Convex half-spaces

by

Marek Lassak (Bydgoszcz)

Abstract. We study convex half-spaces (i.e., convex set with convex complements) of Euclidean space \( \mathbb{R}^n \). It is proved that the family of all convex half-spaces of \( \mathbb{R}^n \) is a sequentially compact topological Fréchet space with respect to the set theoretical limit of sets \( \text{Lim} \).

Our results are applied in the next paper of this issue.

Our terminology follows [4]. Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space with the scalar product \( \langle v_1, v_2 \rangle \). The symbols \( K', K, \text{ rel } K, \text{ int } K \) denote, respectively, the complement of \( K \subset \mathbb{R}^n \), the closure of \( K \), the relative interior of \( K \), and the interior of \( K \). The vector with origin \( x \) and the end-point \( y \) is denoted by \( v_{xy} \). We use the following limits of sequences of sets: the inferior limit

\[
\liminf_{i \to \infty} A_i = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j,
\]

the superior limit

\[
\limsup_{i \to \infty} A_i = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j,
\]

and (if \( \liminf_{i \to \infty} A_i = A = \limsup_{i \to \infty} A_i \)) the limit \( \lim_{i \to \infty} A_i = A \). Obviously, if \( A_i, i = 1, 2, \ldots, \) are convex, then \( \liminf_{i \to \infty} A_i \) is convex and (if it exists) \( \lim_{i \to \infty} A_i \) is convex.

Definition 1. We call a set \( K \subset \mathbb{R}^n \) a convex half-space if both \( K \) and its complement \( K' \) are convex sets. We call \( K \) and \( K' \) complementary convex half-spaces.

Obviously, all open half-spaces and closed half-spaces (in the usual sense), the empty set \( \emptyset \) and the set \( \mathbb{R}^n \) are convex half-spaces of \( \mathbb{R}^n \). Also any semispaces at \( x \in \mathbb{R}^n \) (i.e., a maximal convex set which does not include \( x \)) is a convex half-space [1], [2].

For any \( x \in \mathbb{R}^n \) and for any unit perpendicular vectors \( v_1, \ldots, v_k \), where \( 1 \leq k \leq n \), we define the following sets

\[
B_x(v_1, \ldots, v_k) = \bigcup_{i=1}^{k} \{ y \in \mathbb{R}^n; \langle v_j, v_{xy} \rangle = 0 \text{ for } j < i \text{ and } \langle v_i, v_{xy} \rangle > 0 \},
\]

\[
P_x(v_1, \ldots, v_k) = B_x(v_1, \ldots, v_k) \cup \{ y \in \mathbb{R}^n; \langle v_j, v_{xy} \rangle = 0 \text{ for } j = 1, \ldots, k \}.
\]
Writing \( B_n(v_1, \ldots, v_k) \) or \( P_n(v_1, \ldots, v_k) \) we shall tacitly assume that \( x \in \mathbb{R}^n \), \( 1 \leq k \leq n \) and that \( v_1, \ldots, v_k \) are unit perpendicular vectors.

Let us recall that any cone excluding its vertex is called a blunt cone and any cone including its vertex is called a pointed cone.

The set \( B_n(v_1, \ldots, v_k) \) is a blunt cone with vertex \( x \) as the union of half-planes

\[
H_i = \{ y \in \mathbb{R}^n : \langle y, v_{i} \rangle = 0 \} \quad \text{for} \quad j < i \quad \text{and} \quad \langle y, v_{i} \rangle > 0,
\]

\( i = 1, \ldots, k \), which are blunt cones with vertex \( x \).

Let \( a \) and \( b \) be arbitrary points of \( B_n(v_1, \ldots, v_k) \). There exist numbers \( g < k \) and \( h < k \) such that \( a \in H_g \) and \( b \in H_h \). Let \( g \leq h \). Since \( b \in H_h \) and \( a \in H_g \) we have (see [4], Theorem 6.1, p. 45)

\[
((1-\lambda)a + \lambda b, 0 \leq \lambda < 1) \subseteq H_g \subseteq B_n(v_1, \ldots, v_k).
\]

Therefore \( B_n(v_1, \ldots, v_k) \) is convex. Similarly, \( P_n(v_1, \ldots, v_k) \) is a pointed convex cone with vertex \( x \).

From the obvious equalities

\[
(B_n(v_1, \ldots, v_k)) = P_n(-v_1, \ldots, -v_k)
\]

and

\[
(P_n(v_1, \ldots, v_k)) = B_n(-v_1, \ldots, -v_k)
\]

we infer that \( B_n(v_1, \ldots, v_k) \) and \( P_n(v_1, \ldots, v_k) \) are convex half-spaces.

The above considerations clarify the terms used in the next definition.

**Definition 2.** The sets \( B_n(v_1, \ldots, v_k) \) and the set \( \emptyset \) will be called blunt convex half-spaces with vertex \( x \). The sets \( P_n(v_1, \ldots, v_k) \) and the set \( \mathbb{R}^n \) will be called pointed convex half-spaces with vertex \( x \).

**Theorem 1.** Convex half-spaces of \( \mathbb{R}^n \) have the following properties

1. Any two complementary convex half-spaces have the forms

   \( B_n(v_1, \ldots, v_k) \) and \( P_n(-v_1, \ldots, -v_k) \), or the forms \( \emptyset \) and \( \mathbb{R}^n \).

2. \( B_n(v_1, \ldots, v_k) = B_n(v_1, \ldots, v_k) \) (respectively: \( P_n(v_1, \ldots, v_k) = P_n(v_1, \ldots, v_k) \)) if and only if \( \langle v_i, v_j \rangle = 0 \) for \( i = 1, \ldots, k \).

3. If \( K \) is a convex half-space, then any translate of \( K \) contains \( K \) or is contained in \( K \) (see [3]). More exactly: \( B_n(v_1, \ldots, v_k) \subset B_n(v_1, \ldots, v_k) \) (analogously: \( P_n(v_1, \ldots, v_k) \subset P_n(v_1, \ldots, v_k) \)) if and only if \( y \in B_n(v_1, \ldots, v_k) \).

4. Any arbitrary convex disjoint sets can be supplemented to complementary convex half-spaces [3].

5. If a convex set \( C \) is the union of two convex disjoint sets, then the sets have the forms \( C \cap B_n(v_1, \ldots, v_k) \) and \( C \cap P_n(-v_1, \ldots, -v_k) \), where \( x \in C \) or the forms \( \emptyset \) and \( C \).

6. A set \( K \) is a convex half-space if and only if one of the sets \( K, K^* \) is a maximal convex set disjoint with a plane. This plane is that of all vertices of the convex half-space \( K \).

7. The sets of the form \( B_n(v_1, \ldots, v_k) \) are the only semispaces at \( x \).

8. Any nonempty convex half-space is the convex hull of a sequence of points.

**Proof.** We show the first property. For \( n = 1 \) it is obvious. Assume it holds for \( \mathbb{R}^{n-1} \) and consider two complementary convex half-spaces \( A \) and \( A' \) of \( \mathbb{R}^n \). Since the cases \( A = \emptyset \) and \( A = \mathbb{R}^n \) are trivial, we assume \( A \neq \emptyset \) and \( A 
eq \mathbb{R}^n \). Consequently, \( A \neq \mathbb{R}^n \) because \( r \bar{A} = \mathbb{R} \) (Theorem 6.3 of [4], p. 46).

We shall show that \( \bar{A} \) is a closed half-space. Assume the contrary. Since \( A \) is convex, \( \bar{A} \) is convex too. Being convex and closed, \( \bar{A} \) is the intersection of a family of closed half-spaces ([4], Theorem 11.5, p. 99). Since \( A \neq \mathbb{R}^n \) and \( \bar{A} \) is not a closed half-space, the family contains closed half-spaces \( Q_1 \) and \( Q_2 \) such that \( Q_1 \neq Q_2 \) is not a translate of \( Q_1 \). Hence \( A \cap Q_1 \cap Q_2 
eq \emptyset \). Hence \( A \subset Q_1 \cap Q_2 \). Hence \( \bar{A} \cap Q_1 \cup Q_2 \). Since \( A \neq Q_1 \cup Q_2 \), there exists \( a_i \in Q_i \) and \( a_i \in A \) such that a lies in the segment joining \( a_i \) and \( a_j \). From \( a \in A \cap A \) we conclude that \( A \neq A \). The contradiction shows that \( \bar{A} \) is a closed half-space.

Similarly, \( \bar{A} \) is a closed half-space. Obviously, \( H = \bar{A} \cap \bar{A} \) is the bounding hyperplane of \( \bar{A} \) and of \( A' \). Consequently, \( A = C_1 \cup (A \cap H) \) and \( A' = C_2 \cup (A' \cap H) \) where \( C_1, C_2 \) are open half-spaces bounded by \( H \). From the inductive hypothesis we conclude that \( A \) and \( A' \) have the stipulated forms.

The other properties easily follow from the definition of sets \( B_n(v_1, \ldots, v_k) \) and \( P_n(v_1, \ldots, v_k) \), and from the first property. Properties 2 and 3 are obvious. Recurrently, with the help of the classic separation theorem, we get property 4. It implies properties 5-7. The last property holds because any convex half-space is the union of a finite number of half-planes, and since any half-plane is the convex hull of a sequence of points.

**Theorem 2.** The family \( \mathcal{F} \) of all convex half-spaces of \( \mathbb{R}^n \) is a sequentially compact topological Fréchet space with respect to the limit of sets \( \text{Lim} \).

**Proof.** Our theorem asserts that the following conditions hold:

1. If \( A_i \in \mathcal{F} \) for \( i = 1, 2, \ldots \), and \( \text{Lim} A_i = A \), then \( A \in \mathcal{F} \).
2. If \( A_i \in \mathcal{F} \) for \( i = 1, 2, \ldots \), then \( \text{Lim} A_i = A \).
3. If \( A_i \in \mathcal{F} \) for \( i = 1, 2, \ldots \), and \( \text{Lim} A_i = A \), then for any subsequence \( A_{i_j} \), \( j = 1, 2, \ldots \), we have \( \text{Lim} A_{i_j} = A \).
4. If the sequence \( A_i \in \mathcal{F} \), \( i = 1, 2, \ldots \), does not converge to \( A \in \mathcal{F} \), then there exists a subsequence such that no subsequence of it is convergent to \( A \).
(5) if \( A_j \in \mathcal{F} \) then \( \lim_{i \to \infty} A_{ij} = A_i \) and \( \lim_{j \to \infty} A_j = A \) for \( i, j = 1, 2, \ldots \), then there are subsequences \( i_n \) and \( j_n \) such that \( A_{i_n j_n} = A \).

(6) any sequence of sets from \( \mathcal{F} \) contains a convergent subsequence.

It can be easily shown that conditions (2), (3), and (4) hold for arbitrary sets.

We shall show (1). Let \( A_i, i = 1, 2, \ldots \), be convex half-spaces of \( \mathbb{R}^n \) and let the limit \( \lim_{i \to \infty} A_i = A \) exist. Hence the limit \( \lim_{i \to \infty} A_i' = (\lim_{i \to \infty} A_i)' = A' \) exists.

Since \( A_i \) and \( A_i' \), \( i = 1, 2, \ldots \), are convex, \( A \) and \( A' \) are also convex. Therefore, \( A \) is a convex half-space.

We shall prove (5). It results from (1) that the sets \( A_i, i = 1, 2, \ldots \), and the set \( A \) in (5) are convex half-spaces. Let \( A \neq \emptyset \) and \( A \neq \mathbb{R}^n \). By part 8 of Theorem 1 the set \( A \) is the convex hull of a sequence of points \( x_1, x_2, \ldots \). The complement \( A' \) is also the convex hull of a sequence of points \( y_1, y_2, \ldots \).

Since \( \lim_{i \to \infty} A_i = A \), there exists a number \( i_n \) such that \( A_{i_n} \) contains the points \( x_1, x_2, \ldots, x_n \) and does not contain \( y_1, y_2, \ldots, y_n \). Since \( \lim_{m \to \infty} A_{i_n m} = A_{i_n} \), there exists a number \( j_m \) such that \( A_{i_n j_m} \) contains the points \( x_1, x_2, \ldots, x_m \) and does not contain \( y_1, y_2, \ldots, y_m \). Therefore

\[
(x_1, x_2, \ldots) \in \lim_{m \to \infty} A_{i_n j_m}.
\]

\[
(y_1, y_2, \ldots) \in (\lim_{m \to \infty} A_{i_n j_m})'.
\]

Since \( \lim_{m \to \infty} A_{i_n m} \) and \( (\lim_{m \to \infty} A_{i_n j_m})' \) are convex, the inclusions

\[
\lim_{m \to \infty} A_{i_n m} = \operatorname{conv} \{x_1, x_2, \ldots\} = A,
\]

\[
(\lim_{m \to \infty} A_{i_n j_m})' = \operatorname{conv} \{y_1, y_2, \ldots\} = A'.
\]

hold. Thus \( \lim_{m \to \infty} A_{i_n j_m} = A \). If \( A = \emptyset \) or \( A = \mathbb{R}^n \) the considerations are similar.

Finally, we recurrently show (6). For \( \mathbb{R}^1 \) it is obvious. Assume (6) holds in \( \mathbb{R}^{n-1} \) and consider the space \( \mathbb{R}^n \). The case where a sequence of convex half-spaces of \( \mathbb{R}^n \) contains infinitely many of sets \( \emptyset \) or \( \mathbb{R}^n \) is obvious. In the opposite case, select a subsequence \( A_i, i = 1, 2, \ldots \), of convex half-spaces different from \( \emptyset \) and \( \mathbb{R}^n \). Let \( x_i \) be a vertex of \( A_i, i = 1, 2, \ldots \). The space \( \mathbb{R}^n \) can be viewed as a hyperplane of an \((n+1)\)-dimensional space \( \mathbb{R}^{n+1} \), take a point \( x \in \mathbb{R}^{n+1} \setminus \mathbb{R}^n \). Let \( L_x \) be the line passing through \( x_i \) and \( x, i = 1, 2, \ldots \). Put

\[
C_i = A_i + L_x = \{a + b; a \in A_i, b \in L_x\}, \quad i = 1, 2, \ldots
\]

Obviously, \( A_i \subseteq C_i \cap \mathbb{R}^n, i = 1, 2, \ldots \). Note that \( C_i \) is a convex half-space of \( \mathbb{R}^{n+1} \) different both from \( \emptyset \) and \( \mathbb{R}^{n+1} \) and that x is a vertex of \( C_i, i = 1, 2, \ldots \).

Let \( u_i \) denote a unit vector of \( \mathbb{R}^{n+1} \) perpendicular to the hyperplane bounding \( C_i \) and directed towards \( C_i \). One can select a subsequence \( u_{i_j}, j = 1, 2, \ldots \), which converges (in the usual sense) to a unit vector \( u \).

Let \( G = B_k(u) \) in the notation of \( \mathbb{R}^{n+1} \). Obviously,

\[
G = \liminf_{j \to \infty} C_{i_j} = \limsup_{j \to \infty} C_{i_j} = G.
\]

Consequently,

\[
F \subseteq \liminf_{k \to \infty} A_{i_k} \subseteq \limsup_{k \to \infty} A_{i_k} \subseteq F \cup H,
\]

where \( F = G \cap \mathbb{R}^n \) and \( H \) is the bounding hyperplane of \( F \). By the inductive hypothesis, a subsequence \( A_{i_{k_j}}, j = 1, 2, \ldots \), can be selected in such a way that the limit \( \lim_{k \to \infty} (A_{i_{k_j}} \cap H) = K \) exist. Note that

\[
\liminf_{k \to \infty} A_{i_k} = \liminf_{k \to \infty} (A_{i_{k_j}} \cap F) \lor \liminf_{k \to \infty} (A_{i_{k_j}} \cap H) = F \lor K.
\]

Similarly, \( \limsup_{k \to \infty} A_{i_k} = F \lor K \). Thus the limit \( \lim_{k \to \infty} A_{i_k} \) exists.

**Corollary 1.** Let \( x \in \mathbb{R}^n \). any \( k \in \{1, \ldots, n\} \). The families

\[
[B_k(v_1, \ldots, v_m); k \leq m \leq n], \quad [P_s(v_1, \ldots, v_k); k \leq m \leq n]
\]

and the family of all convex half-spaces with the vertex \( x \) are sequentially compact topological Fréchet spaces with respect to the limit \( Lim \).

**Corollary 2.** For any convex set \( C \subseteq \mathbb{R}^n \), the family of all convex subsets \( D \) of \( C \) such that \( C \setminus D \) is also convex is a sequentially compact topological Fréchet space with respect to the limit \( Lim \).

**Theorem 3.** Let \( B_k(u) = B_k(v_1, \ldots, v_k) \) (analogously: \( P_s(u) = P_s(v_1, \ldots, v_k) \)) if and only if almost all vectors \( u_i \) are positive combinations \( u_i = \lambda_1 v_1 + \ldots + \lambda_k v_k \) and \( \lim_{i \to \infty} (A_{i+k+1} \lor A_i) = 0, j = 1, \ldots, k-1 \).

**Proof.** The equalities \( Lim B_k(u) = B_k(v_1, \ldots, v_k) \) and \( Lim P_s(u) = P_s(v_1, \ldots, v_k) \) are equivalent as the equalities of complementary sets. Therefore, we consider sequences of open half-spaces only. It is sufficient to consider only the case where \( x = 0 \).

For \( \mathbb{R}^1 \) the theorem is obvious. We assume that the theorem holds for \( \mathbb{R}^{n-1} \) and consider the space \( \mathbb{R}^n \).

1. Let \( \lim_{i \to \infty} B_k(u_i) = B_k(v_1, \ldots, v_k) \). Denote by \( R^{n-1} \) the hyperplane
Let \( 2 \leq k \leq n \). Since \( u_i \) is a unit vector and \( v_1, \ldots, v_k \) are unit perpendicular vectors, \( \lambda_1^2 + \ldots + \lambda_k^2 = 1 \). Moreover, since \( \lim_{i \to \infty} (\lambda_{j+1}, \lambda_j) = 0 \) for \( j = 1, \ldots, k-1 \), we have \( \lim_{i \to \infty} \lambda_j = 0 \) for \( j = 2, \ldots, k \) and \( \lim_{i \to \infty} \lambda_1 = 1 \). Therefore \( \lim_{i \to \infty} u_i = v_1 \). Hence

\[
B_0(v_1) \subseteq \inf \sup B_0(u_i) \subseteq P_0(v_1).
\]

Let \( R^{n-1} \) denote the hyperplane bounding the half-space \( B_0(v_1) \). Since almost all combinations \( \lambda_1 v_1 + \ldots + \lambda_k v_k \) are positive, almost all combinations \( \lambda_2 v_2 + \ldots + \lambda_k v_k \) are also positive. Hence almost all sets \( R^{n-1} \cap B_0(u_i) \) are open half-planes of the plane \( R^{n-1} \). Since the vector \( \lambda_2 v_2 + \ldots + \lambda_k v_k \) is perpendicular to the hyperplane \( B_0(u_i) \) and since it is directed towards the side of \( B_0(u_i) \), the vector \( \lambda_2 v_2 + \ldots + \lambda_k v_k \) is perpendicular to the \((n-2)\)-dimensional plane \( K_i \) bounding the half-plane \( R^{n-1} \cap B_0(u_i) \) and directed towards it. Hence the vector \( u_i = \alpha_2 v_2 + \ldots + \alpha_{n-1} v_n \) is unit, perpendicular and directed towards the side of \( R^{n-1} \cap B_0(u_i) \). This is true for almost all numbers \( i = 1, 2, \ldots \). Obviously, almost all the combinations above are positive and \( \alpha_2 \lambda_2 v_2 + \ldots + \alpha_{n-1} v_n \) is unit, perpendicular and directed towards the side of \( R^{n-1} \cap B_0(u_i) \). Therefore, from the inductive hypothesis we conclude that the set \( \lim_{i \to \infty} B_0(u_i) \cap R^{n-1} \) is equal (in the notation of \( R^{n-1} \)) to

\[
B_0(v_2, v_3, \ldots, v_n) \cap R^{n-1}.
\]

This and (*) imply

\[
\inf \sup B_0(u_i) = B_0(v_1, v_2, \ldots, v_n) \subseteq \sup \inf B_0(u_i).
\]

Therefore the limit \( \lim_{i \to \infty} B_0(u_i) \) exists and equals \( B_0(v_1, v_2, \ldots, v_n) \).

The proof is complete.

It would be interesting to discuss decompositions of \( R^n \) onto \( m > 2 \) disjoint convex subsets.

References