On linear homogeneous functional equations in the indeterminate case
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Abstract. Let I be a compact real interval and let ξ be a point of I. Denote by F the set of all continuous functions \( f: I \to \mathbb{R} \) such that \( 0 < f(x) - \xi(x - \xi) < 1 \) for every \( x \in I \setminus \{\xi\} \) and make \( X \) denote the set of all continuous functions \( g: I \to \mathbb{R} \) such that \( g(x) = 0 \) for every \( x \in I \) and \( g(\xi) = 1 \). Considering \( F \) and \( X \) as metric spaces with the uniform convergence metric (they are Baire spaces), we prove that for almost all pairs \( f, g \in F \times X \) (in the sense of the Baire category) the functional equation \( \phi \circ f = g \phi \) has exactly one continuous solution \( \phi: I \to \mathbb{R} \) (just the zero function).

In this paper we deal with the problem of the number of continuous solutions \( \phi \) of the functional equation

\[
\phi \circ f = g \phi.
\]

Let \( I \) be a real interval and let \( \xi \) be a point of \( I \). The theory of continuous solutions of equation (1) has been developed under the following hypotheses (cf. [1], [4]-[6]):

(i) \( f: I \to \mathbb{R} \) is continuous and \( 0 < f(x) - \xi(x - \xi) < 1 \) for \( x \in I \setminus \{\xi\} \).

(ii) \( f \) is strictly increasing in a neighbourhood of \( \xi \).

(iii) \( g: I \to \mathbb{R} \) is continuous and \( g(x) \neq 0 \) for \( x \in I \setminus \{\xi\} \).

Denote by \( f^n, n \in \mathbb{N}_0 \), the \( n \)-th iterate of \( f \).

Remark 1. Under hypothesis (i) \( \xi \) is the unique fixed point of \( f \), and the sequence \( \{f^n(x)\}_{n \in \mathbb{N}_0} \) is monotonic for every \( x \in I \) and converges to \( \xi \) (cf. [5], Th. 0.4).

Let us write

\[
G_n = \prod_{k=0}^{n-1} g \circ f^k, \quad n \in \mathbb{N}_0.
\]

The following three cases are possible (cf. [5], Ch. II, § 2):

(A) There exists a continuous function \( G: I \to \mathbb{R} \) such that \( G(x) = \lim_{n \to \infty} G_n(x) \neq 0 \) for every \( x \in I \).

(B) There exists a continuous function \( G: I \to \mathbb{R} \) such that \( G(x) = 0 \) for every \( x \in I \).

(C) There exists a continuous function \( G: I \to \mathbb{R} \) such that \( G(x) = \lim_{n \to \infty} G_n(x) = 0 \) for every \( x \in I \).

(1) By \( \mathbb{N}_0 \) we denote the set of all non-negative integers.
(B) \( \lim_{x \to -} G_n(x) = 0 \) uniformly on a subinterval of \( I \).

(C) Neither (A) nor (B) occurs.

The above three conditions determine the number of continuous solutions of equation (1). Namely, Theorem 2.2 from [5] (cf. also [1]) states that:

If conditions (i), (ii) and (A) are fulfilled, then equation (1) has in \( I \) a unique one parameter family of continuous solutions. These solutions are given by \( \varphi(x) = cG(x) \), where \( c \) is an arbitrary real constant.

If conditions (i)-(ii) and (B) are fulfilled, then equation (1) has in \( I \) a continuous solution depending on an arbitrary function.

If conditions (i), (ii) and (C) are fulfilled, then the zero function is the unique continuous solution of equation (1) in \( I \).

Note that under assumptions (i) and (ii) case (A) may occur only if \( g(\xi) = 1 \) (otherwise the infinite product defining the sequence \( \{G_n(x)\}_{n=0}^{\infty} \) diverges), but if \( g(\xi) = 1 \), then all the three cases (A), (B) and (C) can actually occur (see Lemma 1). If \( g(\xi) < 1 \), then case (B) occurs (cf. [5], Th. 2.3). If \( g(\xi) > 1 \), then case (C) occurs (cf. [5], Th. 2.4). The case \( g(\xi) = 1 \) is called indeterminate.

In this paper we shall show that the in the indeterminate case for almost all equations (all in the sense of the Baire category) of form (1) case (C) holds.

Let \( I \) be a compact interval. In the sequel the space of all functions \( f \) fulfilling (ii) is denoted by \( F \) whereas \( X^+ \) denotes the space of all functions \( g \) fulfilling (ii) and the condition \( g(\xi) = 1 \). We may treat \( F \) and \( X^+ \) as metric spaces, enveloping them with the uniform convergence metric. Observe that they are Baire spaces. Namely, if we endow the space of all real functions defined and continuous in \( I \) with the uniform convergence metric, then \( F \) and \( X^+ \), as is easy to show, are its \( G_\alpha \) subsets, and thus, by Theorem of Alexandroff (cf. [3], Th. 12.1), they are topologically complete and consequently (cf. [8], Th. 9.1) they are Baire spaces.

Let us define

\[
A^+ = \{(f, g) \in F \times X^+ : \text{the sequence \( \{G_n(x)\}_{n=0}^{\infty} \) converges pointwise to a function from \( X^+ \)}\},
\]

\[
B^+ = \{(f, g) \in F \times X^+ : \lim_{n \to \infty} G_n(x) = 0 \text{ uniformly on \( I \)}\},
\]

\[
C^+ = F \times X^+ \setminus (A^+ \cup B^+)
\]

and, for every \( E \subset F \times X^+ \) and \( f \in F \), put \( E_f = \{g \in X^+ : (f, g) \in E\} \).

We start with the following lemma.

**Lemma 1.** If \( f \in F \), then \( A_f^+ \), \( B_f^+ \) and \( C_f^+ \) are non-empty sets.

**Proof.** We may assume without loss of generality that \( \xi \) is the left endpoint of \( I \). Since the function identically equal to one belongs to \( A_f^+ \), we have \( A_f^+ \neq \emptyset \).

Let \( f \) be a strictly increasing function belonging to \( F \) and such that \( f \leq f^2 \). Choose a sequence \( \{\alpha_n\}_{n=0}^{\infty} \) of positive numbers such that \( \{\alpha_n\}_{n=0}^{\infty} \) increases to one and \( \sum_{n=0}^{\infty} \alpha_n = 0 \). Further, fix an \( x_0 \in I \setminus \{\xi\} \). In virtue of Remark 1 we can find a decreasing function \( g \in X^+ \) such that \( g[F^k(x_0)] = \alpha_k \) for \( k \in \mathbb{N} \). Since \( f \) is strictly increasing, so is \( f^k \) and \( f^k \leq f^\xi \) for every \( k \in \mathbb{N} \). Thus, for every \( x \in [f(x_0), x_0] \) and \( n \in \mathbb{N} \), we get

\[
0 \leq \prod_{k=0}^{n-1} g[F^k(x)] - \prod_{k=0}^{n-1} g[F^k(x)] < \prod_{k=0}^{n-1} g[F^{k+1}(x)] = \prod_{k=0}^{n-1} \alpha_k.
\]

Consequently, \( \lim_{n \to \infty} G_n(x) = 0 \) uniformly in \( [f(x_0), x_0] \subset I \), and so \( g \in B_f^+ \).

Finally, we shall show that \( C_f^+ \) is a non-void set. Indeed, let us take into account the function \( g \) constructed above and observe that the function \( g_1 = f^2 - g \) is an element of \( X^+ \). Since \( g \leq 1 \) and \( \prod_{k=0}^{n} g[F^k(x_0)] = 0 \), we have

\[
\sum_{k=0}^{\infty} |g[F^k(x_0)] - 1| = \sum_{k=0}^{\infty} g_1[F^k(x_0)] = \sum_{k=0}^{\infty} (1 - g[F^k(x_0)]) = \infty.
\]

Consequently, \( \prod_{k=0}^{\infty} g_1[F^k(x_0)] = \infty \) and \( g_1 \notin B_f^+ \). Obviously, since \( g_1 \geq 1 \), \( g_1 \) is not an element of \( B_f^+ \). Thus it belongs to \( C_f^+ \).

**Lemma 2.** Suppose that \( f \in F \). Every two elements of \( X^+ \) coinciding in a neighbourhood of \( \xi \) belong simultaneously to one of the sets \( A_f^+ \), \( B_f^+ \), \( C_f^+ \).

**Proof.** Let \( g, \bar{g} \in X^+ \) coincide in a neighbourhood \( V \) of \( \xi \). On account of Lemma 1 from [3] we have \( f^\bar{g}(\bar{V}) = V \) for every \( \bar{g} \in \mathbb{N} \). Write \( \bar{G}_n(x) = \prod_{k=0}^{n-1} \bar{g}[F^k(x)] \) and note that

\[
\bar{G}_n(x) = \prod_{k=0}^{n-1} \bar{g}[F^k(x)] = \prod_{k=0}^{n-1} \bar{g}[F^k(x)] \prod_{k=0}^{n-1} \bar{g}[F^k(x)]
\]

and

\[
\bar{G}_n(x) = \prod_{k=0}^{n-1} \bar{g}[F^k(x)] = \prod_{k=0}^{n-1} \bar{g}[F^k(x)] \prod_{k=0}^{n-1} \bar{g}[F^k(x)]
\]

for \( x \in I \) and \( n > m \). Making use of this formula, we get our assertion.

Now we pass to our main results.
THEOREM 1. If \( f \in F \), then the sets \( A_f^+ \), \( B_f^+ \) and \( C_f^+ \) are dense in \( X^+ \).

Proof. Fix a \( g \in X^+ \) and a positive real number \( \varepsilon < \inf \{ g(x) : x \in I \} \). In view of Lemma 1 we can find a function \( g_1 \) belonging to \( A_f^+ \). Since \( g \) and \( g_1 \) are continuous and \( g(\xi) = g_1(\xi) \), we have \( |g(x) - g_1(x)| \leq \varepsilon \) for every \( x \) from a closed neighbourhood \( V \) of \( \xi \). On account of Tietze's theorem there exists a continuous function \( \tilde{g} : I \to \mathbb{R} \) such that

\[
\tilde{g}(x) = g_1(x)
\]

and

\[
|g(x) - \tilde{g}(x)| < \varepsilon \quad \text{for} \quad x \in I.
\]

Observe that according to (3) and to the choice of \( \varepsilon \) we have \( \tilde{g}(\xi) = g(\xi) \).

By (3) and Lemma 2 the function \( \tilde{g} \) belongs to \( A_f^+ \), which, jointly with (4), ends the proof of the density of \( A_f^+ \) in \( X^+ \). The reasoning for sets \( B_f^+ \) and \( C_f^+ \) is analogous.

THEOREM 2. If \( f \in F \), then the set \( C_f^+ \) is residual in \( X^+ \).

Proof. It is enough to show that the sets \( A_f^+ \) and \( B_f^+ \) are of the first Baire category. In order to prove that \( A_f^+ \) is of the first category let us define a mapping \( T : X^+ \to X^+ \) by the formula

\[
T \circ f = \varphi \circ f \quad \text{for} \quad \varphi \in X^+.
\]

Since for every \( \varphi \in X^+ \) we have

\[
\prod_{k=0}^{n-1} T \circ f^k = \prod_{k=0}^{n-1} \varphi \circ f^k \circ f^{k+1} = \varphi \circ f^n \neq \varphi,
\]

it follows that \( T(X^+) \subset A_f^+ \). We shall show that \( T(X^+) = A_f^+ \). For a fixed \( g \in A_f^+ \) let \( \{G_n \}_{n \in \mathbb{N}} \) be defined by (2) and put \( G = \lim_n G_n \). Thus \( G \in X^+ \) and, in virtue of Theorem 2.2 from [5], \( G = g \circ G \circ f \). Therefore \( T \circ G = G \circ f \circ f = g \), which proves that \( T(X^+) = A_f^+ \). Hence and from the continuity of \( T \) we infer that \( A_f^+ \) is an analytic set, whence it has the property of Baire (cf. [7], Ch. XIII, § 1). Now, observe that \( A_f^+ \cdot A_f^+ = A_f^+ \) and, as follows from Theorem 1, \( C_f^+ \subset A_f^+ \). Thus, recalling Lemma 9 from [2], we infer that \( A_f^+ \) is the first category.

Now we shall show that \( B_f^+ \) is also of the first category. Let \( \{I_k \}_{k \in \mathbb{N}} \) be a basis of open subsets of \( I \). For \( k \in \mathbb{N} \) define \( B_{f,k} \) as the set of all functions \( g \in X^+ \) such that \( \lim \sup_{n \to \infty} G_n = 0 \) uniformly in \( I_k \). Obviously, \( B_f^+ = \bigcup_{k=1}^{\infty} B_{f,k} \) and it is enough to show that for every fixed positive integer \( k \) the set \( B_{f,k} \) is of the first category. To this end note that

\[
B_{f,k} = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \{ g \in X^+ : \forall x \in I_k, G_n(x) < \frac{1}{i} \},
\]

so \( B_{f,k} \) is an \( F_{\sigma \delta} \) subset of \( X^+ \) and therefore it has the Baire property. Moreover, it is easily seen that \( B_{f,k} \cap B_{f,k} = B_{f,k} \) and it follows from Theorem 1 that \( \text{Int} B_{f,k} = \emptyset \). Hence, making use of Lemma 9 from [2], we infer that every set \( B_{f,k} \) is of the first category and so is \( B_f^+ \).

Applying Theorems 1 and 2, we obtain the following corollary.

THEOREM 3. The sets \( A^+ \), \( B^+ \) and \( C^+ \) are dense in \( F \times X^+ \). Moreover, the set \( C^+ \) is residual in \( F \times X^+ \).

Proof. The density of the sets \( A^+ \), \( B^+ \) and \( C^+ \) follows immediately from the density of all the sections \( A_f^+ \), \( B_f^+ \), \( C_f^+ \). As in the proof of Theorem 2 we can show that the sets \( A^+ \) and \( B^+ \) have the property of Baire. Then we obtain our theorem via Theorem 15.4 from [8] and Theorem 2.

Remarks. In order to get similar results regarding the case where \( g(\xi) = -1 \) write

\[
X^+ = -X^+,
\]

\[
A^+ = \{ (f, g) \in F \times X^+ : \text{the sequence} \ G_{n} \text{converges pointwise to a function from} X^+ \},
\]

\[
B^- = \{ (f, g) \in F \times X^+ : \lim_{n \to \infty} G_n \text{uniformly on a subinterval of} I \},
\]

\[
C^- = F \times X^+ \setminus (A^+ \cup B^-).
\]

In virtue of the remark made at the beginning of our considerations the sets \( A^- \) and \( A_f^- \) for every \( f \in F \) are empty. Moreover, \( B_f^- = -B_f^+ \) and \( C_f^- = -(A_f^+ \cup C_f^+) \). In particular, the sets \( B_f^+ \), \( C_f^- \) for \( f \in F \), \( B^- \) and \( C^- \) have exactly the same properties as those established for the sets \( B_f^+ \), \( C_f^- \), for \( f \in F \), \( B^- \) and \( C^- \), respectively.

Write \( X = X^+ \cup X^- \) and observe that in view of Theorem 2.2 from [5] our results may be stated as follows:

In the indeterminate case almost all equations of the form (1) (which may be identified with elements of \( F \times X \)) have exactly one continuous solution \( \varphi : I \to \mathbb{R} \) (the zero function).

References

Maps of cotriples and a change of rings theorem

by

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Abstract. A well-known theorem on “change of rings” asserts that a ring homomorphism \( f : R \rightarrow S \) induces an inequality \( \dim_R(M) \leq \dim_S(M) + \dim_R(S) \) for each \( S \)-module \( M \). We generalize this result to a categorical setting and obtain an analogous “change of cotriples” inequality induced by a map of cotriples.

Suppose that \( S \) is a ring and \( M \) is a left \( S \)-module; we denote by \( \dim_S(M) \) the projective dimension of \( M \) as an \( S \)-module. A well-known theorem on “change of rings” (see [5, p. 172]) asserts that if \( f : R \rightarrow S \) is a ring homomorphism and \( M \) is an \( S \)-module, then \( M \) and \( S \) are \( R \)-modules in a natural way, and one has the inequality

\[
\dim_R(M) \leq \dim_S(M) + \dim_R(S).
\]

(*)

In this paper we obtain a similar inequality in a more general categorical setting. If \( A \) and \( X \) are additive categories with finite limits and \( G = (G, \epsilon, \delta) \) and \( H = (H, \epsilon', \delta') \) are suitable cotriples on \( A \) and \( X \), respectively, then one can define \( \dim_G(A) \) and \( \dim_H(X) \) via resolutions for all objects \( A \) of \( A \) and \( X \) of \( X \) (see [4]). We show that in this case a “nice” adjunction \((U, F)\) from \( A \) to \( X \) and morphisms of cotriples \( \varphi : FH \Rightarrow GF \) and \( \mu : HU \Rightarrow UG \) induce natural definitions of \( \dim_G(H) \) and \( \dim_H(U) \) and two “change of cotriples” inequalities analogous to (*).

To see the relationship between the “change of rings” theorem and the categorical situation described above, we begin by recalling the following facts. It is well known that if \( E : B \rightarrow A \) is a functor having a left adjoint \( F : A \rightarrow B \), then the adjoint pair \((E, F)\) induces a triple \( T \) on \( A \) and a cotriple \( G \) on \( B \) (see [7, p. 134]). It is also well known (see [3, p. 293]) that given a triple \( T \) or a cotriple \( G \) on a category \( A \) one can construct resolutions in \( A \) which generalize resolutions of modules over a ring in the following way. If \( S \) is a ring, \( s\text{-Mod} \) and \( s\text{-Sets} \) the categories of left \( S \)-modules and sets, respectively, \( E : s\text{-Mod} \rightarrow s\text{-Sets} \) the “underlying set” functor, and \( F : s\text{-Sets} \rightarrow s\text{-Mod} \) the “free \( S \)-module” functor, then \( F \) is left adjoint to \( E \). If \( G = (G_2, \epsilon_2, \delta_2) \) denotes the “free” cotriple on \( s\text{-Mod} \) induced by the adjoint pair \((E_3, F_3)\), then \( G_2 \)-resolutions in \( s\text{-Mod} \) are just projective resolutions. Now suppose that \( R \) and \( S \) are rings, so that we have “free” cotriples \( G_2 = (G_2, \epsilon_2, \delta_2) \) on