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A generalization of Płonka sums

by

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Abstract. In this note we shall consider a method of constructing algebras which is a generalization of Płonka's sum of a semilattice ordered family of algebras. We show that an equational class is closed under the formation of generalized Płonka sums if and only if the equational class under consideration is regular. We provide an example of a generalized Płonka sum that is not equivalent to a Płonka sum.

We shall only consider algebras with finitary operations and without nullary operations.

A semilattice ordered family of sets is a triplet consisting of

(i) a meet semilattice I, with the semilattice order $\leqslant$,

(ii) a family of sets $\langle A_i, i\in I\rangle$,

(iii) a family of mappings $\langle \varphi_{ji}, i,j\in I, i\leqslant j\rangle$ where, for each $i,j\in I$, $i\leqslant j$, φ_{ji} maps A_j into A_i, such that the following conditions are satisfied: for each $i\in I$, φ_{ii} is the identity mapping on A_i, and for all $i,j,k\in I$, with $i\leqslant j\leqslant k$, we have $\varphi_{ji}\varphi_{kj}=\varphi_{ki}$ (see [1], § 21).

Let us now suppose that for each $i\in I$, $\mathfrak{A}_i=\langle A_i; F_i\rangle$ is an algebra. We shall hereby suppose that the algebras $\mathfrak{A}_i$, $i\in I$, are all of type τ, and that the carriers A_i, $i\in I$, are pairwise disjoint. For each $i\in I$ we put $F_i=\langle F_t^{(i)}, t\in T\rangle$. The system

$$\mathfrak{A}=\langle I;\ \langle\mathfrak{A}_i,\ i\in I\rangle;\ \langle\varphi_{ji},\ i,j\in I,\ i\leqslant j\rangle\rangle$$

is of course not a Płonka system (a semilattice ordered family of algebras in the sense of [1], § 21), because in generall the mappings φ_{ji}, $i,j\in I$, $i\leqslant j$, do not give rise to homomorphisms.

We define an algebra $S(\mathfrak{A})$, which we call the sum of the system $\mathfrak{A}$, in the obvious way: the carrier $A=\bigcup_{i\in I} A_i$ of $S(\mathfrak{A})$ is the disjoint union of the carriers of the algebras $\mathfrak{A}_i$, $i\in I$, and the fundamental operations of $S(\mathfrak{A})$ are defined by

$$F_t(a_1,\ldots,a_n)=F_t^{(i_0)}(\varphi_{i_1i_0}(a_1),\ldots,\varphi_{i_ni_0}(a_n))$$

for all $t\in T$, with $a_r\in A_{i_r}$, $r=1,\ldots,n=\tau(t)$, and $i_0=\bigwedge_{r=1}^{n} i_r$. So far the only interesting thing we can say about this sum $S(\mathfrak{A}) = \langle A; \langle F_t, t \in T \rangle \rangle$ is that $S(\mathfrak{A})$ is of type τ.

For every $t \in T$, $i \in I$, $a \in A_i$ and $r \in \{1, \ldots, \tau(t)\}$, let $F^{(i)}_{t,r,a}$ be the $(\tau(t)-1)$-ary operation on A_i which is defined by

$$F^{(i)}_{t,r,a}(b_1, \ldots, b_{\tau(t)-1}) = F^{(i)}_t(b_1, \ldots, b_{r-1}, a, b_r, \ldots, b_{\tau(t)-1}),$$

and let us put

$$F^{(i)}_a = \langle F^{(i)}_{t,r,a}, t \in T, r \in \{1, \ldots, \tau(t)\} \rangle$$

and

$$F^{(i)} = \{F^{(i)}_a \mid a \in A_i\}.$$

Let $\tilde{\mathfrak{A}}_i$ be an algebra with carrier $F^{(i)}$, where the fundamental operations are defined by

$$\bar{F}^{(i)}_t(F^{(i)}_{a_1}, \ldots, F^{(i)}_{a_{\tau(t)}}) = F^{(i)}_{F_t(a_1, \ldots, a_{\tau(t)})}$$

for all $t \in T$. Obviously $\varphi_i\colon \mathfrak{A}_i \to \tilde{\mathfrak{A}}_i$, $a \to F^{(i)}_a$ is a homomorphism of $\mathfrak{A}_i$ onto $\tilde{\mathfrak{A}}_i$ which we shall call the canonical homomorphism of $\mathfrak{A}_i$ onto $\tilde{\mathfrak{A}}_i$.

Remark 1. Let us note that the algebra $\tilde{\mathfrak{A}}_i$ can be defined as the quotient algebra $\mathfrak{A}_i/\theta_i$ where the congruence θ_i is defined in the following way: if $i \in I$ and $a, a' \in A_i$ then $a\theta a'$ if and only if, for each $t \in T$, $b_1, \ldots, b_{\tau(t)-1} \in A_i$ and $r \in \{1, \ldots, \tau(t)\}$,

$$F^{(i)}_t(b_1, \ldots, b_{r-1}, a, b_r, \ldots, b_{\tau(t)-1}) = F^{(i)}_t(b_1, \ldots, b_{r-1}, a', b_r, \ldots, b_{\tau(t)-1}).$$

Our first theorem compares the following conditions, (C1) and (C2), on the sum $S(\mathfrak{A})$ of the system $\mathfrak{A}$:

(C1) Let p be any n-ary polynomial of $S(\mathfrak{A})$ induced by an n-ary polynomial symbol $\boldsymbol{p}$ of type τ. Let $a_r \in A_{i_r}$, $r = 1, \ldots, n$ be any n elements of A, with $p(a_1, \ldots, a_n) \in A_{i_0}$ in $S(\mathfrak{A})$. Then

$$p(a_1, \ldots, a_n) = p(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n)),$$

where $\varphi^*_{i_r i_0}(a_r) = \varphi_{i_r i_0}(a_r)$ if $\boldsymbol{x}_r$ is a variable of $\boldsymbol{p}$ and where $\varphi^*_{i_r i_0}(a_r)$ is any element of A_{i_0} otherwise.

(C2) For any $i, j \in I$, $i \leqslant j$, the mapping $\mathfrak{A}_j \to \tilde{\mathfrak{A}}_i$, $a \to F^{(i)}_{\varphi_{ji}(a)}$ is a homomorphism.

Theorem 1. *Condition* (C2) *for the sum* $S(\mathfrak{A})$ *of the system* $\mathfrak{A}$ *implies* (C1). *If there are no unary operations then* (C1) *and* (C2) *are equivalent.*

Proof. Let us suppose that (C2) holds. Since, for all $i \in I$, φ_{ii} is the identity mapping on A_i, we immediately infer that (C1) holds if $\boldsymbol{p}$ is a variable. Let us now suppose that $\boldsymbol{p} = \boldsymbol{F}_t(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m)$ for $t \in T$, $m = \tau(t)$ and n-ary polynomial symbols $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$ of type τ, and let us suppose that (C1) is satisfied for $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m$. Let $a_r \in A_{i_r}$, $r = 1, \ldots, n$, be any n elements of A with

$p(a_1, \ldots, a_n) \in A_{i_0}$ in $S(\mathfrak{A})$. If $m = \tau(t) = 1$, then both $p_1(a_1, \ldots, a_n)$ and $p(a_1, \ldots, a_n)$ belong to A_{i_0}, and so

$$\begin{aligned} p(a_1, \ldots, a_n) &= F_t(p_1(a_1, \ldots, a_n)) \\ &= F_t\big(p_1(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n))\big) \\ &= p(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n)). \end{aligned}$$

If $m = \tau(t) > 1$, and $p_i(a_1, \ldots, a_n) \in A_{k_i}$, then

$$\begin{aligned} p(a_1, \ldots, a_n) &= F_t(p_1(a_1, \ldots, a_n), \ldots, p_m(a_1, \ldots, a_n)) \\ &= F_t\big(p_1(\varphi^*_{i_1 k_1}(a_1), \ldots, \varphi^*_{i_n k_1}(a_n)), \ldots, p_m(\varphi^*_{i_1 k_m}(a_1), \ldots, \varphi^*_{i_n k_m}(a_n))\big) \\ &= F_t\Big(\varphi_{k_1 i_0}\big(p_1(\varphi^*_{i_1 k_1}(a_1), \ldots, \varphi^*_{i_n k_1}(a_n))\big), \ldots \\ &\qquad \ldots, \varphi_{k_m i_0}\big(p_m(\varphi^*_{i_1 k_m}(a_1), \ldots, \varphi^*_{i_n k_m}(a_n))\big)\Big) \\ &= F^{(i_0)}_{t,1,\varphi_{k_1 i_0}(p_1(\varphi^*_{i_1 k_1}(a_1), \ldots, \varphi^*_{i_n k_1}(a_n)))}\Big(\varphi_{k_2 i_0}\big(p_2(\varphi^*_{i_1 k_2}(a_1), \ldots,)\big) \ldots \\ &\qquad \ldots, \varphi_{k_m i_0}\big(p_m(\ldots, \varphi^*_{i_n k_m}(a_n))\big)\Big) \\ &= F^{(i_0)}_{t,1,p_1(\varphi_{k_1 i_0}\varphi^*_{i_1 k_1}(a_1), \ldots, \varphi_{k_1 i_0}\varphi^*_{i_n k_1}(a_n))}\Big(\varphi_{k_2 i_0}\big(p_2(\varphi^*_{i_1 k_2}(a_1), \ldots,)\big), \ldots \\ &\qquad \ldots, \varphi_{k_m i_0}\big(p_m(\ldots, \varphi^*_{i_n k_m}(a_n))\big)\Big) \\ &= F_t\Big(p_1(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n)), \varphi_{k_2 i_0}\big(p_2(\varphi^*_{i_1 k_2}(a_1), \ldots)\big), \ldots \\ &\qquad \ldots, \varphi_{k_m i_0}\big(p_m(\ldots, \varphi^*_{i_n k_m}(a_n))\big)\Big) \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= F_t\Big(p_1\big(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n), \ldots, p_m(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n))\big)\Big) \\ &= p(\varphi^*_{i_1 i_0}(a_1), \ldots, \varphi^*_{i_n i_0}(a_n)). \end{aligned}$$

It follows by induction that (C2) implies (C1).

Let us now suppose that there are no unary operations and that (C1) holds. Let $s, t \in T$, with $n = \tau(t) > 1$ and $m = \tau(s) > 1$, let $i, j \in I$, $i \leqslant j$, and let $a_1, \ldots, a_n \in A_j$ and $b_1, \ldots, b_{m-1} \in A_i$. Then for any $r \in \{1, \ldots, m\}$ we have

$$\begin{aligned} & F^{(i)}_{s,r,\varphi_{ji}(F_t(a_1, \ldots, a_n))}(b_1, \ldots, b_{m-1}) \\ &\quad = F^{(i)}_s\big(b_1, \ldots, b_{r-1}, \varphi_{ji}(F_t(a_1, \ldots, a_n)), b_r, \ldots, b_{m-1}\big) \\ &\quad = F_s(b_1, \ldots, b_{r-1}, F_t(a_1, \ldots, a_n), b_r, \ldots, b_{m-1}) \\ &\quad = F_s\big(b_1, \ldots, b_{r-1}, F_t(\varphi_{ji}(a_1), \ldots, \varphi_{ji}(a_n)), b_r, \ldots, b_{m-1}\big) \\ &\quad = F^{(i)}_{s,r,F_t(\varphi_{ji}(a_1), \ldots, \varphi_{ji}(a_n))}(b_1, \ldots, b_{m-1}). \end{aligned}$$

Hence

$$F^{(i)}_{\varphi_{ji}(F_t(a_1,\ldots,a_n))} = F^{(i)}_{F_t(\varphi_{ji}(a_1),\ldots,\varphi_{ji}(a_n))} = \bar{F}^{(i)}_t(F^{(i)}_{\varphi_{ji}(a_1)}, \ldots, F^{(i)}_{\varphi_{ji}(a_n)})$$

always holds, and we may conclude that (C2) holds.

If the system $\mathfrak{A}$ satisfies condition (C2) of the foregoing theorem, then we shall call $\mathfrak{A}$ a *generalized Płonka system*, and we shall call $S(\mathfrak{A})$ the *generalized Płonka sum* of the generalized Płonka system $\mathfrak{A}$. It should be obvious that a Płonka sum of a semilattice ordered family of algebras (in the sense of [2], [4]) is a special case of a generalized Płonka sum.

If K is an equational class of algebras of type τ, let $R(K)$ denote the equational class of algebras defined by all the regular identities (in the sense of [4]) holding in K. The equivalence of (i) and (ii) in the following corollary is of course a result of [4].

Corollary 1. *The following are equivalent for the equational class K:*
(i) $R(K) = K$,
(ii) *K is closed for taking Płonka sums,*
(iii) *K is closed for taking generalized Płonka sums.*

If for every algebra $\mathscr{C}$ of an equational class K the canonical homomorphism $\mathscr{C} \to \bar{\mathscr{C}}$ is an isomorphism, then of course Płonka sums coincide with generalized Płonka sums, and in this case Corollary 1 does not contain any new information. The equational classes of lattices, monoids and idempotent semigroups constitute examples for this situation.

Let

$$\mathfrak{A} = \langle I;\ \langle \mathfrak{A}_i,\ i \in I\rangle;\ \langle \varphi_{ji},\ i, j \in I,\ i \leqslant j\rangle\rangle$$

and

$$\mathfrak{A}' = \langle I;\ \langle \mathfrak{A}_i,\ i \in I\rangle;\ \langle \varphi'_{ji},\ i, j \in I,\ i \leqslant j\rangle\rangle$$

be generalized Płonka systems. The sums $S(\mathfrak{A})$ and $S(\mathfrak{A}')$ will be called *equivalent* if $S(\mathfrak{A}) \to S(\mathfrak{A}')$, $a \to a$ is an isomorphism. We shall now give an example of a generalized Płonka sum which is not equivalent to a Płonka sum.

Example 1. Let the generalized Płonka sum $S(\mathfrak{A})$ be equivalent to the Płonka sum $S(\mathfrak{A}')$, where the systems $\mathfrak{A}$ and $\mathfrak{A}'$ are as above, and where for each $t \in T$, $\tau(t) > 1$. Then for all $i, j \in I$, with $i \leqslant j$, all $t \in T$, all $r \in \{1, \ldots, \tau(t)\}$, all $a \in A_j$ and all $b_1, \ldots, b_{\tau(t)-1} \in A_i$ we have

$$\begin{aligned} F^{(i)}_{t,r,\varphi_{ji}(a)}(b_1, \ldots, b_{\tau(t)-1}) &= F^{(i)}_t(b_1, \ldots, b_{r-1}, \varphi_{ji}(a), b_r, \ldots, b_{\tau(t)-1}) \\ &= F_t(b_1, \ldots, b_{r-1}, a, b_r, \ldots, b_{\tau(t)-1}) \\ &= F^{(i)}_t(b_1, \ldots, b_{r-1}, \varphi'_{ji}(a), b_r, \ldots, b_{\tau(t)-1}) \\ &= F^{(i)}_{t,r,\varphi'_{ji}(a)}(b_1, \ldots, b_{\tau(t)-1}), \end{aligned}$$

and therefore

$$F^{(i)}_{t,r,\varphi_{ji}(a)} = F^{(i)}_{t,r,\varphi'_{ji}(a)}.$$

Thus, for all $i, j \in I$, $i \leqslant j$, we have

$$\varphi_i \varphi_{ji} = \varphi_i \varphi'_{ji}.$$

If in particular $\varphi_i \varphi_{ji}$ is an isomorphism of $\mathfrak{A}_j$ onto $\bar{\mathfrak{A}}_i$, then the homomorphism φ'_{ji} must be injective and $\varphi'_{ji}(\mathfrak{A}_j)$ is a transversal in $\mathfrak{A}_i$ with respect to the congruence which is induced on $\mathfrak{A}_i$ by φ_i: $\mathfrak{A}_i$ splits over the congruence which is induced on $\mathfrak{A}_i$ by φ_i. Let us now turn to the equational class K of semigroups. Let $I = \{0, 1\}$ be the two-element semilattice, with $0 < 1$ and let

$$\mathfrak{A} = \langle I;\ \langle \mathfrak{A}_0, \mathfrak{A}_1\rangle;\ \langle \varphi_{00}, \varphi_{10}, \varphi_{11}\rangle\rangle$$

be a generalized Płonka system of semigroups. Using the terminology of [3], III. 3 and III. 4, we can say that $S(\mathfrak{A})$ is a Płonka sum if and only if $S(\mathfrak{A})$ is a retract extension of $\mathfrak{A}_0$, and that $S(\mathfrak{A})$ is a generalized Płonka sum if and only if $S(\mathfrak{A})$ is a strict extension of $\mathfrak{A}_0$. In particular, let the multiplication of $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be given by

$\mathfrak{A}_0$	p_0	q_0	a_0	b_0	o_0
p_0	p_0	p_0	a_0	a_0	o_0
q_0	q_0	q_0	b_0	b_0	o_0
a_0	o_0	o_0	o_0	o_0	o_0
b_0	o_0	o_0	o_0	o_0	o_0
o_0	o_0	o_0	o_0	o_0	o_0

$\mathfrak{A}_1$	p_1	q_1	a_1	o_1
p_1	p_1	p_1	a_1	o_1
q_1	q_1	q_1	a_1	o_1
a_1	o_1	o_1	o_1	o_1
o_1	o_1	o_1	o_1	o_1

Let φ_{10} be given by

$$\varphi_{10}\colon\ p_1 \to p_0,\ q_1 \to q_0,\ a_1 \to a_0,\ o_1 \to o_0.$$

It is easy to see that $\varphi_0 \varphi_{10}$ is an isomorphism of $\mathfrak{A}_1$ onto $\bar{\mathfrak{A}}_0$. Remark that $\{a_0, b_0\}$ is the only non-trivial congruence class of the congruence which is induced on $\mathfrak{A}_0$ by the canonical homomorphism φ_0: $\mathfrak{A}_0 \to \bar{\mathfrak{A}}_0$. Thus, $\mathfrak{A}_0$ does not split over this congruence $\varphi_0^{-1}\varphi_0$. Consequently, $S(\mathfrak{A})$ is a generalized Płonka sum which is not equivalent to a Płonka sum.

We shall now establish a decomposition of generalized Płonka sums as a subdirect product of special ones. Therefore we first have the following theorem.

Theorem 2. *Let $\mathfrak{A}$ be an algebra of type τ, and let I be a semilattice with least element 0. For all $i \in I$, $i \neq 0$, let $\bar{\mathfrak{A}}^{(i)}$ be a subalgebra of $\bar{\mathfrak{A}}$ such that for all $i, j \in I$, with $0 \neq i \leqslant j$, we have $\bar{\mathfrak{A}}^{(j)} \subseteq \bar{\mathfrak{A}}^{(i)}$. If φ is the canonical homomorphism φ: $\mathfrak{A} \to \bar{\mathfrak{A}}$, let φ' be a mapping φ': $\bar{\mathfrak{A}} \to \mathfrak{A}$ such that $\varphi\varphi'$ is the identity* *transformation on* $\mathfrak{A}$. *Let* $\langle \mathscr{B}_i,\ i\in I\rangle$ *be a family of pairwise disjoint algebras, such that for all* $i\in I$, $i\neq 0$, $\bar{\psi}_i\colon \mathscr{B}_i \to \mathfrak{A}^{(i)}$ *is an isomorphism, and* $\mathscr{B}_0 = \mathfrak{A}$. *Then*

$$\mathscr{B} = \langle I;\ \langle \mathscr{B}_i,\ i\in I\rangle;\ \langle \psi_{ji},\ i,j\in I,\ i\leqslant j\rangle\rangle$$

is a generalized Płonka system, where
$\psi_{ji} = \bar{\psi}_i^{-1}\bar{\psi}_j$ *if* $0\neq i\leqslant j$,
ψ_{00} *is the identity transformation on* $\mathscr{B}_0 = \mathfrak{A}$,
$\psi_{j0} = \varphi'\bar{\psi}_j$ *if* $0<j$.

Proof. It is clear that for each $i\in I$, ψ_{ii} is the identity mapping on $\mathscr{B}_i$. Let $i,j,k\in I$ be such that $i<j<k$. If $i\neq 0$, then

$$\psi_{ji}\psi_{kj} = \bar{\psi}_i^{-1}\bar{\psi}_j\bar{\psi}_j^{-1}\bar{\psi}_k = \bar{\psi}_i^{-1}\bar{\psi}_k = \psi_{ki}$$

since $\mathfrak{A}^{(j)} \supseteq \mathfrak{A}^{(k)}$. If $i=0$, then

$$\psi_{j0}\psi_{kj} = \varphi'\bar{\psi}_j\bar{\psi}_j^{-1}\bar{\psi}_k = \varphi'\bar{\psi}_k = \psi_{k0},$$

for the same reason.

For all $i,j\in I$, with $0\neq i\leqslant j$, the mapping ψ_{ji} is an injective homomorphism of $\mathscr{B}_j$ into $\mathscr{B}_i$, and in this case the mapping $\mathscr{B}_j \to \bar{\mathscr{B}}_i$, $a\to F^{(i)}_{\psi_{ji}(a)}$ is clearly a homomorphism. If $j>0$, then the mapping $\mathscr{B}_j\to\bar{\mathscr{B}}_0 = \bar{\mathfrak{A}}$, $a\to F^{(0)}_{\psi_{j0}(a)}$ is precisely the injective homomorphism $\bar{\psi}_j$. We conclude that $\mathscr{B}$ is a generalized Płonka system.

Remark 2. Remark that the generalized Płonka system $\mathscr{B}$ of Theorem 2 has been constructed from one single algebra $\mathfrak{A}$, a semilattice I and injective homomorphisms $\bar{\psi}_i$, $i\in I$, $i\neq 0$. One can show that the generalized Płonka sum $S(\mathscr{B})$ does not depend on the choice of φ'. Indeed, let φ'' be another mapping $\varphi''\colon \mathfrak{A}\to\mathfrak{A}$ such that $\varphi\varphi''$ is the identity transformation on $\mathfrak{A}$, put $\psi'_{ji} = \psi_{ji}$ if $0\neq i\leqslant j$, $\psi'_{00} = \psi_{00}$ and $\psi'_{j0} = \varphi''\bar{\psi}_j$ if $0<j$. Then

$$\mathscr{B}' = \langle I;\ \langle \mathscr{B}_i,\ i\in I\rangle;\ \langle \psi'_{ji},\ i,j\in I,\ i\leqslant j\rangle\rangle$$

is again a generalized Płonka system. We shall use the notations $S(\mathscr{B}) = \langle B;\ \langle F_t,\ t\in T\rangle\rangle$, $S(\mathscr{B}') = \langle B;\ \langle F'_t,\ t\in T\rangle\rangle$. For any $t\in T$, $a_r\in B_{i_r}$, $r=1,\ldots,n=\tau(t)$ and $i_0 = \bigwedge_{r=1}^{n} i_r$ we then have

(1) $$F_t(a_1,\ldots,a_n) = F_t^{(i_0)}(\psi_{i_1 i_0}(a_1),\ldots,\psi_{i_n i_0}(a_n))$$

and

(2) $$F'_t(a_1,\ldots,a_n) = F_t^{(i_0)}(\psi'_{i_1 i_0}(a_1),\ldots,\psi'_{i_n i_0}(a_n)).$$

Clearly, if $i_0>0$, then (1) = (2). Let us now suppose that $i_0=0$. If $i_1=0$, then

$$(2) = F_t^{(0)}(\psi'_{i_1 0}(a_1),\ldots,\psi'_{i_n 0}(a_n)) = F_t^{(0)}(\psi_{i_1 0}(a_1),\psi'_{i_2 0}(a_2),\ldots,\psi'_{i_n 0}(a_n))$$

and if $i_1>0$, then

$$\begin{aligned}(2) &= F^{(0)}_{t,1,\psi'_{i_1 0}(a_1)}(\psi'_{i_2 0}(a_2),\ldots,\psi'_{i_n 0}(a_n))\\ &= F^{(0)}_{t,1,\psi_{i_1 0}(a_1)}(\psi'_{i_2 0}(a_2),\ldots,\psi'_{i_n 0}(a_n))\\ &= F_t^{(0)}(\psi_{i_1 0}(a_1),\psi'_{i_2 0}(a_2),\ldots,\psi'_{i_n 0}(a_n)).\end{aligned}$$

Thus, if $n=1$, then we have (1) = (2). If $n\geqslant 2$ we show by induction, using the same method as before, that

$$(2) = F_t^{(0)}(\psi_{i_1 0}(a_1),\ldots,\psi_{i_r 0}(a_r),\psi'_{i_{r+1} 0}(a_{r+1}),\ldots,\psi'_{i_n 0}(a_n))$$

for all r. We conclude that (1) = (2) in all cases. Thus $S(\mathscr{B})$ and $S(\mathscr{B}')$ are equivalent.

Remark 3. Notice that the system $\mathscr{B}$ which has been constructed in Theorem 2 is almost a Płonka system: all structure mappings ψ_{ji}, except perhaps the structure mappings ψ_{j0}, are injective homomorphisms. $S(\mathscr{B})$ is equivalent to a Płonka sum if and only if one can find a mapping $\varphi''\colon \mathfrak{A}\to\mathfrak{A}$ whose restriction to $\mathfrak{A}^{(i)}$ is an isomorphism for all $i\in I$, $i>0$.

If $\mathfrak{A}$ is an algebra of type τ, and if $\mathscr{O}$ is a one-element algebra of type τ, then we define $\mathfrak{A}^0$ to be the Płonka sum of the Płonka system

$$\langle I;\ \langle \mathfrak{A}_i,\ i\in I\rangle;\ \langle \varphi_{ji},\ i,j\in I,\ i\leqslant j\rangle\rangle$$

where $I=\{1,0\}$ is the two-element semilattice, $\mathfrak{A}_1 = \mathfrak{A}$ and $\mathfrak{A}_0 = \mathscr{O}$ (see also [2]). We shall say that a generalized Płonka sum is special if it is of the form $S(\mathscr{B})$ or $S(\mathscr{B})^0$, where $\mathscr{B}$ is a generalized Płonka system which has been constructed in Theorem 2.

Theorem 3. *Every generalized Płonka sum is a subdirect product of special generalized Płonka sums.*

Proof. Let us consider $S(\mathfrak{A})$, where $\mathfrak{A}$ is the generalized Płonka system

$$\mathfrak{A} = \langle I;\ \langle \mathfrak{A}_i,\ i\in I\rangle;\ \langle \varphi_{ji},\ i,j\in I,\ i\leqslant j\rangle\rangle.$$

Let i be any element of I, and let $\langle \mathscr{B}_j,\ j\in I, j\geqslant i\rangle$ be a family of pairwise disjoint algebras such that for all $j\in I$, $j>i$, $\bar{\psi}_j\colon \mathscr{B}_j\to\varphi_i\varphi_{ji}\mathfrak{A}_j$ is an isomorphism, and $\mathscr{B}_i = \mathfrak{A}_i$. For all $j\in I$, $j>i$ we use the notation $\mathscr{B}_j = \langle B_j;\ \langle G_t^{(j)},\ t\in T\rangle\rangle$. Let φ'_i be any mapping $\varphi'_i\colon \mathfrak{A}_i\to\mathfrak{A}_i$ such that $\varphi_i\varphi'_i$ is the identity transformation on $\mathfrak{A}_i$. Let $[i)$ be the principal filter generated by i in I, and consider the generalized Płonka system

$$\mathscr{B}^{(i)} = \langle [i);\ \langle \mathscr{B}_j,\ j\in[i)\rangle;\ \langle \psi_{kj},\ k,j\in[i),\ j\leqslant k\rangle\rangle$$

where
$\psi_{kj} = \bar{\psi}_j^{-1}\bar{\psi}_k$ if $i\neq j<k$,
ψ_{ii} is the identity transformation on $\mathscr{B}_i = \mathfrak{A}_i$,
$\psi_{ji} = \varphi'_i\bar{\psi}_j$ if $i<j$.

Then $S(\mathscr{B}^{(i)})$ is a special generalized Płonka sum. Let us consider the mapping

$$\sigma_i\colon \bigcup_{j\geqslant i} A_j \to \bigcup_{j\geqslant i} B_j,$$

$$a \to \begin{cases} \bar{\psi}_j^{-1}\varphi_i\varphi_{ji}(a) & \text{if } a\in A_j,\ j>i,\\ a & \text{if } a\in A_i.\end{cases}$$

This mapping is certainly surjective. We shall show that σ_i is a homomorphism of the subalgebra of $S(\mathfrak{A})$ whose carrier is $\bigcup_{j\geqslant i} A_j$ onto $S(\mathscr{B}^{(i)})$. Therefore, let us consider $t\in T$, $a_r\in A_{i_r}$, $r=1,\ldots,n=\tau(t)$, and $i_0 = \bigwedge_{r=1}^{n} i_r \geqslant i$. We use the notation $S(\mathscr{B}^{(i)}) = \langle B^{(i)};\ \langle G_t,\ t\in T\rangle\rangle$. We put $\varphi_{i_r i}(a_r) = \varphi_{i_0 i}\varphi_{i_r i_0}(a_r) = b_r$ for all r. Let us first suppose that $i_0 > i$. Then

$$\begin{aligned}\sigma_i F_t(a_1,\ldots,a_n) &= \bar{\psi}_{i_0}^{-1}\varphi_i\varphi_{i_0 i}F_t(a_1,\ldots,a_n)\\ &= \bar{\psi}_{i_0}^{-1}\varphi_i\varphi_{i_0 i}F_t^{(i_0)}(\varphi_{i_1 i_0}(a_1),\ldots,\varphi_{i_n i_0}(a_n))\\ &= \bar{\psi}_{i_0}^{-1}\bar{F}_t^{(i)}(\varphi_i\varphi_{i_0 i}\varphi_{i_1 i_0}(a_1),\ldots,\varphi_i\varphi_{i_0 i}\varphi_{i_n i_0}(a_n))\\ &= \bar{\psi}_{i_0}^{-1}\bar{F}_t^{(i)}(\varphi_i\varphi_{i_1 i}(a_1),\ldots,\varphi_i\varphi_{i_n i}(a_n))\\ &= G_t^{(i_0)}(\bar{\psi}_{i_0}^{-1}\varphi_i\varphi_{i_1 i}(a_1),\ldots,\bar{\psi}_{i_0}^{-1}\varphi_i\varphi_{i_n i}(a_n))\\ &= G_t^{(i_0)}(\bar{\psi}_{i_0}^{-1}\bar{\psi}_{i_1}\bar{\psi}_{i_1}^{-1}\varphi_i\varphi_{i_1 i}(a_1),\ldots,\bar{\psi}_{i_0}^{-1}\bar{\psi}_{i_n}\bar{\psi}_{i_n}^{-1}\varphi_i\varphi_{i_n i}(a_n))\\ &= G_t^{(i_0)}(\psi_{i_1 i_0}\sigma_i(a_1),\ldots,\psi_{i_n i_0}\sigma_i(a_n))\\ &= G_t(\sigma_i(a_1),\ldots,\sigma_i(a_n)).\end{aligned}$$

Next, suppose that $i_0 = i$. If $i_1 = i$, then $\sigma_i(a_1) = a_1 = b_1$, and thus $G_t(\sigma_i(a_1),\ldots,\sigma_i(a_n)) = G_t(b_1,\sigma_i(a_2),\ldots,\sigma_i(a_n))$. Let us now suppose that $i_1 > i$; then $n=\tau(t)\geqslant 2$ and

$$\begin{aligned}G_t(\sigma_i(a_1),\ldots,\sigma_i(a_n)) &= F_t^{(i)}(\psi_{i_1 i}\sigma_i(a_1),\ldots,\psi_{i_n i}\sigma_i(a_n))\\ &= F_t^{(i)}(\varphi_i'\bar{\psi}_{i_1}\bar{\psi}_{i_1}^{-1}\varphi_i\varphi_{i_1 i}(a_1),\psi_{i_2 i}\sigma_i(a_2),\ldots,\psi_{i_n i}\sigma_i(a_n))\\ &= F_t^{(i)}(\varphi_i'\varphi_i(b_1),\psi_{i_2 i}\sigma_i(a_2),\ldots,\psi_{i_n i}\sigma_i(a_n))\\ &= F_{t,1,\varphi_i'\varphi_i(b_1)}^{(i)}(\psi_{i_2 i}\sigma_i(a_2),\ldots,\psi_{i_n i}\sigma_i(a_n))\\ &= F_{t,1,b_1}^{(i)}(\psi_{i_2 i}\sigma_i(a_2),\ldots,\psi_{i_n i}\sigma_i(a_n))\\ &= F_t^{(i)}(b_1,\psi_{i_2 i}\sigma_i(a_2),\ldots,\psi_{i_n i}\sigma_i(a_n)).\end{aligned}$$

We can now show by induction, using the same method as before, that

$$G_t(\sigma_i(a_1),\ldots,\sigma_i(a_n)) = F_t^{(i)}(b_1,\ldots,b_n).$$

Obviously

$$F_t^{(i)}(b_1,\ldots,b_n) = \sigma_i F_t^{(i)}(b_1,\ldots,b_n) = \sigma_i F_t(a_1,\ldots,a_n).$$

We conclude that σ_i is a homomorphism.

Let Θ_i be a congruence on $S(\mathfrak{A})$ which is defined as follows:

$$a\,\Theta_i\,b \quad\text{if}\quad a\in A_k,\ b\in A_m,\ k,m\notin[i) \text{ or}$$
$$a,b\in A_j,\ j\in[i) \text{ and } \sigma_i(a)=\sigma_i(b).$$

Then $S(\mathfrak{A})/\Theta_i \cong S(\mathscr{B}^{(i)})$ if i is the least element of I and $S(\mathfrak{A})/\Theta_i \cong S(\mathscr{B}^{(i)})^0$ otherwise. For each i, Θ_i separates the elements of A_i, and thus $\bigcap_{i\in I}\Theta_i$ is the identity. We conclude that $S(\mathfrak{A})$ is a subdirect product of the algebras $S(\mathfrak{A})/\Theta_i$ which are all special generalized Płonka sums.

The foregoing theorem implies that if a generalized Płonka sum is subdirectly irreducible, then it must be a special generalized Płonka sum. From [2] we know that a subdirectly irreducible Płonka sum of algebras from the equational class K must be of the form $\mathfrak{A}$ or $\mathfrak{A}^0$, where $\mathfrak{A}$ is subdirectly irreducible in K. The situation is much more complex in the case of generalized Płonka sums. We show this by the following example.

Example 2. Let $I=\{1,0\}$ be the two-element semilattice. Let $\mathfrak{A}_0$ and $\mathfrak{A}_1$ be the groupoids with the following multiplication tables:

$\mathfrak{A}_0$	a_0	b_0	c_0	d_0
a_0	c_0	c_0	b_0	c_0
b_0	c_0	c_0	b_0	c_0
c_0	a_0	a_0	d_0	b_0
d_0	c_0	c_0	a_0	c_0

$\mathfrak{A}_1$	a_1	c_1	d_1
a_1	c_1	a_1	c_1
c_1	a_1	d_1	a_1
d_1	c_1	a_1	c_1

Let $\varphi_{10}\colon \mathfrak{A}_1\to\mathfrak{A}_0$ be given by

$$\varphi_{10}\colon a_1\to a_0,\ c_1\to c_0,\ d_1\to d_0.$$

Let us consider the generalized Płonka system

$$\mathfrak{A} = \langle I;\ \langle\mathfrak{A}_0,\mathfrak{A}_1\rangle;\ \langle\varphi_{00},\varphi_{10},\varphi_{11}\rangle\rangle.$$

Then $S(\mathfrak{A})$ is a special generalized Płonka sum. One can show that every non-trivial congruence on $S(\mathfrak{A})$ contains the pair (a_0,b_0). Therefore $S(\mathfrak{A})$ is subdirectly irreducible. From [2] it already follows that $S(\mathfrak{A})$ cannot be equivalent to a Płonka sum. Let K be the equational class of groupoids defined by the equation

$$(x^2(x^2)^2)^2 = (x^2(x^2)^2)(y^2(y^2)^2).$$

One can easily check that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ belong to K, whence $S(\mathfrak{A})$ belongs to $R(K)$. Theorem III of [5] guarantees that $S(\mathfrak{A})$ is a semilattice of groupoids which belong to K, and one may check that the semilattice decomposition of $S(\mathfrak{A})$ into $\mathfrak{A}_0$ and $\mathfrak{A}_1$ is the only semilattice decomposition of $S(\mathfrak{A})$ into groupoids which belong to K. Therefore $S(\mathfrak{A})$ cannot be a Płonka sum of groupoids which belong to K. This also provides an example of an equational class K where $R(K)$ properly contains the class of all algebras which are Płonka sums of algebras in K.

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Fixed points and nonexpansive retracts in locally convex spaces*

by

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Abstract. Locally convex topological vector spaces can be normed over a topological semifield. Using this norm, Banach operators and nonexpansive mappings are defined and several fixed point theorems are proven. Also, it is shown for strictly convex spaces that, under suitable conditions, the fixed point set of a nonexpansive map is a nonexpansive retract.

0. Introduction. The concept of a topological semifield was introduced by Antonovskii, Boltyanskii and Sarymsakov [1]. They observed that it is possible to define a semifield valued "norm" for certain topological vector spaces; in particular the class of Hausdorff locally convex spaces. The aim of the present paper is to prove fixed point theorems in this class of spaces for Banach operators and nonexpansive mappings. Also we show that for strictly convex spaces, under suitable conditions, the fixed point set of a nonexpansive mapping is a nonexpansive retract.

These results extend those of Bahtin [2], Cain and Nashed [5], Hicks and Kubicek [9], Chandler and Faulkner [6], Bruck [3], [4] and others.

Let Δ be a nonempty set and $R^{\Delta} = \prod_{\alpha\in\Delta} R_\alpha$ be the product of the real line with the product topology. Addition and multiplications in R^{Δ} are defined pointwise. A partial ordering is defined by the cone $R^{\Delta}_{+} = \{f\colon f(\alpha) \geqslant 0,\ \alpha\in\Delta\}$. A general introduction to the space R^{Δ} may be found in [1].

If E is a real locally convex space, whose topology is generated by a family $\{\varrho_\alpha\colon \alpha\in\Delta\}$ of continuous seminorms, then the function $\varrho\colon E\to R^{\Delta}_{+}$ defined by $[\varrho(x)](\alpha) = \varrho_\alpha(x)$, $x\in E$, $\alpha\in\Delta$, satisfies

(1) $$\varrho(x)\geqslant 0,$$

(2) $$\varrho(\lambda x) = |\lambda|\,\varrho(x),$$

(3) $$\varrho(x+y)\leqslant \varrho(x)+\varrho(y)$$

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