On linear homogeneous functional equations in the indeterminate case

by

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Abstract. Let I be a compact real interval and let ξ be a point of I. Denote by F the set of all continuous functions f: $I \to \boldsymbol{R}$ such that $0 < (f(x)-\xi)/(x-\xi) < 1$ for every $x \in I \setminus \{\xi\}$ and make X denote the set of all continuous functions g: $I \to \boldsymbol{R}$ such that $g(x) \neq 0$ for every $x \in I$ and $|g(\xi)| = 1$. Considering F and X as metric spaces with the uniform convergence metric (they are Baire spaces), we prove that for almost all pairs $(f, g) \in F \times X$ (all in the sense of the Baire category) the functional equation $\varphi \circ f = g\varphi$ has exactly one continuous solution φ: $I \to \boldsymbol{R}$ (just the zero function).

In this paper we deal with the problem of the number of continuous solutions φ of the functional equation

$$\varphi \circ f = g\varphi. \tag{1}$$

Let I be a real interval and let ξ be a point of I. The theory of continuous solutions of equation (1) has been developed under the following hypotheses (cf. [1], [4]–[6]):

(i) f: $I \to \boldsymbol{R}$ is continuous and $0 < (f(x)-\xi)/(x-\xi) < 1$ for $x \in I \setminus \{\xi\}$.

(ii) f is strictly increasing in a neighbourhood of ξ.

(iii) g: $I \to \boldsymbol{R}$ is continuous and $g(x) \neq 0$ for $x \in I \setminus \{\xi\}$.

Denote by f^n, $n \in N_0$(1), the nth iterate of f.

Remark 1. Under hypothesis (i) ξ is the unique fixed point of f, and the sequence $\{f^n(x)\}_{n \in N_0}$ is monotonic for every $x \in I$ and converges to ξ (cf. [5], Th. 0.4).

Let us write

$$G_n = \prod_{k=0}^{n-1} g \circ f^k, \qquad n \in N_0. \tag{2}$$

The following three cases are possible (cf. [5], Ch. II, § 2):

(A) There exists a continuous function G: $I \to \boldsymbol{R}$ such that $G(x) = \lim_{n \to \infty} G_n(x)$ and $G(x) \neq 0$ for every $x \in I$.

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(1) By N_0 we denote the set of all non-negative integers.

(B) $\lim_{n\to\infty} G_n = 0$ uniformly on a subinterval of I.

(C) Neither (A) nor (B) occurs.

The above three cases determine the number of continuous solutions of equation (1). Namely, Theorem 2.2 from [5] (cf. also [1]) states that:

If conditions (i), (iii) and (A) are fulfilled, then equation (1) has in I a unique one-parameter family of continuous solutions. These solutions are given by $\varphi(x) = c/G(x)$, where c is an arbitrary real constant.

If conditions (i)–(iii) and (B) are fulfilled, then equation (1) has in I a continuous solution depending on an arbitrary function.

If conditions (i), (iii) and (C) are fulfilled, then the zero function is the unique continuous solution of equation (1) in I.

Note that under assumptions (i) and (iii) case (A) may occur only if $g(\xi) = 1$ (otherwise the infinite product defining the sequence $\{G_n\}_{n\in N_0}$ diverges), but if $g(\xi) = 1$, then all the three cases (A), (B) and (C) can actually occur (see Lemma 1). If $|g(\xi)| < 1$, then case (B) occurs (cf. [5], Th. 2.3). If $|g(\xi)| > 1$, then case (C) occurs (cf. [5], Th. 2.4). The case $|g(\xi)| = 1$ is called indeterminante.

In this paper we shall show that in the indeterminate case for almost all equations (all in the sense of the Baire category) of form (1) case (C) holds.

Let I be a compact interval. In the sequel the space of all functions f fulfilling (i) is denoted by F whereas X^+ denotes the space of all functions g fulfilling (iii) and the condition $g(\xi) = 1$. We may treat F and X^+ as metric spaces, endowing them with the uniform convergence metric. Observe that they are Baire spaces. Namely, if we endow the space of all real functions defined and continuous in I with the uniform convergence metric, then F and X^+, as is easy to show, are its G_δ subsets, and thus, by Theorem of Alexandroff (cf. [8], Th. 12.1), they are topologically complete and consequently (cf. [8], Th. 9.1) they are Baire spaces.

Let us define

$$A^+ = \{(f, g)\in F\times X^+:\ \text{the sequence } \{G_n\}_{n\in N} \text{ converges pointwise to a function from } X^+\},$$
$$B^+ = \{(f, g)\in F\times X^+:\ \lim_{n\to\infty} G_n = 0 \text{ uniformly on a subinterval on } I\},$$
$$C^+ = F\times X^+\setminus(A^+\cup B^+)$$

and, for every $E\subset F\times X^+$ and $f\in F$, put $E_f = \{g\in X^+:\ (f, g)\in E\}$.

We start with the following lemma.

LEMMA 1. *If $f\in F$, then A_f^+, B_f^+ and C_f^+ are non-empty sets.*

Proof. We may assume without loss of generality that ξ is the left endpoint of I. Since the function identically equal to one belongs to A_f^+, we have $A_f^+ \neq \emptyset$.

Let $\bar f$ be a strictly increasing function belonging to F and such that $\bar f \leqslant f$ (2). Choose a sequence $\{a_n\}_{n\in N}$ of positive numbers such that $\{a_n\}_{n\in N}$ increases to one and $\prod_{n=1}^{\infty} a_n = 0$. Further, fix an $x_0\in I\setminus\{\xi\}$. In virtue of Remark 1 we can find a decreasing function $g\in X^+$ such that $g[\bar f^k(x_0)] = a_k$ for $k\in N$. Since $\bar f$ is strictly increasing, so is $\bar f^n$ and $\bar f^n \leqslant f^n$ for every $n\in N_0$. Thus, for every $x\in[\bar f(x_0), x_0]$ and $n\in N$, we get

$$0 \leqslant \prod_{k=0}^{n-1} g[f^k(x)] \leqslant \prod_{k=0}^{n-1} g[\bar f^k(x)] \leqslant \prod_{k=0}^{n-1} g[\bar f^{k+1}(x_0)] = \prod_{k=1}^{n} a_k.$$

Consequently, $\lim_{n\to\infty} G_n = 0$ uniformly in $[\bar f(x_0), x_0]\subset I$, and so $g\in B_f^+$.

Finally, we shall show that C_f^+ is a non-void set. Indeed, let us take into account the function g constructed above and observe that the function $g_1 = 2-g$ is an element of X^+. Since $g\leqslant 1$ and $\prod_{k=0}^{\infty} g[f^k(x_0)] = 0$, we have

$$\sum_{k=0}^{\infty} |g_1[f^k(x_0)]-1| = \sum_{k=0}^{\infty} (g_1[f^k(x_0)]-1) = \sum_{k=0}^{\infty} (1-g[f^k(x_0)])$$
$$= \sum_{k=0}^{\infty} |g[f^k(x_0)]-1| = \infty.$$

Consequently, $\prod_{k=0}^{\infty} g_1[f^k(x_0)] = \infty$ and $g_1\notin A_f^+$. Obviously, since $g_1\geqslant 1$, g_1 is not an element of B_f^+. Thus it belongs to C_f^+.

LEMMA 2. *Suppose that $f\in F$. Every two elements of X^+ coinciding in a neighbourhood of ξ belong simultaneously to one of the sets A_f^+, B_f^+, C_f^+.*

Proof. Let $g, \bar g\in X^+$ coincide in a neighbourhood V of ξ. On account of Lemma 1 from [3] we have $f^m(I)\subset V$ for an $m\in N$. Write $\bar G_n = \prod_{k=0}^{n-1} \bar g\circ f^k$ and note that

$$\bar G_n(x) = \prod_{k=0}^{n-1} \bar g[f^k(x)] = \prod_{k=0}^{m-1} \bar g[f^k(x)] \prod_{k=m}^{n-1} \bar g[f^k(x)]$$
$$= \bar G_m(x) \prod_{k=m}^{n-1} g[f^k(x)] = [\bar G_m(x)/G_m(x)]\, G_n(x)$$

for $x\in I$ and $n > m$. Making use of this formula, we get our assertion.

Now we pass to our main results.

(2) We may take for instance

$$\bar f(x) = \xi + \frac{1}{d}(x-\xi)\,(\min\{f(t):\ t\in I,\ t\geqslant x\}-\xi) \quad \text{for} \quad x\in I,$$

where $d = \sup\{x-\xi:\ x\in I\}$.

THEOREM 1. *If $f \in F$, then the sets A_f^+, B_f^+ and C_f^+ are dense in X^+.*

Proof. Fix a $g \in X^+$ and a positive real number $\varepsilon < \inf\{g(x): x \in I\}$. In view of Lemma 1 we can find a function g_1 belonging to A_f^+. Since g and g_1 are continuous and $g(\xi) = g_1(\xi)$, we have $|g(x) - g_1(x)| < \varepsilon$ for every x from a closed neighbourhood V of ξ. On account of Tietze's theorem there exists a continuous function $\bar{g}: I \to \mathbf{R}$ such that

(3) $$\bar{g}|_V = g_1|_V$$

and

(4) $$|g(x) - \bar{g}(x)| < \varepsilon \quad \text{for} \quad x \in I.$$

Observe that according to (3) and to the choice of ε we have $\bar{g}(\xi) = g_1(\xi) = 1$ and $\bar{g}(x) \geqslant g(x) - \varepsilon > 0$ for every $x \in I$, which shows that $\bar{g}$ belongs to X^+. By (3) and Lemma 2 the function $\bar{g}$ belongs to A_f^+, which, jointly with (4), ends the proof of the density of A_f^+ in X^+. The reasoning for sets B_f^+ and C_f^+ is analogous.

THEOREM 2. *If $f \in F$, then the set C_f^+ is residual in X^+.*

Proof. It is enough to show that the sets A_f^+ and B_f^+ are of the first Baire category. In order to prove that A_f^+ is of the first category let us define a mapping $T: X^+ \to X^+$ by the formula

$$T\varphi = \varphi/\varphi \circ f \quad \text{for} \quad \varphi \in X^+.$$

Since for every $\varphi \in X^+$ we have

$$\prod_{k=0}^{n-1} T\varphi \circ f^k = \prod_{k=0}^{n-1} \varphi \circ f^k / \varphi \circ f^{k+1} = \varphi/\varphi \circ f^n \underset{n}{\to} \varphi,$$

it follows that $T(X^+) \subset A_f^+$. We shall show that $T(X^+) = A_f^+$. For a fixed $g \in A_f^+$ let $\{G_n\}_{n \in N}$ be defined by (2) and put $G = \lim_{n\to\infty} G_n$. Thus $G \in X^+$ and, in virtue of Theorem 2.2 from [5], $G = gG \circ f$. Therefore $TG = G/G \circ f = g$, which proves that $T(X^+) = A_f^+$. Hence and from the continuity of T we infer that A_f^+ is an analytic set, whence it has the property of Baire (cf. [7], Ch. XIII, § 1). Now, observe that $A_f^+ \cdot A_f^+ = A_f^+$ and, as follows from Theorem 1, int $A_f^+ = \emptyset$. Thus, recalling Lemma 9 from [2], we infer that A_f^+ is of the first category.

Now we shall show that B_f^+ is also of the first category. Let $\{I_n\}_{n \in N}$ be a basis of open subsets of I. For $k \in N$ define $B_{f,k}^+$ as the set of all functions $g \in X^+$ such that $\lim_{n\to\infty} G_n = 0$ uniformly in I_k. Obviously, $B_f^+ = \bigcup_{k=1}^{\infty} B_{f,k}^+$ and it is enough to show that for every fixed positive integer k the set $B_{f,k}^+$ is of the first category. To this end note that

$$B_{f,k}^+ = \bigcap_{i=1}^{\infty} \bigcup_{m=1}^{\infty} \{g \in X^+: \underset{x \in I_k}{\forall}\ \underset{n \geqslant m}{\forall}\ G_n(x) \leqslant 1/i\},$$

so $B_{f,k}^+$ is an $F_{\sigma,\delta}$ subset of X^+ and therefore it has the Baire property. Moreover, it is easily seen that $B_{f,k}^+ \cdot B_{f,k}^+ \subset B_{f,k}^+$ and it follows from Theorem 1 that int $B_{f,k}^+ = \emptyset$. Hence, making use of Lemma 9 from [2], we infer that every set $B_{f,k}^+$ is of the first category and so is B_f^+.

Applying Theorems 1 and 2, we obtain the following corollary.

THEOREM 3. *The sets A^+, B^+ and C^+ are dense in $F \times X^+$. Moreover, the set C^+ is residual in $F \times X^+$.*

Proof. The density of the sets A^+, B^+ and C^+ follows immediately from the density of all the sections A_f^+, B_f^+, C_f^+. As in the proof of Theorem 2 we can show that the sets A^+ and B^+ have the property of Baire. Then we obtain our theorem via Theorem 15.4 from [8] and Theorem 2.

Remarks. In order to get similar results regarding the case where $g(\xi) = -1$ write

$X^- = -X^+$,

$A^- = \{(f, g) \in F \times X^-$: the sequence $\{G_n\}_{n \in N}$ converges pointwise to a function from $X^-\}$,

$B^- = \{(f, g) \in F \times X^-: \lim_{n\to\infty} G_n = 0$ uniformly on a subinterval of $I\}$,

$C^- = F \times X^- \setminus (A^- \cup B^-)$.

In virtue of the remark made at the beginning of our considerations the sets A^- and A_f^- for every $f \in F$ are empty. Moreover, $B_f^- = -B_f^+$ and $C_f^- = -(A_f^+ \cup C_f^+)$. In particular, the sets B_f^-, C_f^- for $f \in F$, B^- and C^- have exactly the same properties as those established for the sets B_f^+, C_f^+ for $f \in F$, B^+ and C^+, respectively.

Write $X = X^+ \cup X^-$ and observe that in view of Theorem 2.2 from [5] our results may be stated as follows:

In the indeterminate case almost all equations of the form (1) (which may be identified with elements of $F \times X$) have exactly one continuous solution $\varphi: I \to \mathbf{R}$ (the zero function).

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Maps of cotriples and a change of rings theorem

by

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Abstract. A well-known theorem on "change of rings" asserts that a ring homomorphism $f\colon R \to S$ induces an inequality $\dim_R(M) \leqslant \dim_S(M) + \dim_R(S)$ for each S-module M. We generalize this result to a categorical setting and obtain an analogous "change of cotriples" inequality induced by a map of cotriples.

Suppose that S is a ring and M is a left S-module; we denote by $\dim_S(M)$ the projective dimension of M as an S-module. A well-known theorem on "change of rings" (see [5, p. 172]) asserts that if $f\colon R \to S$ is a ring homomorphism and M is an S-module, then M and S are R-modules in a natural way, and one has the inequality

$$\dim_R(M) \leqslant \dim_S(M) + \dim_R(S). \qquad (*)$$

In this paper we obtain a similar inequality in a more general categorical setting. If $\boldsymbol{A}$ and $\boldsymbol{X}$ are additive categories with finite limits and $\boldsymbol{G} = (G, \varepsilon, \delta)$ and $\boldsymbol{H} = (H, \varepsilon', \delta')$ are suitable cotriples on $\boldsymbol{A}$ and $\boldsymbol{X}$, respectively, then one can define $\dim_{\boldsymbol{G}}(A)$ and $\dim_{\boldsymbol{H}}(X)$ via resolutions for all objects A of $\boldsymbol{A}$ and X of $\boldsymbol{X}$ (see [4]). We show that in this case a "nice" adjunction (U, F) from $\boldsymbol{A}$ to $\boldsymbol{X}$ and morphisms of cotriples $\tau\colon FH \Rightarrow GF$ and $\mu\colon HU \Rightarrow UG$ induce natural definitions of $\dim_{\boldsymbol{H}}(\boldsymbol{G})$ and $\dim_{\boldsymbol{G}}(\boldsymbol{H})$ and two "change of cotriples" inequalities analogous to $(*)$.

To see the relationship between the "change of rings" theorem and the categorical situation described above, we begin by recalling the following facts. It is well known that if $E\colon \boldsymbol{B} \to \boldsymbol{A}$ is a functor having a left adjoint $F\colon \boldsymbol{A} \to \boldsymbol{B}$, then the adjoint pair (E, F) induces a triple $\boldsymbol{T}$ on $\boldsymbol{A}$ and a cotriple $\boldsymbol{G}$ on $\boldsymbol{B}$ (see [7, p. 134]). It is also well known (see [1, p. 290]) that given a triple $\boldsymbol{T}$ or a cotriple $\boldsymbol{G}$ on a category $\boldsymbol{A}$ one can construct resolutions in $\boldsymbol{A}$ which generalize resolutions of modules over a ring in the following way. If S is a ring, ${}_S\boldsymbol{Mod}$ and $\boldsymbol{Sets}$ the categories of left S-modules and sets, respectively, $E_S\colon {}_S\boldsymbol{Mod} \to \boldsymbol{Sets}$ the "underlying set" functor, and $F_S\colon \boldsymbol{Sets} \to {}_S\boldsymbol{Mod}$ the "free S-module" functor, then F_S is left adjoint to E_S. If $\boldsymbol{G}_S = (G_S, \varepsilon_S, \delta_S)$ denotes the "free" cotriple on ${}_S\boldsymbol{Mod}$ induced by the adjoint pair (E_S, F_S), then $\boldsymbol{G}_S$-resolutions in ${}_S\boldsymbol{Mod}$ are just projective resolutions. Now suppose that R and S are rings, so that we have "free" cotriples $\boldsymbol{G}_R = (G_R, \varepsilon_R, \delta_R)$ on