$ in which $P_0, P_1, \ldots$ are $\mathbf{G}$-projective. A morphism $f\colon A \to B$ in $\mathbf{A}$ is a *$\mathbf{G}$-isomorphism* if $f_*\colon \mathbf{A}(P, A) \to \mathbf{A}(P, B)$ is an isomorphism in $\mathbf{Ab}$ for every $\mathbf{G}$-projective P. We note that if $0 \to A \to B \xrightarrow{f} C$ is a $\mathbf{G}$-exact sequence in $\mathbf{A}$, then the unique map $s\colon A \to \ker(f)$ coming from the universal property of the kernel is a $\mathbf{G}$-isomorphism (equivalently, $G(s)$ is an isomorphism). Hence if G reflects isomorphisms, any $\mathbf{G}$-exact sequence $0 \to A \to B \xrightarrow{f} C \to \ldots$ has $A \cong \ker(f)$, and any short $\mathbf{G}$-exact sequence $0 \to A \to B \to P \to 0$ with P a $\mathbf{G}$-projective is in fact split exact. Thus we obtain:

LEMMA 1.1 (*$\mathbf{G}$-Schanuel Lemma*). *Suppose that G reflects isomorphims. If P and P' are $\mathbf{G}$-projective and if*

$$0 \to A \to P \to B \to 0 \quad \textit{and} \quad 0 \to A' \to P' \to B \to 0$$

are short $\mathbf{G}$-exact sequences in $\mathbf{A}$, then $P' \oplus A \cong P \oplus A'$. ∎

If we then define the *G-projective dimension* of an object A of $\mathbf{A}$, denoted by $\dim_G(A)$, to be $\leqslant k$ if A has a G-resolution $0 \to P_k \to P_{k-1} \to \ldots \to P_1 \to P_0 \to A \to 0$, we get the usual comparison theorem for resolutions when G reflects isomorphisms:

PROPOSITION 1.2. *Suppose that G reflects isomorphisms. Then for every object A of $\mathbf{A}$, the following statements are equivalent*:

(a) $\dim_G(A) \leqslant k$.

(b) *If* $\ldots \to Y_k \to Y_{k-1} \xrightarrow{g} Y_{k-2} \to \ldots \to Y_1 \to Y_0 \to A \to 0$ *is any G-resolution of A, then* $\ker(g)$ *is G-projective.* ∎

We now define *chain complex, chain homotopy, chain equivalence*, and *mapping cylinder* in the standard fashion (see [8, Ch. II]) and obtain the following results:

PROPOSITION 1.3. *Suppose that G reflects isomorphisms and that $0 \to M' \to M \to M'' \to 0$ is a short G-exact sequence in $\mathbf{A}$. Then*

(a) *If* $\dim_G(M'') > \dim_G(M)$, *then* $\dim_G(M'') = \dim_G(M')+1$.

(b) $\dim_G(M) \leqslant \sup\{\dim_G(M'), \dim_G(M'')\}$.

(c) *For any positive integer n, if $\dim_G(M) < n$ and $\dim_G(M'') < n$, then* $\dim_G(M') < n$. ∎

We refer the reader to [1] for the construction and properties of the cohomology groups $H^k_G(A, B) = H^k_G(A, \mathbf{A}(-, B))$ for every pair of objects A and B of $\mathbf{A}$. We define the *G-cohomological dimension* of an object A of $\mathbf{A}$, denoted $\mathrm{cohd}_G(A)$, to be $\leqslant n$ if $H^k_G(A, B) = 0$ for all $k > n$ and for every object B of $\mathbf{A}$. Then we have the following relationship between $\dim_G$ and cohd_G:

THEOREM 1.4. *If G reflects isomorphisms, then $\mathrm{cohd}_G(A) \leqslant \dim_G(A)$ for every object A of $\mathbf{A}$; and if ε_X is the coequalizer of $G\varepsilon_X$ and ε_{GX} for every object X of $\mathbf{A}$, then $\mathrm{cohd}_G(A) = \dim_G(A)$ for every object A of $\mathbf{A}$.* ∎

If G does not reflect isomorphisms, we consider the class Σ of G-isomorphisms and construct the "category of fractions" $\Sigma^{-1}\mathbf{A}$ as in [3].

PROPOSITION 1.5. *$\mathbf{G}$, together with the canonical functor $r\colon \mathbf{A} \to \Sigma^{-1}\mathbf{A}$, induces a cotriple $\bar{\mathbf{G}}$ on $\Sigma^{-1}\mathbf{A}$ whose functor reflects isomorphisms.* ∎

We then define $\dim_G(A) = \dim_{\bar{G}}(rA)$ for every object A of $\mathbf{A}$.

THEOREM 1.6. *Suppose that for every object A of $\mathbf{A}$ the morphisms $G\varepsilon_A$ and ε_{GA} have a coequalizer which is preserved by G. Then $\Sigma^{-1}\mathbf{A}$ is equivalent to a full reflective subcategory $\mathbf{D}$ of $\mathbf{A}$, $\bar{\mathbf{G}}$ is $\mathbf{G}$ restricted to $\mathbf{D}$, and $\mathrm{cohd}_G(A) = \mathrm{cohd}_{\bar{G}}(rA) = \dim_{\bar{G}}(rA) = \dim_G(A)$ for every object A of $\mathbf{A}$.*

§ 2. Maps of cotriples on a category.

Suppose that $\mathbf{A}$ is an additive category with finite limits, $\mathbf{G} = (G, \varepsilon, \delta)$ and $\mathbf{H} = (H, \varepsilon', \delta')$ are cotriples on $\mathbf{A}$, and $\tau\colon H \Rightarrow G$ is a natural transformation for which $\varepsilon \cdot \tau = \varepsilon'$ and $\delta \cdot \tau = G\tau \cdot \tau H \cdot \delta'$. (That is, consider the special case in which $\mathbf{A} = \mathbf{X}$ and $F\colon \mathbf{X} \to \mathbf{A}$ is the identity functor on $\mathbf{A}$; thus τ is a morphism of cotriples.) Suppose further that G and H reflect isomorphisms. Then one easily verifies the following two lemmas:

LEMMA 2.1. *If $A \in |\mathbf{A}|$ is $\mathbf{H}$-projective, then A is $\mathbf{G}$-projective. In particular, HA is $\mathbf{G}$-projective for every object A of $\mathbf{A}$.* ∎

LEMMA 2.2. *If $B\colon \ldots \to B_n \to B_{n-1} \to \ldots \to B_1 \to B_0 \to A \to 0$ is a $\mathbf{G}$-exact sequence in $\mathbf{A}$, then B is $\mathbf{H}$-exact.* ∎

For each object A of $\mathbf{A}$ and each integer $n \geqslant 1$, let $(\tau^n)_A\colon H^n A \to G^n A$ denote the composition

$$H^n A \xrightarrow{H^{n-1}\tau_A} H^{n-1} GA \xrightarrow{H^{n-2}\tau_{GA}} H^{n-2} G^2 A \to \ldots \to HG^{n-1} A \xrightarrow{\tau_{G^{n-1}A}} G^n A,$$

and let

$$\sum_{i=0}^{n} (-1)^i G^{n-i} \varepsilon_{G^i A} = (d_n)_A\colon G^{n+1} A \to G^n A$$

and

$$\sum_{i=0}^{n} (-1)^i H^{n-i} \varepsilon'_{H^i A} = (d'_n)_A\colon H^{n+1} A \to H^n A.$$

From the naturality of τ, ε, and ε' we obtain:

LEMMA 2.3. *For every $n > 0$ and every object A of $\mathbf{A}$, the diagram*

$$\begin{array}{ccc} H^{n+1} A & \xrightarrow{(d'_n)_A} & H^n A \\ {\scriptstyle (\tau^{n+1})_A}\downarrow & & \downarrow{\scriptstyle (\tau^n)_A} \\ G^{n+1} A & \xrightarrow[(d_n)_A]{} & G^n A \end{array}$$

commutes. ∎

We also observe that if for some $n > 0$ τ^n is an isomorphism with inverse γ_n, then τ^{n+1} is an isomorphism with inverse $\gamma_{n+1} = \delta'_{H^{n-1}} \cdot \gamma_n \cdot \varepsilon_{G^n}$. By induction we obtain:

LEMMA 2.4. *If τ^n is an isomorphism for some $n > 0$, then τ^k is an isomorphism for all $k > n$.* ∎

COROLLARY 2.5. *If τ^n is an isomorphism for some $n > 0$, then for all $k > n$ the diagram*

$$\begin{array}{ccc} G^{k+1} A & \xrightarrow{(d_k)_A} & G^k A \\ {\scriptstyle (\gamma_{k+1})_A}\downarrow & & \downarrow{\scriptstyle (\gamma_k)_A} \\ H^{k+1} A & \xrightarrow[(d'_k)_A]{} & H^k A \end{array}$$

commutes. ∎

We now define the *dimension of $\mathbf{H}$ with respect to $\mathbf{G}$*, denoted by $\dim_G(\mathbf{H})$, to be the smallest positive integer n for which $\tau^{n+1}\colon H^{n+1} \Rightarrow G^{n+1}$

is an isomorphism. If τ^k is never an isomorphism for $k>0$, we say that $\dim_{\boldsymbol G}(\boldsymbol H)=\infty$. We now state the main result of this section:

THEOREM 2.6. *For every object A of $\boldsymbol A$,*

$$\dim_{\boldsymbol G}(A)\leqslant \dim_{\boldsymbol H}(A)+\dim_{\boldsymbol G}(\boldsymbol H).$$

For the proof of 2.6, we consider several cases. Clearly if $\dim_{\boldsymbol G}(\boldsymbol H)=0$, then τ is an isomorphism and $\dim_{\boldsymbol H}(A)=\dim_{\boldsymbol G}(A)$; and if $\dim_{\boldsymbol G}(\boldsymbol H)=\infty$, then the result is trivially true. So suppose that $1\leqslant\dim_{\boldsymbol G}(\boldsymbol H)=k<\infty$.

LEMMA 2.6.1. *If $\dim_{\boldsymbol H}(A)=n>k$, then $\dim_{\boldsymbol G}(A)\leqslant\dim_{\boldsymbol H}(A)$.*

Proof of 2.6.1. Suppose that

$$0\to K'_n=\ker(d'_{n-1})_A\xrightarrow{(i'_n)_A}H^nA\xrightarrow{(d'_{n-1})_A}H^{n-1}A\to\dots$$
$$\dots\to H^2A\xrightarrow{(d'_1)_A}HA\xrightarrow{\varepsilon'_A}A\to 0$$

is an $\boldsymbol H$-projective resolution of A and consider the $\boldsymbol G$-exact sequence

$$0\to K_n=\ker(d_{n-1})_A\xrightarrow{(i_n)_A}G^nA\xrightarrow{(d_{n-1})_A}G^{n-1}A\to\dots$$
$$\dots\to G^2A\xrightarrow{(d_1)_A}GA\xrightarrow{\varepsilon_A}A\to 0.$$

From 2.3, 2.5, and the universal properties of K_n and K'_n, we obtain $K_n\cong K'_n$; then by 2.1 we have that K_n is $\boldsymbol G$-projective. ∎

LEMMA 2.6.2. *If $\dim_{\boldsymbol H}(A)=n\leqslant k$, then*

$$\dim_{\boldsymbol G}(A)\leqslant\dim_{\boldsymbol H}(A)+\dim_{\boldsymbol G}(\boldsymbol H).$$

Proof of 2.6.2. If $\dim_{\boldsymbol H}(A)=0$, then $\dim_{\boldsymbol G}(A)=0$ by 2.1; so assume that $\dim_{\boldsymbol H}(A)\geqslant 1$. Since $k\geqslant n$, it follows from [4, Theorem 1.6] that the sequence

$$0\to\ker(d'_n)_A=K'_{k+1}\to H^{k+1}A\xrightarrow{(d'_k)_A}H^kA\to\dots\to H^2A\xrightarrow{(d'_1)_A}HA\xrightarrow{\varepsilon'_A}A\to 0$$

is an $\boldsymbol H$-projective resolution of A; in particular, K'_{k+1} is $\boldsymbol H$-projective. Consider the $\boldsymbol G$-exact sequence

$$0\to\ker(d_k)_A=K_{k+1}\to G^{k+1}A\xrightarrow{(d_k)_A}G^kA\to\dots\to G^2A\xrightarrow{(d_1)_A}GA\xrightarrow{\varepsilon_A}A\to 0.$$

By 2.3, 2.5, and the universal properties of K_{k+1} and K'_{k+1}, we have that $K_{k+1}\cong K'_{k+1}$; then from 2.1 we see that K_{k+1} is $\boldsymbol G$-projective. Hence $\dim_{\boldsymbol G}(A)\leqslant k+1\leqslant\dim_{\boldsymbol G}(\boldsymbol H)+\dim_{\boldsymbol H}(A)$.

This completes the proof of Theorem 2.6. ∎

If G and H do not reflect isomorphisms, we construct (as in § 1) the cotriples $\bar{\boldsymbol G}$ and $\bar{\boldsymbol H}$, whose functors do reflect isomorphisms. We then define $\dim_{\boldsymbol G}(A)=\dim_{\bar{\boldsymbol G}}(rA)$ and $\dim_{\boldsymbol H}(A)=\dim_{\bar{\boldsymbol H}}(r'A)$ for all objects A of $\boldsymbol A$, where r and r' are the canonical functors from $\boldsymbol A$ into the "categories of fractions" based on the $\boldsymbol G$-isomorphisms and $\boldsymbol H$-isomorphisms, respectively, and we let $\dim_{\boldsymbol G}(\boldsymbol H)=\dim_{\bar{\boldsymbol G}}(\bar{\boldsymbol H})$. It follows easily from the construction that 2.1–2.6 extend to $\bar{\boldsymbol G}$ and $\bar{\boldsymbol H}$.

§ 3. Maps of cotriples on different categories. Suppose that $\boldsymbol A$ and $\boldsymbol X$ are additive exact categories (in the sense of [9, Ch. I]), $\boldsymbol G=(G,\varepsilon,\delta)$ and $\boldsymbol H=(H',\varepsilon',\delta')$ are cotriples on $\boldsymbol A$ and $\boldsymbol X$, respectively, $F\colon \boldsymbol X\to\boldsymbol A$ is a faithful, zero-preserving functor and $\tau\colon \boldsymbol H\Rightarrow\boldsymbol G$ is a morphism of cotriples (i.e., a natural transformation $\tau\colon FH\Rightarrow GF$ such that $\varepsilon F\cdot\tau=F\varepsilon'$ and $G\tau\cdot\tau H\cdot F\delta'=\delta F\cdot\tau$). Then one easily proves:

LEMMA 3.1. *If X is an $\boldsymbol H$-projective object of $\boldsymbol X$, then FX is $\boldsymbol G$-projective. In particular, FHX is $\boldsymbol G$-projective for every X in $\boldsymbol X$.* ∎

Using 3.1 and a theorem of Freyd [9, Theorem 7.1] we obtain:

LEMMA 3.2. *If $B\colon\ \dots\to B_n\xrightarrow{f_{n-1}}B_{n-1}\to\dots\xrightarrow{f_1}B_1\xrightarrow{f_0}B_0\xrightarrow{f}X\to 0$ is a sequence in $\boldsymbol X$ for which the sequence*

$$FB\colon\ \dots FB_n\xrightarrow{Ff_{n-1}}FB_{n-1}\to\dots\xrightarrow{Ff_1}FB_1\xrightarrow{Ff_0}FB_0\xrightarrow{Ff}FX\to 0$$

is $\boldsymbol G$-exact in $\boldsymbol A$, then B is $\boldsymbol H$-exact in $\boldsymbol X$. ∎

Suppose now that F has a right adjoint U. Let $m\colon 1_{\boldsymbol X}\Rightarrow UF$ and $p\colon FU\Rightarrow 1_{\boldsymbol A}$ denote the unit and counit, respectively, of the adjunction. For each object A of $\boldsymbol A$ and each integer $n\geqslant 1$, let $(\tau U)^n_A\colon (FHU)^n_A\to(GFU)^n_A$ denote the composition

$$(FHU)^nA\xrightarrow{(FHU)^{n-1}\tau_{UA}}(FHU)^{n-1}GFUA\xrightarrow{(FHU)^{n-2}\tau_{UGFUA}}$$
$$(FHU)^{n-2}(GFU)^2A\to\dots\to(FHU)(GFU)^{n-1}A\xrightarrow{\tau_{U(GFU)^{n-1}A}}(GFU)^nA,$$

and let

$$\sum_{i=1}^{n}(-1)^i(FHU)^{n-i}FHm_{HU(FHU)^{i-1}A}\cdot(FHU)^{n-i}F\delta'_{U(FHU)^{i-1}A}$$
$$=(\partial'_n)_A\colon (FHU)^nA\to(FHU)^{n+1}A;$$

define $(\partial_n)_A\colon (GFU)^nA\to(GFU)^{n+1}A$ similarly. By a straightforward (although messy) naturality argument we obtain:

LEMMA 3.3. *For every $n>0$ and every object A of $\boldsymbol A$, the diagram*

$$\begin{array}{ccc}(FHU)^nA & \xrightarrow{(\partial'_n)_A} & (FHU)^{n+1}A\\ {\scriptstyle(\tau U)^n_A}\downarrow & & \downarrow{\scriptstyle(\tau U)^{n+1}_A}\\ (GFU)^nA & \xrightarrow{(\partial_n)_A} & (GFU)^{n+1}A\end{array}$$

commutes. ∎

We say that $\boldsymbol H$ *intertwines* with the adjoint pair (U,F) if the composition

$$HUFHUA\xrightarrow{\delta_{UFHUA}}HHUFHUA\xrightarrow{HHUF\varepsilon'_{UA}}HHUFUA\xrightarrow{Hm_{HUFUA}}$$
$$HUFHUFA\xrightarrow{HUFHUp_A}HUFHUA$$

is the identity morphism on $HUFHUA$ for every object A of $\boldsymbol A$. (For example, if $f\colon R\to S$ is a ring homomorphism, then the "free" cotriple on ${}_S\mathbf{Mod}$ intertwines with the adjoint pair $(\mathrm{Hom}(S, -), S\otimes_R -)$.)

LEMMA 3.4. *Suppose that* $\boldsymbol{H}$ *intertwines with* (U, F) *and that* $(\tau U)^n$ *is an isomorphism for some* $n>0$. *Then* $(\tau U)^k$ *is an isomorphism for all* $k>n$.

Proof. Let $(\gamma U)_n$ denote the inverse of $(\tau U)^n$, and let $(\gamma U)_{n+1}\colon (FHU)^{n+1}\Rightarrow(GFU)^{n+1}$ be defined on objects by $((\gamma U)_{n+1})_A=$ the composition

$$(GFU)^{n+1}A\xrightarrow{(GFU)^n\varepsilon_{FUA}}(GFU)^nFUA\xrightarrow{(GFU)^n p_A}(GFU)^nA\xrightarrow{((\gamma U)_n)_A}(FHU)^nA$$
$$\xrightarrow{(FHU)^{n-1}F\delta'_{UA}}(FHU)^{n-1}FHHUA\xrightarrow{(FHU)^{n-1}FHm_{HUA}}(FHU)^{n+1}A.$$

Then $(\gamma U)_{n+1}$ is the inverse of $(\tau U)^{n+1}$. The conclusion of the lemma follows by induction. ∎

COROLLARY 3.5. *If* $\boldsymbol{H}$ *intertwines with* (U, F) *and* $(\tau U)^n$ *is an isomorphism for some* $n>0$, *then for every* $k>n$ *and every object* A *of* $\mathbf{A}$, *the diagram*

$$\begin{array}{ccc}(GFU)^nA & \xrightarrow{(\partial_n)_A} & (GFU)^{n+1}A\\ {\scriptstyle((\gamma U)_n)_A}\downarrow & & \downarrow{\scriptstyle((\gamma U)_{n+1})_A}\\ (FHU)^nA & \xrightarrow[(\partial'_n)_A]{} & (FHU)^{n+1}A\end{array}$$

commutes. ∎

For each integer $n\geqslant 1$ and for each object A of $\mathbf{A}$ and X of $\mathbf{X}$, let $(d_n)_A\colon G^{n+1}A\to G^nA$ and $(d'_n)_X\colon H^{n+1}X\to H^nX$ be defined as in § 2 and let $(\tau_n)_X\colon FH^nX\to G^nFX$ denote the composition

$$FH^nX\xrightarrow{\tau_{H^{n-1}X}}GFH^{n-1}X\xrightarrow{G\tau_{H^{n-2}X}}G^2FH^{n-2}X\to\dots$$
$$\dots\to G^{n-1}FHX\xrightarrow{G^{n-1}\tau_X}G^nFX.$$

(Observe that $\tau_1=\tau$.)

LEMMA 3.6. *For every object* X *of* $\mathbf{X}$ *and every* $n>0$, *the diagram*

$$\begin{array}{ccc}FH^{n+1}X & \xrightarrow{F(d'_n)_X} & FH^nX\\ {\scriptstyle(\tau_{n+1})_X}\downarrow & & \downarrow{\scriptstyle(\tau_n)_X}\\ G^{n+1}FX & \xrightarrow[(d_n)_{FX}]{} & G^nFX\end{array}$$

commutes.

The proof is by induction, using naturality and the fact that τ is a morphism of cotriples. ∎

We observe that if τ is an isomorphism, then τ_n is an isomorphism for all $n\geqslant 1$, and we have the following:

COROLLARY 3.7. *If* τ *is an isomorphism, then for every object* X *of* $\mathbf{X}$ *and every* $n\geqslant 1$, *the diagram*

$$\begin{array}{ccc}G^{n+1}Fx & \xrightarrow{(d_n)_{FX}} & G^nFX\\ {\scriptstyle(\tau_{n+1})_X^{-1}}\downarrow & & \downarrow{\scriptstyle(\tau_n)_X^{-1}}\\ FH^{n+1}X & \xrightarrow[F(d'_n)_X]{} & FH^nX\end{array}$$

commutes. ∎

Assume (unless stated otherwise) for the remainder of this section that $\boldsymbol{H}$ intertwines with (U, F) and that G and H reflect isomorphisms.

PROPOSITION 3.8. *Suppose that* τ *is an isomorphism and* X *is an object of* $\mathbf{X}$ *for which* $\dim_{\boldsymbol{H}}(X)\leqslant m$. *Then* $\dim_{\boldsymbol{G}}(FX)\leqslant m$.

Proof. Let

$$0\to K'_m\xrightarrow{i'_m}H^mX\xrightarrow{(d'_{m-1})_X}H^{m-1}X\xrightarrow{(d'_{m-2})_X}\dots\to H^2X\xrightarrow{(d'_1)_X}HX\xrightarrow{\varepsilon'_X}X\to 0$$

be an $\boldsymbol{H}$-projective resolution of X, and consider the diagram

$$\begin{array}{ccccccccccccc}0\to FK'_m & \xrightarrow{F(i'_m)} & FH^mX & \xrightarrow{F(d'_{m-1})_X} & FH^{m-1}X & \to\dots & \xrightarrow{F(d'_2)_X} & FH^2X & \xrightarrow{F(d'_1)_X} & FHX & \xrightarrow{F(\varepsilon'_X)} & FX\to 0\\ & & {\scriptstyle(\tau_m)_X}\downarrow\uparrow{\scriptstyle(\tau_m)_X^{-1}} & & {\scriptstyle(\tau_{m-1})_X}\downarrow\uparrow{\scriptstyle(\tau_{m-1})_X^{-1}} & & & {\scriptstyle(\tau_2)_X}\downarrow\uparrow{\scriptstyle(\tau_2)_X^{-1}} & & {\scriptstyle\tau}\downarrow\uparrow{\scriptstyle\tau_X^{-1}} & & \| \\ 0\to K_m & \xrightarrow[i_m]{} & G^mFX & \xrightarrow[(d_{m-1})_{FX}]{} & G^{m-1}FX & \to\dots & \xrightarrow[(d_2)_{FX}]{} & G^2FX & \xrightarrow[(d_1)_{FX}]{} & GFX & \xrightarrow[\varepsilon_{FX}]{} & FX\to 0\end{array}$$

in which all the squares commute (by 3.6 and 3.7), the bottom row is $\boldsymbol{G}$-exact, and $FHX, FH^2X, \dots, FH^mX, FK'_m$ are $\boldsymbol{G}$-projective (by 3.1). From 3.6, 3.7, and the universal properties of K_m and FK'_m, we obtain $K_m\cong FK'_m$; hence K_m is $\boldsymbol{G}$-projective and $\dim_{\boldsymbol{G}}(FX)\leqslant m$. ∎

We now define the *dimension of* $\boldsymbol{H}$ *with respect to* $\boldsymbol{G}$, denoted by $\dim_{\boldsymbol{G}}(\boldsymbol{H})$, to be the smallest non-negative integer n for which $(TU)^n\tau\colon (FHU)^nFH\Rightarrow(GFU)^nGF$ is an isomorphism. ($(\tau U)^n\tau=$ the composition $(FHU)^nFH\xrightarrow{(\tau U)^nFU}(GFU)^nFH\xRightarrow{(GFU)^n\tau}(GFU)^nGF$.) We note that this definition coincides with the definition in § 2 for the special case $\mathbf{A}=\mathbf{X}$, $U=F=$ the identity functor on $\mathbf{A}$. If $(\tau U)^n\tau$ is never an isomorphism for any $n\geqslant 0$, we say that $\dim_{\boldsymbol{G}}(\boldsymbol{H})=\infty$.

We now state the main result of this section:

THEOREM 3.12. *For every object* X *of* $\mathbf{X}$,

$$\dim_{\boldsymbol{G}}(FX)\leqslant\dim_{\boldsymbol{H}}(X)+\dim_{\boldsymbol{G}}(\boldsymbol{H}).$$

The proof of 3.12 consists of several parts. We begin with:

LEMMA 3.9. *If* $\dim_{\boldsymbol{G}}(\boldsymbol{H})=n$, *then* τ_{n+1} *is an isomorphism.*

Proof. By induction on n. For $n=0$, the result follows from the definition of $\dim_{\boldsymbol{G}}(\boldsymbol{H})$. For $n=1$, let X be an object of $\mathbf{X}$ and consider the commutative diagram

$$\begin{array}{ccccccc}FH^2X & \xrightarrow{\tau_{HX}} & GFHX & = & GFHX & \xrightarrow{G\tau_X} & G^2FX\\ {\scriptstyle FHm_{HX}}\downarrow & & \downarrow{\scriptstyle GFm_{HX}} & \nearrow{\scriptstyle GpFHX} & & \nearrow{\scriptstyle GpGFX} & \\ FHUFHX & \xrightarrow[(\tau U)_{FHX}]{} & GFUFHX & \xrightarrow[GFU\tau_X]{} & GFUGFX & & \end{array}$$

in which the bottom row is $[(\tau U)\tau]_X$ and the top row is $(\tau_2)_X$. Now if $[(\tau U)\tau]_X$ is an isomorphism, we obtain from naturality and the cotriple identities that $(\tau_2)_X$ is an isomorphism. For $n = k > 1$, we construct, similarly, the commutative diagram:

$$\begin{array}{ccc}
FH^{k+1}X & \xrightarrow{(\tau_{k+1})X} & G^{k+1}FX \\
{\scriptstyle FHm_{H^kX}}\downarrow & & \uparrow{\scriptstyle Gp_{G^kFX}} \\
FHUFH^kX & & (GFU)G^kFX \\
{\scriptstyle FHUFHm_{H^{k-1}X}}\downarrow & & \uparrow \\
(FHU)^2FH^{k-1}X & & \vdots \\
\downarrow & & \uparrow \\
\vdots & & (GFU)^{k-2}G^3FX \\
& & \uparrow{\scriptstyle (GFU)^{k-2}Gp_{G^2FX}} \\
\downarrow & & (GFU)^{k-1}G^2FX \\
{\scriptstyle (FHU)^{k-1}FHm_{HX}}\downarrow & & \uparrow{\scriptstyle (GFU)^{k-1}Gp_{GFX}} \\
(FHU)^kFHX & \xrightarrow{((\tau U)^k\tau)X} & (GFU)^kGFX
\end{array}$$

Now if $(\tau U)^k\tau$ is an isomorphism, it follows from naturality and the cotriple identities that τ_{k+1} is an isomorphism. ∎

THEOREM 3.10. *Suppose that* $\dim_{\boldsymbol{G}}(\boldsymbol{H}) = k \geqslant 0$ *and suppose that* X *is an object of* $\boldsymbol{X}$ *for which* $\dim_{\boldsymbol{H}}(X) = n > k$. *Then* $\dim_{\boldsymbol{G}}(FX) \leqslant \dim_{\boldsymbol{H}}(X) + \dim_{\boldsymbol{G}}(\boldsymbol{H})$.

Proof. If $k = 0$, the result follows directly from 3.8. Suppose that $k \geqslant 1$. Let

$$0 \to K'_n \xrightarrow{i'_n} H^nX \xrightarrow{(d'_{n-1})X} H^{n-1}X \to \ldots \to H^2X \xrightarrow{(d'_1)} HX \xrightarrow{\varepsilon'X} X \to 0$$

be an $\boldsymbol{H}$-projective resolution of X and consider the diagram

$$\begin{array}{ccccccccccccc}
0 \to FK'_n & \xrightarrow{F(i'_n)} & FH^nX & \xrightarrow{F(d'_{n-1})X} & FH^{n-1}X & \to \ldots \to & FH^2X & \xrightarrow{F(d'_1)X} & FHX & \xrightarrow{F\varepsilon'X} & FX \to 0 \\
& & \downarrow{\scriptstyle (\tau_n)X} & & \downarrow{\scriptstyle (\tau_{n-1})X} & & \downarrow{\scriptstyle (\tau_2)X} & & \downarrow{\scriptstyle \tau X} & & \| \\
0 \to K_n & \xrightarrow{i_n} & G^nFX & \xrightarrow[(d_{n-1})FX]{} & G^{n-1}FX & \to \ldots \to & G^2FX & \xrightarrow[(d_1)FX]{} & GFX & \xrightarrow[\varepsilon FX]{} & FX \to 0
\end{array}$$

in which the bottom row is $\boldsymbol{G}$-exact, all the squares commute (by 3.6), and $FHX, FH^2X, \ldots, FH^nX, FK'_n$ are $\boldsymbol{G}$-projective (by 3.1). It follows from 3.9 (and from 3.4, if $n > k+1$) that $(\tau_n)_X$ is an isomorphism, whence (by the same argument as in 3.8)

$$\dim_{\boldsymbol{G}}(FX) \leqslant n = \dim_{\boldsymbol{H}}(X) < \dim_{\boldsymbol{H}}(X) + \dim_{\boldsymbol{G}}(\boldsymbol{H}). \quad ∎$$

THEOREM 3.11. *Suppose that* $\dim_{\boldsymbol{G}}(\boldsymbol{H}) = k \geqslant 0$ *and suppose that* X *is an object of* $\boldsymbol{X}$ *for which* $\dim_{\boldsymbol{H}}(X) = n$ *where* $0 \leqslant n \leqslant k$. *Then* $\dim_{\boldsymbol{G}}(FX) \leqslant \dim_{\boldsymbol{H}}(X) + \dim_{\boldsymbol{G}}(\boldsymbol{H})$.

Proof. If $k = 0$ or $n = 0$, then the result follows from 3.1. So suppose $1 \leqslant n \leqslant k$. Let

$$0 \to K'_{k+1} \xrightarrow{i'_k} H^{k+1}X \xrightarrow{(d'_k)X} H^kX \to \ldots \to H^2X \xrightarrow{(d'_1)X} HX \xrightarrow{\varepsilon'X} X \to 0$$

be an $\boldsymbol{H}$-projective resolution of X (not the shortest one, of course), and consider the diagram

$$\begin{array}{ccccccccccccc}
0 \to FK'_{k+1} & \xrightarrow{Fi'_k} & FH^{k+1}X & \xrightarrow{F(d'_k)X} & FH^kX & \to \ldots \to & FH^2X & \xrightarrow{F(d'_1)X} & FHX & \xrightarrow{F\varepsilon'X} & FX \to 0 \\
& & {\scriptstyle (\tau_{k+1})X}\downarrow\ \uparrow{\scriptstyle (\tau_{k+1})_X^{-1}} & & \downarrow{\scriptstyle (\tau_k)X} & & \downarrow{\scriptstyle (\tau_2)X} & & \downarrow{\scriptstyle \tau X} & & \| \\
0 \to K_{k+1} & \xrightarrow{i_k} & G^{k+1}FX & \xrightarrow[(d_k)FX]{} & G^kFX & \to \ldots \to & G^2FX & \xrightarrow[(d_1)FX]{} & GFX & \xrightarrow[\varepsilon FX]{} & FX \to 0
\end{array}$$

in which the squares commute, the bottom row is $\boldsymbol{G}$-exact, and $FHX, FH^2X, \ldots, FH^{k+1}X$, and FK'_{k+1} are all $\boldsymbol{G}$-projective. As before, we have that K_{k+1} is $\boldsymbol{G}$-projective and hence that $\dim_{\boldsymbol{G}}(FX) \leqslant k+1$. But $k+1 \leqslant \dim_{\boldsymbol{G}}(\boldsymbol{H}) + \dim_{\boldsymbol{H}}(X)$. ∎

THEOREM 3.12. *For every object* X *of* $\boldsymbol{X}$,

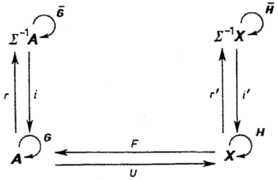
$$\dim_{\boldsymbol{G}}(FX) \leqslant \dim_{\boldsymbol{H}}(X) + \dim_{\boldsymbol{G}}(\boldsymbol{H}).$$

Proof. 3.10 and 3.11. ∎

For the case in which G and H do not reflect isomorphisms, we construct (dually to [4, § 3] categories $\Sigma^{-1}\boldsymbol{A}$ and $\Sigma^{-1}\boldsymbol{X}$ and cotriples $\bar{\boldsymbol{G}} = (\bar{G}, \bar{\varepsilon}, \bar{\delta})$ on $\Sigma^{-1}\boldsymbol{A}$ and $\bar{\boldsymbol{H}} = (\bar{H}, \bar{\varepsilon}', \bar{\delta}')$ on $\Sigma^{-1}\boldsymbol{X}$ whose functors reflect isomorphisms. The following proposition is the dual of [4, 3.7 and 3.8]:

PROPOSITION 3.13. *Suppose that for every object* A *of* $\boldsymbol{A}$ *the morphisms* $G\varepsilon_A$ *and* ε_{GA} *have a coequalizer* k_A *which is preserved by* G. *Then* $\Sigma^{-1}\boldsymbol{A}$ *is equivalent to a full reflective subcategory of* $\boldsymbol{A}$, $\bar{\boldsymbol{G}}$ *is* $\boldsymbol{G}$ *restricted to* $\Sigma^{-1}\boldsymbol{A}$, *and* $\dim_{\boldsymbol{G}}(A) = \dim_{\bar{\boldsymbol{G}}}(rA)$ *for every object* A *of* $\boldsymbol{A}$, *where* r: $\boldsymbol{A} \to \Sigma^{-1}\boldsymbol{A}$ *is the reflector. Similarly, if for every object* X *of* $\boldsymbol{X}$ *the morphisms* $H\varepsilon'_X$ *and* ε'_{HX} *have a coequalizer* k'_X *which is preserved by* H, *then* $\Sigma^{-1}\boldsymbol{X}$ *is equivalent to a full reflective subcategory of* $\boldsymbol{X}$, $\bar{\boldsymbol{H}}$ *is* $\boldsymbol{H}$ *restricted to* $\Sigma^{-1}\boldsymbol{X}$, *and* $\dim_{\boldsymbol{H}}(X) = \dim_{\bar{\boldsymbol{H}}}(r'X)$ *for every object* X *of* $\boldsymbol{X}$, *where* r': $\boldsymbol{X} \to \Sigma^{-1}\boldsymbol{X}$ *is the reflector.* ∎

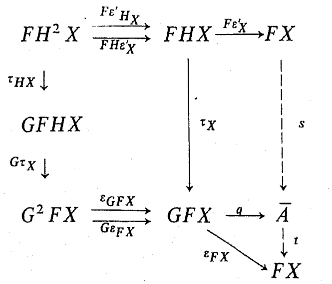
We refer the reader to [4] and [10] for examples of categories and cotriples satisfying the hypotheses of 3.13. Assume for the remainder of this section that the hypotheses of 3.13 are satisfied. Then we have the following picture:



In order to extend 3.12 to this more general situation, we need to construct a functor $\bar{F}$: $\Sigma^{-1}\boldsymbol{X} \to \Sigma^{-1}\boldsymbol{A}$ having the same properties as F and a righ adjoint $\bar{U}$: $\Sigma^{-1}\boldsymbol{A} \to \Sigma^{-1}\boldsymbol{X}$ to $\bar{F}$ such that $\bar{\boldsymbol{H}}$ intertwines with $(\bar{U}, \bar{F})$. That F restricted to $\Sigma^{-1}X$ is such an $\bar{F}$ is shown in the following:

PROPOSITION 3.14. *If X is an object of $\Sigma^{-1}\mathbf{X}$, then FX is an object of $\Sigma^{-1}\mathbf{A}$.*

Proof. It follows from the proof of 3.13 that X is in $\Sigma^{-1}\mathbf{X}$ iff ε'_X is the coequalizer of ε'_{FX} and $F\varepsilon'_X$. Since F has a right adjoint, F preserves colimits; so $F\varepsilon'_X$ is the coequalizer of $F\varepsilon'_{HX}$ and $FH\varepsilon'_X$. In order to show that ε_{FX} is the coequalizer of ε_{GFX} and $G\varepsilon_{FX}$ (and hence that FX is in $\Sigma^{-1}\mathbf{A}$), we let $q\colon GFX \to \bar{A}$ be the coequalizer of ε_{GFX} and $G\varepsilon_{FX}$ and consider the diagram:



in which the rows are coequalizer diagrams and s and t are the unique maps (coming from the universal property of FX and $\bar{A}$) for which the right hand square and triangle commute. One easily shows that $t\cdot s = 1_{FX}$ and $s\cdot t = 1_{\bar{A}}$; hence t is an isomorphism and ε_{FX} is the desired coequalizer. ■

It follows immediately from 3.14 that $\bar{F} = rFi'$ is a functor from $\Sigma^{-1}\mathbf{X}$ to $\Sigma^{-1}\mathbf{A}$ (namely, F restricted to $\Sigma^{-1}\mathbf{X}$) and that $\bar{F}$, like F, is a faithful, zero-preserving functor.

PROPOSITION 3.15. *Let $\bar{U} = r'Ui\colon \Sigma^{-1}\mathbf{A} \to \Sigma^{-1}\mathbf{X}$. Then $\bar{U}$ is right adjoint to $\bar{F}$.*

Proof. For all objects A of $\Sigma^{-1}\mathbf{A}$ and X of $\Sigma^{-1}\mathbf{X}$ we have

$$\begin{aligned}\Sigma^{-1}\mathbf{A}(\bar{F}X, A) &\cong \Sigma^{-1}\mathbf{A}(rFi'X, A)\\ &\cong \mathbf{A}(Fi'X, iA)\\ &\cong \mathbf{X}(i'X, UiA)\\ &\cong \Sigma^{-1}\mathbf{X}(X, r'UiA)\\ &\cong \Sigma^{-1}\mathbf{X}(X, \bar{U}A). \ ■\end{aligned}$$

Since $\bar{F}$ and $\bar{\mathbf{H}}$ are restrictions of F and $\mathbf{H}$, it is clear that $\bar{\mathbf{H}}$ intertwines with $(\bar{U}, \bar{F})$. Let $\bar{\tau}\colon \bar{F}\bar{H} \Rightarrow \bar{G}\bar{F}$ denote the morphism of cotriples induced by the restrictions of F, $\mathbf{G}$, and $\mathbf{H}$, and define $\dim_{\mathbf{G}}(\mathbf{H}) = \dim_{\bar{\mathbf{G}}}(\bar{\mathbf{H}})$ = the smallest non-negative integer n for which $(\bar{\tau}\bar{U})^n\bar{\tau}$ is an isomorphism. Then we have extended 3.12 to:

THEOREM 3.16. *Suppose that the hypotheses of 3.13 are satisfied. Then for every object X of $\mathbf{X}$,*

$$\dim_{\bar{\mathbf{G}}}(rFX) \leqslant \dim_{\bar{\mathbf{H}}}(r'X) + \dim_{\bar{\mathbf{G}}}(\mathbf{H}). \ ■$$

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Accepté par la Rédaction le 9. 11. 1981