3. Let $\mathcal{G}$ denote the family of all open B-convex half-spaces. By the preceding part of the theorem $\mathcal{C}_B$ is the smallest family which satisfies conditions (M), (U), (H) and the inclusion $\mathcal{C}_B \supset \mathcal{G}$. From Theorem 13 we get $\mathcal{C}_B = \mathcal{G}_{TL'M}$. Since $\mathcal{G}_T = \mathcal{G}$, $\mathcal{G}_{L'}$ is an intersection basis of the family $\mathcal{C}_B$. Consequently, we can repeat the proof of Theorem 18 in [9] for any natural number n. We get the last conclusion of Theorem 26.

The proof is complete.

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Solenoids and inverse limits of sequences of arcs with open bonding maps

by

P. Krupski (Wrocław)

Abstract. The class $\mathcal{K}$ of inverse limits of sequences of arcs with open bonding maps is characterized as the class of chainable continua with property K, with one or two end-points and with arcs as proper subcontinua. Also it is proved that each monotone image of X, where X is from $\mathcal{K}$ or from the class of solenoids, is homeomorphic to X.

Introduction. Let us denote by $\mathcal{K}$ the class of inverse limits of sequences of arcs with open bonding maps.

In this paper we establish some analogies between solenoids and class $\mathcal{K}$. Next, we give a characterization of continua from $\mathcal{K}$ as chainable continua with property K, with one or two end-points and with arcs as proper subcontinua. This answers Problem 2 in [7] and corresponds to the characterization of solenoids in [7]. Finally, it is shown that a monotone image of X from $\mathcal{K}$ is homeomorphic to X and that the same holds for solenoids. Thus both of these classes provide examples of the continua which J. J. Charatonik asks about in [4].

Preliminaries. By a continuum we mean a compact, connected, metric nondegenerate space. Denote by I the interval $[0, 1]$. For each integer $s \geqslant 1$ let w_s denote the map of I onto I such that $w_s(i/s) = 0$ if i is even, $w_s(i/s) = 1$ if i is odd, where $0 \leqslant i \leqslant s$ and w_s is linear on each interval $[i/s, (i+1)/s)$ for $0 \leqslant i < s$.

It is known from [9, Lemma 1, page 453 and Theorem 7, page 455] that class $\mathcal{K}$ is topologically equal to the class of inverse limits of sequences $\{I, f_i\}$, where, for each i, $f_i = w_s$ for some s. So, each continuum $K \in \mathcal{K}$ is determined by a sequence of natural numbers $(s_1, s_2, \ldots)$ such that $K = \operatorname{inv\,lim}\{I, w_{s_k}\}$. We will denote such K by $K(s_1, s_2, \ldots)$.

A chain (circular chain) is a finite collection of open sets $\{U_1, \ldots, U_m\}$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$ ($|i-j| \leqslant 1$ or $i = 1$ and $j = m$). A subchain of a chain $\mathcal{U}$ between links U_i and U_j will be denoted by $\mathcal{U}(i, j)$.

A chain $\mathcal{U}^2 = \{U_1^2, \ldots, U_m^2\}$ refines a chain $\mathcal{U}^1 = \{U_1^1, \ldots, U_k^1\}$ if there is a function α: $\{1, \ldots, m\} \to \{1, \ldots, k\}$ such that $U_i^2 \subset U_{\alpha(i)}^1$ for every i.

A chain $\mathcal{U}^2$ is of *type* s in a chain $\mathcal{U}^1$ if $\mathcal{U}^2$ refines $\mathcal{U}^1$ and if there is a sequence of integers

$$1 = a_0 < a_1 < \ldots < a_s = m$$

such that $\alpha(a_i) = 1$ for even i, $\alpha(a_i) = k$ for odd i and α is monotone on each interval $[a_i, a_{i+1}]$, $0 \leqslant i < s$.

It is clear that a continuum X is homeomorphic to $K(s_1, s_2, \ldots)$ if and only if there exist chains $\mathcal{U}^1, \mathcal{U}^2, \ldots$ covering X such that mesh $\mathcal{U}^n < \delta_n$, where $\delta_n \to 0$ if $n \to \infty$ and $\mathcal{U}^{n+1}$ is of type s_n in $\mathcal{U}^n$ for each n.

Let a chain $\mathcal{U}^2$ refine a chain $\mathcal{U}^1$. We say (see [7]) that a subchain $\mathcal{U}^2(i, k)$ is a *fold in* $\mathcal{U}^1$ if there exist numbers $i < j < k$ and i', j' such that

$$\bigcup \mathcal{U}^2(i, k) \subset \bigcup \mathcal{U}^1(i', j'),$$

$$U_i^2 \cup U_k^2 \subset U_{i'}^1, \qquad U_j^2 \subset U_{j'}^1,$$

$$\bigcup \mathcal{U}^2(i, k) \setminus (U_{i'}^1 \cap U_{j'}^1) \neq \emptyset.$$

Furthermore, we will consider only maximal folds $\mathcal{U}^2(i, k)$ for which the subchains $\mathcal{U}^2(i, j)$ and $\mathcal{U}^2(j, k)$ contain no fold in $\mathcal{U}^1$. The link U_j^2 will be called extremal.

It is easy to see that a chainable continuum K belongs to the class $\mathcal{K}$ if and only if there exists a sequence of chains $\mathcal{U}^1, \mathcal{U}^2, \ldots$ covering K, with mesh $\mathcal{U}^n$ tending to 0, such that $\mathcal{U}^{n+1}$ refines $\mathcal{U}^n$ and the first and last links of $\mathcal{U}^{n+1}$ as well as the extremal links of each fold of $\mathcal{U}^{n+1}$ in $\mathcal{U}^n$ are contained in the first or the last link of $\mathcal{U}^n$.

By a solenoid $\Sigma(s_1, s_2, \ldots)$ we mean the inverse limit of a sequence of unit circles S of complex numbers z such that $|z| = 1$ with bonding maps $\alpha_k(z) = z^{s_k}$.

Each solenoid is a topological group; denote its identity by e.

Recall (see [10]) that X has property K at a point $a \in X$ if for each subcontinuum $A \subset X$ containing a and for each sequence of points $a_n \in X$ converging to a there exists a sequence of subcontinua $A_n \subset X$, converging to A, such that $a_n \in A_n$. A continuum X has property K if it has property K at each point.

A point x is an *end-point* of a chainable continuum X if for every $\varepsilon > 0$ there exists an ε-chain covering X, the first link of which contains x. Equivalently (see [2, p. 661]), x is an end-point of X if, for each two subcontinua of X containing x, one of the subcontinua contains the other.

Natural connections between solenoids and continua of class $\mathcal{K}$. For a convenient description, let

$$S = \{\exp(\pi i t)\colon t \in [-1, 1]\}, \qquad A = \{\exp(\pi i t)\colon t \in [0, 1]\}$$

and represent a continuum $K(s_1, s_2, \ldots)$ as the inverse limit of $\{A, \varphi_k\}$, where $\varphi_k(\exp(\pi i t)) = \exp(\pi i w_{s_k}(t))$ for $t \in [0, 1]$.

Define a map $\beta\colon S \to A$ by $\beta(\exp(\pi i t)) = \exp(\pi i |t|)$ for $t \in [-1, 1]$. Then the following diagram commutes:

$$\begin{array}{ccc} S & \xleftarrow{\alpha_k} & S \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xleftarrow{\varphi_k} & A \end{array}$$

and we obtain a map $\bar{\beta} = \beta \times \beta \times \ldots$ of $\Sigma(s_1, s_2, \ldots)$ onto $K(s_1, s_2, \ldots)$ (cf. Bellamy [1] for the case of a diadic solenoid). Since all the maps in the diagram are open, $\bar{\beta}$ is open. Moreover (cf. [1, p. 314]), we get

LEMMA 1. *$\bar{\beta}(x) = \bar{\beta}(y)$ if and only if $x = y$ or $x = y^{-1}$.*

From Lemma 1, since a solenoid is a topological group, it follows that if x is of order greater than 2, then $\bar{\beta}$ is one-to-one in some open neighbourhood of x. So, we have

LEMMA 2. *$\bar{\beta}$ is a local homeomorphism at each point different from e and from an element of order 2.*

Remark. It is easily seen that $\Sigma(s_1, s_2, \ldots)$ has an element of order 2 if and only if at most finitely many numbers in the sequence $(s_1, s_2, \ldots)$ are even and that $\Sigma(s_1, s_2, \ldots)$ contains at most one such element.

LEMMA 3. *A point of $K(s_1, s_2, \ldots)$ is an end-point if and only if it is the image under $\bar{\beta}$ of e or of an element of order 2.*

Proof. Since e has the coordinates $(1, 1, \ldots)$ in the inverse limit $\Sigma(s_1, s_2, \ldots)$, $\bar{\beta}(e)$ has the coordinates $(1, 1, \ldots)$ in the inverse limit $K(s_1, s_2, \ldots)$. If e' is of order 2, then e' has the coordinates $(1, \ldots, 1, -1, -1, \ldots)$ in $\Sigma(s_1, s_2, \ldots)$ and $\bar{\beta}(e')$ has the coordinates $(1, \ldots, 1, -1, -1, \ldots)$ in $K(s_1, s_2, \ldots)$. In both cases, by the definition of an end-point, $\bar{\beta}(e)$ and $\bar{\beta}(e')$ are end-points. However, if $e \neq x \neq e'$ then, by Lemma 2, $\bar{\beta}(x)$ belongs to the interior of an arc, and so $\bar{\beta}(x)$ is not an end-point.

Recall that a sequence of natural numbers $(s_1, s_2, \ldots)$ is called a *factorant* of another sequence of natural numbers $(p_1, p_2, \ldots)$ if there exists an i_0 such that for every $i \geqslant i_0$ there is a natural number m such that $s_{i_0} \cdot s_{i_0+1} \cdot \ldots \cdot s_i$ is a factor of $p_1 \cdot p_2 \cdot \ldots \cdot p_m$ (see [5, p. 236]).

LEMMA 4. *The set of sequences of natural numbers such that none of them is a factorant of another is uncountable.*

Proof[1]. It suffices to show that there are uncountably many sequences of natural numbers such that every two of them have a finite intersection. Indeed, assign to a sequence $(a_1, a_2, \ldots)$, where $a_i = 0, 1$, the sequence $(a_1, a_1 a_2, a_1 a_2 a_3, \ldots)$, where $a_1 a_2 \ldots a_i$ denotes the natural number

(1) The author thanks Mr. W. Charatonik for the proof.

written in a position system and observe that this correspondence is one-to-one.

H. Cook proved in [5] that $\Sigma(s_1, s_2, \ldots)$ is a continuous image of $\Sigma(p_1, p_2, \ldots)$ if and only if $(s_1, s_2, \ldots)$ is a factorant of $(p_1, p_2, \ldots)$. Hence, Lemma 4 implies the following theorem.

THEOREM 1. *There are uncountably many solenoids such that none of them is a continuous image of another.*

An essential difference between solenoids and class $\mathscr{K}$ is that each member of $\mathscr{K}$, with the exception of an arc, can be continuously mapped onto each other one (see [9, p. 455]). However, W. Dębski has recently proved in [6]

THEOREM 2. *There are uncountably many continua from class $\mathscr{K}$ such that none of them is an open image of another.*

A characterization.

LEMMA 5. *If X is a chainable continuum with property K, with one end-point and with arcs as proper subcontinua, then there exists such a sequence of chains $\mathscr{V}^n$ covering X, with mesh $\mathscr{V}^n$ tending to 0, that $\mathscr{V}^{n+1}$ refines $\mathscr{V}^n$ and the first and last links of $\mathscr{V}^{n+1}$ are contained in the first link of $\mathscr{V}^n$ for each $n = 1, 2, \ldots$*

Proof. Let a be the end-point of X. There exist chains

$$\mathscr{U}^n = \{U_1^n, \ldots, U_{r_n}^n\}, \quad n = 1, 2, \ldots,$$

covering X and such that mesh $\mathscr{U}^n \to 0$, $\mathscr{U}^{n+1}$ refines $\mathscr{U}^n$, $a \in U_1^{n+1} \subset U_1^n$ for every n.

First, note that $\mathrm{cl}(U_{r_n}^n)$ tends to $\{a\}$ if $n \to \infty$. In fact, suppose $\mathrm{cl}(U_{r_n}^n)$ tends to $\{b\}$, $b \neq a$. Take nondegenerate arcs bc and bd in X such that $bc \cap bd = \{b\}$, and points $b_n \in U_{r_n}^n$, $b_n \neq b$. By property K, there exist sequences of arcs C_n and D_n in X converging to bc and bd, respectively, and such that $b_n \in C_n \cap D_n$. But

$$C_n \subset \bigcup \mathscr{U}^n(j_n, r_n) \quad \text{and} \quad D_n \subset \bigcup \mathscr{U}^n(k_n, r_n)$$

for some j_n, k_n such that $C_n \cap U_{j_n}^n \neq \emptyset$, $D_n \cap U_{k_n}^n \neq \emptyset$.

Since, for every n,

$$\bigcup \mathscr{U}^n(j_n, r_n) \subset \bigcup \mathscr{U}^n(k_n, r_n)$$

or conversely

$$\bigcup \mathscr{U}^n(k_n, r_n) \subset \bigcup \mathscr{U}^n(j_n, r_n)$$

and

$$\operatorname*{Lim}_{n\to\infty} \mathrm{cl}\big(\bigcup \mathscr{U}^n(j_n, r_n)\big) = \operatorname*{Lim}_{n\to\infty} C_n = bc,$$

$$\operatorname*{Lim}_{n\to\infty} \mathrm{cl}\big(\bigcup \mathscr{U}^n(k_n, r_n)\big) = \operatorname*{Lim}_{n\to\infty} D_n = bd,$$

we obtain $bc \subset bd$ or $bd \subset bc$, a contradiction.

Further, consider, for each n, the family of subchains $\mathscr{U}^n(1, i)$ for which $U_{r_m}^m \subset U_i^n$ for some $m > n$, and let $\mathscr{U}^n(1, i_n)$ be the maximal element of the family with respect to inclusion.

Let $\mathscr{V}^n$ be a chain consisting of $V_1^n = \bigcup \mathscr{U}^n(1, i_n)$ and of links of $\mathscr{U}^n$ if they are not contained in V_1^n. It is clear that $\mathscr{V}^{n+1}$ refines $\mathscr{V}^n$ and the first and last links of $\mathscr{V}^{n+1}$ are contained in the first link of $\mathscr{V}^n$. Also mesh $\mathscr{V}^n \to 0$. Indeed, if there is a sequence V_1^n tending to X, then, since $V_1^{n+1} \subset V_1^n$, we have $V_1^n = X$, $n = 1, 2, \ldots$, whence, for each n, $U_{r_m}^m \subset U_{r_n}^n$ and thus X has two end-points. However, if there is a sequence V_1^n tending to a nondegenerate arc ab, then

$$\operatorname*{Lim}_{n\to\infty} \mathrm{cl}\big(\bigcup \mathscr{U}^n(1, i_n)\big) = ab.$$

Since $\mathrm{cl}(U_{r_n}^n)$ tends to $\{a\}$, $\mathrm{cl}(U_{i_n}^i)$ tends to $\{a\}$. Thus there is a sequence of points x_n converging to b such that $x_n \in U_{j_n}^n$, for $1 < j_n < i_n$. We have

$$\operatorname*{Lim}_{n\to\infty} \mathrm{cl}\big(\bigcup \mathscr{U}^n(j_n, i_n)\big) = ab.$$

By property K, there is such a sequence of arcs X_n in X converging to an arc bc, where $c \notin ab$, that $x_n \in X_n$. But we can assume that X_n intersects all the links of $\mathscr{U}^n(j_n, i_n)$. Hence $bc \supset ab$, a contradiction. The proof of Lemma 5 is thus complete.

Let us observe that if a continuum X has two end-points a, b and each proper subcontinuum of X is an arc, then X is irreducible between a and b. Indeed, if Y is a proper subcontinuum of X containing a and b, Y is an arc ab. Note that the composant of a in X is equal to ab and, since it is dense in X, we have $Y = ab = \mathrm{cl}(ab) = X$, a contradiction. Thus, using Theorem 14 in [2, p. 661] we get

LEMMA 6. *If X is a chainable continuum with two end-points a and b such that each proper subcontinuum of X is an arc, then a and b are the opposite end-points of X* (i.e. *for each $\varepsilon > 0$ there is an ε-chain $\{U_1, \ldots, U_k\}$ covering X and such that $a \in U_1$ and $b \in U_k$*).

LEMMA 7. *If X is a chainable continuum with property K, with one or two end-points and with arcs as proper subcontinua, then there is such a sequence of chains $\mathscr{V}^n$ covering X, with mesh $\mathscr{V}^n$ tending to 0, that $\mathscr{V}^{n+1}$ refines $\mathscr{V}^n$ and* *the first and last links of $\mathscr{V}^{n+1}$, as well as the extremal links of any fold of $\mathscr{V}^{n+1}$ in $\mathscr{V}^n$, are contained in the first or last links of $\mathscr{V}^n$.*

Proof. By Lemma 5 (in the case of one end-point) and by Lemma 6 (in the case of two end-points), there exist chains $\mathscr{U}^n = \{U_1^n, \ldots, U_{r_n}^n\}$ covering X such that mesh $\mathscr{U}^n \to 0$, if $n \to \infty$, and for each $n \geqslant 1$

$\mathscr{U}^{n+1}$ refines $\mathscr{U}^n$,

$U_1^{n+1} \subset U_1^n$,

$U_{r_{n+1}}^{n+1} \subset U_1^n$, if X has one end-point or

$U_{r_{n+1}}^{n+1} \subset U_{r_n}^n$, if X has two end-points.

First of all, we will construct a sequence of chains $\mathscr{W}^n$ covering X such that

mesh $\mathscr{W}^n \to 0$, if $n \to \infty$,

$\mathscr{W}^{n+1}$ refines $\mathscr{W}^n$,

the first and last links of $\mathscr{W}^{n+1}$ are contained in the first or last links of $\mathscr{W}^n$,

in these chains there is no sequence of folds with diameters tending to 0

(by the diameter of a subchain we understand the diameter of the union of its links).

For each n, consider the family $\mathfrak{A}_n$ of subchains $\mathscr{U}^n(i, j)$ for which there is a fold $\mathscr{U}^m(s, t)$ in $\mathscr{U}^{m-1}$ for some $m > n$, with diam $\mathscr{U}^m(s, t) < 1/n$, intersecting all the links of $\mathscr{U}^n(i, j)$. Order $\mathfrak{A}_n$ by inclusion.

Next, define a family $\mathscr{B}_n$:

the elements of $\mathscr{B}_n$ are the links of $\mathscr{U}^n$ which are not contained in any maximal element of $\mathfrak{A}_n$ and the unions of links of maximal elements from $\mathfrak{A}_n$.

It is easy to see that mesh $\mathscr{B}_n \to 0$ and $\mathscr{B}_{n+1}$ refines $\mathscr{B}_n$. However, $\mathscr{B}_n$ need not be a chain. But by a standard consolidation of $\mathscr{B}_n$ we can obtain the desired chain $\mathscr{W}^n$. To this end, fix n and let $\mathscr{B}_n = \{B_1, \ldots, B_k\}$ with a natural order inherited from the order of links of the chain $\mathscr{U}^n$. Define a sequence of integers

$$1 \leqslant p(1) < p(2) < \ldots < p(i) = k$$

such that

$$B_1 \cap B_{p(1)} \neq \emptyset,\ B_{p(1)+1} \cap B_1 = \emptyset,$$

$$B_{p(2)} \cap B_{p(1)} \neq \emptyset,\ B_{p(2)+1} \cap B_{p(1)} = \emptyset,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$B_{p(i)} \cap B_{p(i-1)} \neq \emptyset.$$

Put

$$W_1 = B_1 \cup \ldots \cup B_{p(1)},$$

$$W_2 = B_{p(1)+1} \cup \ldots \cup B_{p(2)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$W_i = B_{p(i-1)+1} \cup \ldots \cup B_{p(i)}.$$

Since $\mathscr{U}^n$ is a chain, we have

$$W_1 = B_1 \cup B_{p(1)},$$

$$W_2 = B_{p(1)+1} \cup B_{p(2)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$W_i = B_{p(i-1)+1} \cup B_{p(i)}.$$

Define $\mathscr{W}^n = \{W_1, \ldots, W_i\}$. The properties of the chains $\mathscr{W}^n$ follow easily from the properties of $\mathscr{B}_n$.

Now, for the sequence $\mathscr{W}^n$, there exists a positive number ε such that each fold in this sequence is of diameter greater than ε.

Denote $\mathscr{W}^n = \{W_1^n, \ldots, W_{s_n}^n\}$. Consider a family $\mathscr{C}_n$ of subchains $\mathscr{W}^n(1, i)$ and $\mathscr{W}^n(j, s_n)$ such that, for some $m > n$, there is a fold $\mathscr{W}^m(s, t)$ in $\mathscr{W}^{m-1}$ which has an extremal link in W_i^n (W_j^n respectively) and which does not meet $\bigcup \mathscr{W}^n(1, i-1)$ ($\bigcup \mathscr{W}^n(j+1, s_n)$ respectively). Denote the maximal subchains from $\mathscr{C}^n$ by $\mathscr{W}^n(1, i_n)$ and $\mathscr{W}^n(j_n, s_n)$.

Let us define a chain $\mathscr{V}^n$:

the elements of $\mathscr{V}^n$ are the unions $\bigcup \mathscr{W}^n(1, i_n)$, $\bigcup \mathscr{W}^n(j_n, s_n)$ and the links of $\mathscr{W}^n$ which are not contained either in $\mathscr{W}^n(1, i_n)$ or in $\mathscr{W}^n(j_n, s_n)$.

We will show mesh $\mathscr{V}^n \to 0$ if $n \to \infty$. Otherwise, we can assume that there exists a sequence of $V_{k_n}^n \in \mathscr{V}^n$ converging to a nondegenerate subcontinuum of X. So,

$$V_{k_n}^n = \bigcup \mathscr{W}^n(1, i_n) \quad \text{or} \quad V_{k_n}^n = \bigcup \mathscr{W}^n(j_n, s_n),$$

and $V_{k_n}^n$ tends to X or to a nondegenerate arc A in X. It suffices to consider only the case $V_{k_n}^n = \bigcup \mathscr{W}^n(1, i_n)$ (the other case is analogous). This means that $V_{k_n}^n$ is the first link in $\mathscr{V}^n$, i.e. $k_n = 1$.

If V_1^n tends to X, then, since $V_1^{n+1} \subset V_1^n$, we have $V_1^n = X$, for $n = 1, 2, \ldots$ Hence $i_n = s_n$. But, for large n, diam $W_{s_n}^n < \varepsilon$, and thus $W_{s_n}^n$ contains no fold of diameter greater than ε, contrary to the definition of $\mathscr{W}^n(1, s_n)$.

If V_1^n tends to A, then A is an arc ab, where a is the end-point of X belonging to W_1^n, $n = 1, 2, \ldots, b \neq a$. Observe that if X is not an arc, b is not an end-point. By the definition of V_1^n, there exists a fold $\mathscr{W}^{k_n}(i, j)$ in $\mathscr{W}^{k_n-1}$, $k_n > n$, such that an extremal link $W_m^{k_n}$ is contained in $W_{i_n}^n$ and $\bigcup \mathscr{W}^{k_n}(i, j) \subset \bigcup \mathscr{W}^n(i_n, s_n)$.

We will show that $W_m^{k_n}$ tends to $\{a\}$ if $n \to \infty$. Otherwise, $W_m^{k_n}$ tends to $\{c\}$, $c \neq a$. Note that $c \in ab$. Take points $c_n \in W_m^{k_n}$ such that $c_n \neq c$ for every $n = 1, 2, \ldots, c_n \to c$, if $n \to \infty$, and let xy be an arc of diameter less than ε, $c \in xy$, $x \neq c \neq y$.

By property K, there exist arcs X_n, Y_n, $n = 1, 2, \ldots$, converging to the arcs xc and cy respectively, such that $c_n \in X_n \cap Y_n$. We can assume diam $X_n < \varepsilon$ and diam $Y_n < \varepsilon$ for $n = 1, 2, \ldots$ Hence, by the definition of a fold and since folds have diameters greater than ε,

$$X_n \cup Y_n \subset \bigcup \mathscr{W}^{k_n}(i, j) \subset \bigcup \mathscr{W}^n(i_n, s_n).$$

Let $X_n \subset \bigcup \mathscr{W}^n(i_n, u)$, where u is between i_n and s_n and X_n meets every link of $\mathscr{W}^n(i_n, u)$. Similarly $Y_n \subset \bigcup \mathscr{W}^n(i_n, v)$, where v is between i_n and s_n and Y_n meets every link of $\mathscr{W}^n(i_n, v)$. Then $\mathrm{cl}(\bigcup \mathscr{W}^n(i_n, u))$ converges to xc and $\mathrm{cl}(\bigcup \mathscr{W}^n(i_n, v))$ converges to cy. But, for each n,

$$\bigcup \mathscr{W}^n(i_n, u) \subset \bigcup \mathscr{W}^n(i_n, v),$$

or conversely

$$\bigcup \mathscr{W}^n(i_n, v) \subset \bigcup \mathscr{W}^n(i_n, u),$$

whence $xc \subset cy$ or $cy \subset xc$, which is impossible in view of the choice of xy.

So, $W_m^{k_n}$ tends to $\{a\}$, and thus $W_{i_n}^n$ tends to $\{a\}$. Therefore there exists a sequence of points $b_n \in W_{m_n}^n$, $1 < m_n < i_n$, converging to b. We have

$$\operatorname*{Lim}_{n\to\infty} \mathrm{cl}(\bigcup \mathscr{W}^n(m_n, i_n)) = ab.$$

By property K, there is a sequence of arcs Z_n converging to an arc bd, where $d \notin ab$, such that $b_n \in Z_n$. But we can assume that Z_n intersects all the links of $\mathscr{W}^n(m_n, i_n)$. Hence $bd \supset ab$; this contradiction completes the argumentation that mesh $\mathscr{V}^n \to 0$ if $n \to \infty$.

The remaining properties of the sequence $\mathscr{V}^n$ follow easily from its definition. The proof of Lemma 7 is complete.

LEMMA 8. *Suppose that each proper subcontinuum of a continuum X is an arc, Y is a chainable continuum and f is a continuous map of X onto Y. Then each proper subcontinuum of Y is an arc.*

Proof. Let Y' be a proper subcontinuum of Y. Since the map f is weakly confluent (see [8, Theorem 4]), there exists a proper subcontinuum X' of X such that $f(X') = Y'$. Since X' is an arc, Y' is locally connected; observe also that Y' is chainable. Therefore Y' is an arc.

THEOREM 3. *A continuum X belongs to class $\mathscr{K}$ if and only if X is a chainable continuum with property K, with one or two end-points and with arcs as proper subcontinua.*

Proof. If $X \in \mathscr{K}$, then X is a chainable continuum which is, by Lemma 2, an open image of a corresponding solenoid. Since solenoids are topological groups, they have property K (see [10, 2.6 Corollary, page 294]). Hence, since open maps preserve this property (see [10, 4.5 Corollary, page 297]), X also has property K. Furthermore, each proper subcontinuum of a solenoid is an arc, whence, by Lemma 8, proper subcontinua of X are arcs. Finally, continua from class $\mathscr{K}$ have one or two end-points (Lemma 3). The converse implication is an immediate consequence of Lemma 7.

Monotone images. It is easy to see, from Theorem 3, that a monotone image of a continuum from $\mathscr{K}$ belongs to $\mathscr{K}$. However, using an idea from Bing's proof that a monotone image of a chainable continuum is chainable [3, Theorem 3, p. 47], one can obtain a considerably stronger result.

THEOREM 4. *If $K \in \mathscr{K}$ and f is a monotone map of K, then $f(K)$ is homeomorphic to K.*

Proof. Following Bing, we first construct for every $\varepsilon > 0$ a chain $\mathscr{V}$ covering $f(K)$ with mesh $\mathscr{V} < \varepsilon$. So, let $\varepsilon > 0$ and $\delta > 0$ be such that if $\varrho(x, x') < \delta$ then $\varrho'(f(x), f(x')) < \varepsilon/5$, where ϱ and ϱ' are metrics in K and $f(K)$ respectively.

Let $\mathscr{U} = \{U_1, \ldots, U_k\}$ be a chain covering K with mesh $\mathscr{U} < \delta$. It is easily seen that δ can be chosen so small that there exists a sequence of integers $t_1, \ldots, t_j$ such that

(*) $1 = t_1 < t_2 < \ldots < t_j = k$ and for each $i = 1, \ldots, j-1$ the preimage $f^{-1}(y)$ of a point $y \in f(K)$ intersects U_{t_i} and $U_{t_{i+1}}$ but no preimage intersects U_{t_i} and $U_{t_{i+1}+1}$

(since otherwise, for each δ, a preimage $f^{-1}(y)$ intersects all the links of $\mathscr{U}$ and $f(K)$ is degenerate).

Let $V(i, j) = \{y \in f(K)\colon f^{-1}(y) \subset \bigcup \mathscr{U}(i, j)\}$. Then

$$\mathscr{V} = \{V(t_1, t_5), V(t_4, t_8), \ldots, V(t_{3d+1}, t_j)\}, \quad \text{where} \quad j-4 \leqslant 3d+1 \leqslant j-2,$$

is the required ε-chain covering $f(K)$.

Call the chains $\mathscr{V}$ and $\mathscr{U}$ in this construction corresponding chains.

Since $K \in \mathscr{K}$, there exists a sequence of natural numbers $(s_1, s_2, \ldots)$ such that $K = K(s_1, s_2, \ldots)$. Hence, we can consider, for $n = 1, 2, \ldots$, corresponding chains $\mathscr{V}^n$, $\mathscr{U}^n$ such that

mesh $\mathscr{V}^n < 1/n$ for each n,

mesh $\mathscr{U}^n < \delta_n$, where $\delta_n \to 0$ if $n \to \infty$,

$\mathscr{U}^{n+1}$ is of type s_n in $\mathscr{U}^n$.

The chain $\mathscr{V}^{n+1}$ need not refine $\mathscr{V}^n$ but we can replace it by a chain $\mathscr{W}^{n+1}$ which does refine $\mathscr{V}^n$. Namely, a link of $\mathscr{W}^{n+1}$ is a link of $\mathscr{V}^{n+1}$ if it is contained in a link of $\mathscr{V}^n$; or, if a link V of $\mathscr{V}^{n+1}$ is not contained in any link of $\mathscr{V}^n$, we take as links of $\mathscr{W}^{n+1}$ the nonempty intersections of V with links of $\mathscr{V}^n$; the links of $\mathscr{W}^{n+1}$ are ordered in such a way that the first and last links are contained in the first and last links of $\mathscr{V}^{n+1}$ respectively. So, we assume further without loss of generality that $\mathscr{V}^{n+1}$ refines $\mathscr{V}^n$ for every n.

To get the conclusion of our theorem, it suffices to show that $\mathscr{V}^{n+1}$ is of type s_n in $\mathscr{V}^n$ for every n.

Fix n and use the following notation:

$$\mathscr{U}^{n+1} = \{U_1^2, \ldots, U_m^2\}, \qquad \mathscr{U}^n = \{U_1^1, \ldots, U_k^1\},$$

$$\mathscr{V}^{n+1} = \{V_1^2, \ldots, V_{m'}^2\}, \qquad \mathscr{V}^n = \{V_1^1, \ldots, V_{k'}^1\}$$

and let α be the function induced by the refinement $\mathscr{U}^{n+1}$ of $\mathscr{U}^n$.

There exist integers $1 = a_0 < a_1 < \ldots < a_{s_n} = m$ such that $\alpha(a_i) = 1$ for even i, $\alpha(a_i) = k$ for odd i and α increases (decreases) on $[a_i, a_{i+1}]$ for even (odd) i.

Let

$$1 = t_1^1 < t_2^1 < \ldots < t_{j_1}^1 = k \quad \text{and} \quad 1 = t_1^2 < t_2^2 < \ldots < t_{j_2}^2 = m$$

be two sequences satisfying $(*)$ for $\mathscr{U}^n$ and $\mathscr{U}^{n+1}$ respectively. For every i there exist p_i and r_i such that

$$t_{p_i}^2 \leqslant a_i \leqslant t_{r_i}^2 \quad \text{and} \quad V^2(t_{p_i}^2, t_{r_i}^2) \in \mathscr{V}^{n+1}.$$

It follows that

$$U_{a_i}^2 \subset \bigcup \mathscr{U}^{n+1}(t_{p_i}^2, t_{r_i}^2) \subset \begin{cases} \bigcup \mathscr{U}^n(1, t_5^1) & \text{for even } i, \\ \bigcup \mathscr{U}^n(t_{j_1-4}^1, k) & \text{for odd } i. \end{cases}$$

Hence $V^2(t_{p_i}^2, t_{r_i}^2)$ is contained in the first link $V_1^1 = V^1(1, t_5^1)$ of $\mathscr{V}^n$ for even i; for odd i, it is contained in the union $V_{k'-1}^1 \cup V_{k'}^1$ of the last two links of $\mathscr{V}^n$, and hence in $V_{k'}^1$, since $\mathscr{V}^{n+1}$ refines $\mathscr{V}^n$.

Denote by b_i the number of $V^2(t_{p_i}^2, t_{r_i}^2)$ in $\mathscr{V}^{n+1}$. We have

$$1 = b_1 \leqslant b_2 \leqslant \ldots \leqslant b_{s_n} = m',$$

and, in fact, this sequence is strictly increasing. Indeed, the equality $b_i = b_{i+1}$ yields

$$t_{p_i}^2 \leqslant a_i \leqslant t_{r_i}^2 \quad \text{and} \quad t_{p_i}^2 \leqslant a_{i+1} \leqslant t_{r_i}^2,$$

whence

$$U_{a_i}^2 \cup U_{a_{i+1}}^2 \subset \bigcup \mathscr{U}^{n+1}(t_{p_i}^2, t_{r_i}^2) \subset \left(\bigcup \mathscr{U}^n(1, t_5^1) \cap \bigcup \mathscr{U}^n(t_{j_1-4}^1, k)\right)$$

and hence

$$\emptyset \neq V^2(t_{p_i}^2, t_{r_i}^2) \subset V_1^1 \cap V_{k'}^1,$$

a contradiction for sufficiently large n.

Finally, one easily verifies that the function α' induced by the refinement $\mathscr{V}^{n+1}$ of $\mathscr{V}^n$ increases (decreases) on $[b_i, b_{i+1}]$ for even (odd) i. The proof of Theorem 4 is thus complete.

We shall now prove an analogon of Theorem 4 for solenoids. Firstly, let us observe that by a slight modification of the proof of Theorem 3 in [3, p. 47] and by applying it to circular chains instead of chains we obtain

LEMMA 9. *A monotone image of a circularly chainable continuum is circularly chainable.*

THEOREM 5. *A monotone image of a solenoid Σ is homeomorphic to Σ.*

Proof. Method I. We can proceed as if the proof of Theorem 4, replacing chains by circular chains and a continuum $K(s_1, s_2, \ldots)$ by the solenoid $\Sigma(s_1, s_2, \ldots)$.

Method II. Recall the following theorems.

A. Σ is a solenoid if and only if Σ is a circularly chainable continuum with property K such that each point x of Σ belongs to an arc with ends different from x (see [7, Theorem 1]).

B. Suppose that Σ_1 and Σ_2 are two solenoids and there exist an upper semi-continuous mapping f of Σ_1 onto Σ_2 such that, for each point x of Σ_1, $f(x)$ is a proper subcontinuum of Σ_2, and an upper semi-continuous mapping g of Σ_2 onto Σ_1 such that, for each point x of Σ_2, $g(x)$ is a proper subcontinuum of Σ_1. Then Σ_1 and Σ_2 are homeomorphic (see [5, Theorem 9, p. 239]).

Since monotone maps preserve property K ([10, 4.4 Corollary, p. 296]), it follows from Lemma 9 and from A that a monotone image of a solenoid is a solenoid and from B we get the conclusion.

Open problems. Theorem 4 is of course not true for open maps. Nevertheless, an open image of a chainable continuum is chainable, open maps preserve property K and the property of being an end-point, and, since each proper subcontinuum of a continuum from $\mathscr{K}$ is an arc, each proper subcontinuum of an open image of the continuum is also an arc. So, the following question arises:

1. Does each open image of any continuum from $\mathscr{K}$ belong to $\mathscr{K}$?

Concerning solenoids we have shown that an open image of a solenoid may belong to $\mathscr{K}$. Thus, we pose the following question.

2. Is it true that each open image of any solenoid is a solenoid or belongs to class $\mathscr{K}$?

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INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

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A generalization of Płonka sums

by
E. Graczyńska (Wrocław) and **F. Pastijn** (Gent)

Abstract. In this note we shall consider a method of constructing algebras which is a generalization of Płonka's sum of a semilattice ordered family of algebras. We show that an equational class is closed under the formation of generalized Płonka sums if and only if the equational class under consideration is regular. We provide an example of a generalized Płonka sum that is not equivalent to a Płonka sum.

We shall only consider algebras with finitary operations and without nullary operations.

A semilattice ordered family of sets is a triplet consisting of

(i) a meet semilattice I, with the semilattice order $\leqslant$,

(ii) a family of sets $\langle A_i,\ i\in I\rangle$,

(iii) a family of mappings $\langle \varphi_{ji},\ i,j\in I,\ i\leqslant j\rangle$ where, for each $i,j\in I$, $i\leqslant j$, φ_{ji} maps A_j into A_i, such that the following conditions are satisfied: for each $i\in I$, φ_{ii} is the identity mapping on A_i, and for all $i,j,k\in I$, with $i\leqslant j\leqslant k$, we have $\varphi_{ji}\varphi_{kj}=\varphi_{ki}$ (see [1], § 21).

Let us now suppose that for each $i\in I$, $\mathfrak{A}_i=\langle A_i;\ F_i\rangle$ is an algebra. We shall hereby suppose that the algebras $\mathfrak{A}_i$, $i\in I$, are all of type τ, and that the carriers A_i, $i\in I$, are pairwise disjoint. For each $i\in I$ we put $F_i=\langle F_t^{(i)},\ t\in T\rangle$. The system

$$\mathfrak{A}=\langle I;\ \langle\mathfrak{A}_i,\ i\in I\rangle;\ \langle\varphi_{ji},\ i,j\in I,\ i\leqslant j\rangle\rangle$$

is of course not a Płonka system (a semilattice ordered family of algebras in the sense of [1], § 21), because in generall the mappings φ_{ji}, $i,j\in I$, $i\leqslant j$, do not give rise to homomorphisms.

We define an algebra $S(\mathfrak{A})$, which we call the sum of the system $\mathfrak{A}$, in the obvious way: the carrier $A=\bigcup_{i\in I} A_i$ of $S(\mathfrak{A})$ is the disjoint union of the carriers of the algebras $\mathfrak{A}_i$, $i\in I$, and the fundamental operations of $S(\mathfrak{A})$ are defined by

$$F_t(a_1,\ldots,a_n)=F_t^{(i_0)}(\varphi_{i_1 i_0}(a_1),\ldots,\varphi_{i_n i_0}(a_n))$$

for all $t\in T$, with $a_r\in A_{i_r}$, $r=1,\ldots,n=\tau(t)$, and $i_0=\bigwedge_{r=1}^{n} i_r$. So far the only