Convex half-spaces

by

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Abstract. We study convex half-spaces (i.e., convex set with convex complements) of Euclidean space R^n. It is proved that the family of all convex half-spaces of R^n is a sequentially compact topological Fréchet space with respect to the set theoretical limit of sets Lim.

Our results are applied in the next paper of this issue.

Our terminology follows [4]. Let R^n denote the n-dimensional Euclidean space with the scalar product $\langle v_1, v_2\rangle$. The symbols K', $\bar{K}$, ri K, int K denote, respectively, the complement of $K \subset R^n$, the closure of K, the relative interior of K, and the interior of K. The vector with origin x and the end-point y is denoted by v_{xy}. We use the following limits of sequences of sets: the inferior limit

$$\operatorname*{Lim\,inf}_{i\to\infty} A_i = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j,$$

the superior limit

$$\operatorname*{Lim\,sup}_{i\to\infty} A_i = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j,$$

and (if $\operatorname*{Lim\,inf}_{i\to\infty} A_i = A = \operatorname*{Lim\,sup}_{i\to\infty} A_i$) the limit $\operatorname*{Lim}_{i\to\infty} A_i = A$. Obviously, if A_i, $i = 1, 2, \ldots$, are convex, then $\operatorname*{Lim\,inf}_{i\to\infty} A_i$ is convex and (if it exists) $\operatorname*{Lim}_{i\to\infty} A_i$ is convex.

Definition 1. We call a set $K \subset R^n$ a *convex half-space* if both K and its complement K' are convex sets. We call K and K' *complementary convex half-spaces*.

Obviously, all open half-spaces and closed half-spaces (in the usual sense), the empty set $\emptyset$ and the set R^n are convex half-spaces of R^n. Also any semispace at $x \in R^n$ (i.e., a maximal convex set which does not include x) is a convex half-space [1], [2].

For any $x \in R^n$ and for any unit perpendicular vectors $v_1, \ldots, v_k$, where $1 \leqslant k \leqslant n$, we define the following sets

$$B_x(v_1, \ldots, v_k) = \bigcup_{i=1}^{k} \{y \in R^n;\ \langle v_j, v_{xy}\rangle = 0 \text{ for } j < i \text{ and } \langle v_i, v_{xy}\rangle > 0\},$$

$$P_x(v_1, \ldots, v_k) = B_x(v_1, \ldots, v_k) \cup \{y \in R^n;\ \langle v_j, v_{xy}\rangle = 0 \text{ for } j = 1, \ldots, k\}.$$

Writing $B_x(v_1, \ldots, v_k)$ or $P_x(v_1, \ldots, v_k)$ we shall tacitly assume that $x \in R^n$, $1 \leqslant k \leqslant n$ and that $v_1, \ldots, v_k$ are unit perpendicular vectors.

Let us recall that any cone excluding its vertex is called a blunt cone and any cone including its vertex is called a pointed cone.

The set $B_x(v_1, \ldots, v_k)$ is a blunt cone with vertex x as the union of half-planes

$$H_i = \{y \in R^n;\ \langle v_j, v_{xy}\rangle = 0 \text{ for } j < i \text{ and } \langle v_i, v_{xy}\rangle > 0\},$$

$i = 1, \ldots, k$, which are blunt cones with vertex x.

Let a and b be arbitrary points of $B_x(v_1, \ldots, v_k)$. There exist numbers $g \leqslant k$ and $h \leqslant k$ such that $a \in H_g$ and $b \in H_h$. Let $g \leqslant h$. Since $b \in \bar{H}_g$ and $a \in H_g = \operatorname{ri} H_g$, we have (see [4], Theorem 6.1, p. 45)

$$\{(1-\lambda)a + \lambda b;\ 0 \leqslant \lambda < 1\} \subset H_g \subset B_x(v_1, \ldots, v_k).$$

Therefore $B_x(v_1, \ldots, v_k)$ is convex. Similarly, $P_x(v_1, \ldots, v_k)$ is a pointed convex cone with vertex x.

From the obvious equalities

$$(B_x(v_1, \ldots, v_k))' = P_x(-v_1, \ldots, -v_k)$$

and

$$(P_x(v_1, \ldots, v_k))' = B_x(-v_1, \ldots, -v_k)$$

we infer that $B_x(v_1, \ldots, v_k)$ and $P_x(v_1, \ldots, v_k)$ are convex half-spaces.

The above considerations clarify the terms used in the next definition.

Definition 2. The sets $B_x(v_1, \ldots, v_k)$ and the set $\emptyset$ will be called *blunt convex half-spaces with vertex* x. The sets $P_x(v_1, \ldots, v_k)$ and the set R^n will be called *pointed convex half-spaces with vertex* x.

Theorem 1. *Convex half-spaces of R^n have the following properties*

1. *Any two complementary convex half-spaces have the forms $B_x(v_1, \ldots, v_k)$ and $P_x(-v_1, \ldots, -v_k)$, or the forms $\emptyset$ and R^n.*

2. $B_x(v_1, \ldots, v_k) = B_y(v_1, \ldots, v_k)$ *(respectively: $P_x(v_1, \ldots, v_k) = P_y(v_1, \ldots, v_k)$) if and only if $\langle v_i, v_{xy}\rangle = 0$ for $i = 1, \ldots, k$.*

3. *If K is a convex half-space, then any translate of K contains K or is contained in K* (see [1])(1). *More exactly: $B_x(v_1, \ldots, v_k) \subset B_y(v_1, \ldots, v_k)$ (analogously: $P_x(v_1, \ldots, v_k) \subset P_y(v_1, \ldots, v_k)$) if and only if $y \notin B_x(v_1, \ldots, v_k)$.*

4. *Two arbitrary convex disjoint sets can be supplemented to complementary convex half-spaces* [3].

5. *If a convex set C is the union of two convex disjoint sets, then the sets have the forms $C \cap B_x(v_1, \ldots, v_k)$ and $C \cap P_x(-v_1, \ldots, -v_k)$, where $x \in C$, or the forms $\emptyset$ and C.*

6. *A set K is a convex half-space if and only if one of the sets K, K' is a maximal convex set disjoint with a plane. This plane is that of all vertices of the convex half-space K.*

7. *The sets of the form $B_x(v_1, \ldots, v_n)$ are the only semispaces at x.*

8. *Any nonempty convex half-space is the convex hull of a sequence of points.*

(1) The question whether the inverse holds has been answered negatively by Dr. A. Prószyński with the help of the Zorn–Kuratowski lemma.

Proof. We show the first property. For $n = 1$ it is obvious. Assume it holds for R^{n-1} and consider two complementary convex half-spaces A and A' of R^n. Since the cases $A = \emptyset$ and $A = R^n$ are trivial, we assume $A \neq \emptyset$ and $A \neq R^n$. Consequently, $\bar{A} \neq R^n$ because $\operatorname{ri} A = \operatorname{ri} \bar{A}$ (Theorem 6.3 of [4], p. 46).

We shall show that $\bar{A}$ is a closed half-space. Assume the contrary. Since A is convex, $\bar{A}$ is convex too. Being convex and closed, $\bar{A}$ is the intersection of a family of closed half-spaces ([4], Theorem 11.5, p. 99). Since $\bar{A} \neq R^n$ and $\bar{A}$ is not a closed half-space, the family contains closed half-spaces Q_1 and Q_2 such that Q_2 is not a translate of Q_1. Hence $A \subset \bar{A} \subset Q_1 \cap Q_2$ and $A' \supset Q_1' \cup Q_2'$. Take $a \in A$. Since $a \in Q_1 \cap Q_2$, there exist $a_1 \in Q_1'$ and $a_2 \in Q_2'$ such that a lies in the segment joining a_1 and a_2. From $a_1 \in A'$, $a_2 \in A'$ and $a \notin A'$ we conclude that A' is not convex. The contradiction shows that $\bar{A}$ is a closed half-space.

Similarly, $\overline{A'}$ is a closed half-space. Obviously, $H = \bar{A} \cap \overline{A'}$ is the bounding hyperplane of $\bar{A}$ and of $\overline{A'}$. Consequently, $A = C_1 \cup (A \cap H)$ and $A' = C_2 \cup (A' \cap H)$ where C_1, C_2 are open half-spaces bounded by H. From the inductive hypothesis we counclude that A and A' have the stipulated forms.

The other properties easily follows from the definition of sets $B_x(v_1, \ldots, v_k)$ and $P_x(v_1, \ldots, v_k)$, and from the first property. Properties 2 and 3 are obvious. Recurrently, with the help of the classic separation theorem, we get property 4. It implies properties 5–7. The last property holds because any convex half-space is the union of a finite number of half-planes and since any half-plane is the convex hull of a sequence of points.

Theorem 2. *The family $\mathcal{T}$ of all convex half-spaces of R^n is a sequentially compact topological Fréchet space with respect to the limit of sets* Lim.

Proof. Our theorem asserts that the following conditions hold:

(1) if $A_i \in \mathcal{T}$ for $i = 1, 2, \ldots$ and $\operatorname*{Lim}_{i\to\infty} A_i = A$, then $A \in \mathcal{T}$,

(2) if $A_i = A \in \mathcal{T}$ for $i = 1, 2, \ldots$, then $\operatorname*{Lim}_{i\to\infty} A_i = A$,

(3) if $A_i \in \mathcal{T}$ for $i = 1, 2, \ldots$ and $\operatorname*{Lim}_{i\to\infty} A_i = A$, then for any subsequence A_{i_j}, $j = 1, 2, \ldots$, we have $\operatorname*{Lim}_{j\to\infty} A_{i_j} = A$,

(4) if the sequence $A_i \in \mathcal{T}$; $i = 1, 2, \ldots$, does not converge to $A \in \mathcal{T}$, then there exists a subsequence such that no subsequence of it is convergent to A,

(5) if $A_{ij} \in \mathcal{T}$, $\operatorname{Lim}_{j\to\infty} A_{ij} = A_i$ and $\operatorname{Lim}_{i\to\infty} A_i = A$ for $i, j = 1, 2, \ldots$, then there are subsequences i_m and j_m such that $\operatorname{Lim}_{m\to\infty} A_{i_m j_m} = A$,

(6) any sequence of sets from $\mathcal{T}$ contains a convergent subsequence.

It can be easily shown that conditions (2), (3) and (4) hold for arbitrary sets.

We shall show (1). Let A_i, $i = 1, 2, \ldots$, be convex half-spaces of R^n and let the limit $\operatorname{Lim}_{i\to\infty} A_i = A$ exist. Hence the limit $\operatorname{Lim}_{i\to\infty} A_i' = (\operatorname{Lim}_{i\to\infty} A_i)' = A'$ exists. Since A_i and A_i', $i = 1, 2, \ldots$, are convex, A and A' are also convex. Therefore, A is a convex half-space.

We shall prove (5). It results from (1) that the sets A_i, $i = 1, 2, \ldots$, and the set A in (5) are convex half-spaces. Let $A \neq \emptyset$ and $A \neq R^n$. By part 8 of Theorem 1 the set A is the convex hull of a sequence of points $x_1, x_2, \ldots$ The complement A' is also the convex hull of a sequence of points $y_1, y_2, \ldots$ Since $\operatorname{Lim}_{i\to\infty} A_i = A$, there exists a number i_m such that A_{i_m} contains the points $x_1, \ldots, x_m$ and does not contain $y_1, \ldots, y_m$. Since $\operatorname{Lim}_{j\to\infty} A_{i_m j} = A_{i_m}$, there exists a number j_m such that $A_{i_m j_m}$ contains the points $x_1, \ldots, x_m$ and does not contain $y_1, \ldots, y_m$. Therefore

$$\{x_1, x_2, \ldots\} \subset \operatorname{Lim}_{m\to\infty} A_{i_m j_m},$$

$$\{y_1, y_2, \ldots\} \subset \operatorname{Lim}_{m\to\infty} A_{i_m j_m}' = (\operatorname{Lim}_{m\to\infty} A_{i_m j_m})'.$$

Since $\operatorname{Lim}_{m\to\infty} A_{i_m j_m}$ and $(\operatorname{Lim}_{m\to\infty} A_{i_m j_m})'$ are convex, the inclusions

$$\operatorname{Lim}_{m\to\infty} A_{i_m j_m} \supset \operatorname{conv}\{x_1, x_2, \ldots\} = A,$$

$$(\operatorname{Lim}_{m\to\infty} A_{i_m j_m})' \supset \operatorname{conv}\{y_1, y_2, \ldots\} = A'$$

hold. Thus $\operatorname{Lim}_{m\to\infty} A_{i_m j_m} = A$. If $A = \emptyset$ or $A = R^n$ the considerations are similar.

Finally, we recurrently show (6)[2]. For R^1 it is obvious. Assume (6) holds in R^{n-1} and consider the space R^n. The case where a sequence of convex half-spaces of R^n contains infinitely many of sets $\emptyset$ or R^n is obvious. In the opposite case, select a subsequence A_i, $i = 1, 2, \ldots$, of convex half-spaces different from $\emptyset$ and R^n. Let x_i be a vertex of A_i, $i = 1, 2, \ldots$ The space R^n can be viewed as a hypersurface of an $(n+1)$-dimensional

(2) The author thanks Dr. J. Cichoń for a considerable simplification of a previous version of the proof of (6).

space R^{n+1}. Take a point $x \in R^{n+1} \setminus R^n$. Let L_i be the line passing through x_i and x, $i = 1, 2, \ldots$ Put

$$C_i = A_i + L_i = \{a+b;\ a \in A_i,\ b \in L_i\}, \quad i = 1, 2, \ldots$$

Obviously, $A_i = C_i \cap R^n$, $i = 1, 2, \ldots$ Note that C_i is a convex half-space of R^{n+1} different both from $\emptyset$ and R^{n+1} and that x is a vertex of C_i, $i = 1, 2, \ldots$ Let u_i denote a unit vector of R^{n+1} perpendicular to the hyperplane bounding $\operatorname{int} C_i$ and directed towards $\operatorname{int} C_i$. One can select a subsequence u_{i_j}, $j = 1, 2, \ldots$, which converges (in the usual sense) to a unit vector u. Let $G = B_x(u)$ in the notation of R^{n+1}. Obviously,

$$G \subset \operatorname{Lim\,inf}_{j\to\infty} C_{i_j} \subset \operatorname{Lim\,sup}_{j\to\infty} C_{i_j} \subset \bar{G}.$$

Consequently,

$$F \subset \operatorname{Lim\,inf}_{j\to\infty} A_{i_j} \subset \operatorname{Lim\,sup}_{j\to\infty} A_{i_j} \subset F \cup H,$$

where $F = G \cap R^n$ and H is the bounding hyperplane of F. By the inductive hypothesis, a subsequence $A_{i_{j_k}}$, $k = 1, 2, \ldots$, can be selected in such a way that the limit $\operatorname{Lim}_{k\to\infty}(A_{i_{j_k}} \cap H) = K$ exist. Note that

$$\operatorname{Lim\,inf}_{k\to\infty} A_{i_{j_k}} = \operatorname{Lim\,inf}_{k\to\infty} (A_{i_{j_k}} \cap F) \cup \operatorname{Lim\,inf}_{k\to\infty} (A_{i_{j_k}} \cap H) = F \cup K.$$

Similarly, $\operatorname{Lim\,sup}_{k\to\infty} A_{i_{j_k}} = F \cup K$. Thus the limit $\operatorname{Lim}_{k\to\infty} A_{i_{j_k}}$ exists.

COROLLARY 1. *Let $x \in R^n$ any $k \in \{1, \ldots, n\}$. The families*

$$\{B_x(v_1, \ldots, v_m);\ k \leqslant m \leqslant n\}, \quad \{P_x(v_1, \ldots, v_m);\ k \leqslant m \leqslant n\}$$

and the family of all convex half-spaces with the vertex x are sequentially compact topological Fréchet spaces with respect to the limit Lim.

COROLLARY 2. *For any convex set $C \subset R^n$, the family of all convex subsets D of C such that $C \setminus D$ is also convex is a sequentially compact topological Fréchet space with respect to the limit* Lim.

THEOREM 3. $\operatorname{Lim}_{i\to\infty} B_x(u_i) = B_x(v_1, \ldots, v_k)$ *(analogously: $\operatorname{Lim}_{i\to\infty} P_x(u_i) = P_x(v_1, \ldots, v_k)$) if and only if almost all vectors u_i are positive combinations $u_i = \lambda_{i1} v_1 + \ldots + \lambda_{ik} v_k$ and $\operatorname{Lim}_{i\to\infty} (\lambda_{i,j+1}/\lambda_{ij}) = 0$, $j = 1, \ldots, k-1$.*

Proof. The equalities $\operatorname{Lim}_{i\to\infty} B_x(u_i) = B_x(v_1, \ldots, v_k)$ and $\operatorname{Lim}_{i\to\infty} P_x(-u_i) = P_x(-v_1, \ldots, -v_k)$ are equivalent as the equalities of complementary sets. Therefore, we consider sequences of open half-spaces only. It is sufficient consider only the case where $x = 0$.

For R^1 the theorem is obvious. We assume that the theorem holds for R^{n-1}, and consider the space R^n.

1. Let $\operatorname{Lim}_{i\to\infty} B_0(u_i) = B_0(v_1, \ldots, v_k)$. Denote by R^{n-1} the hyperplane

bounding the half-space $B_0(v_1)$. We get

$$\operatorname*{Lim}_{i\to\infty}(R^{n-1}\cap B_0(u_i)) = R^{n-1}\cap \operatorname*{Lim}_{i\to\infty} B_0(u_i) = R^{n-1}\cap B_0(v_1, \ldots, v_k).$$

Let $k = 1$. Then

$$\operatorname*{Lim}_{i\to\infty}(R^{n-1}\cap B_0(u_i)) = R^{n-1}\cap B_0(v_1) = \emptyset.$$

Since $0\in R^{n-1}$, the sets $R^{n-1}\cap B_0(u_i)$ are either open half-spaces of the space R^{n-1} or empty. If infinitely many of them are open half-spaces of R^{n-1}, then by Corollary 1 one can select a subsequence, with the nonempty limit Lim and therefore $\operatorname*{Lim}_{i\to\infty}(R^{n-1}\cap B_0(u_i))\neq\emptyset$. The contradiction shows that almost all sets $R^{n-1}\cap B_0(u_i)$ are empty. Hence for almost all numbers $i = 1, 2, \ldots$, the equality $B_0(u_i) = B_0(v_1)$ and consequently the equality $u_i = v_1$ hold.

Let $2\leqslant k\leqslant n$. Then

$$\operatorname*{Lim}_{i\to\infty}(R^{n-1}\cap B_0(u_i)) = R^{n-1}\cap B_0(v_1, \ldots, v_k)\neq\emptyset.$$

Hence almost all sets $R^{n-1}\cap B_0(u_i)$ are nonempty. So almost all sets $R^{n-1}\cap B_0(u_i)$ are open half-spaces of R^{n-1}. If $R^{n-1}\cap B_0(u_i)$ is an open half-space of R^{n-1}, then let u_i' denote a unit vector of R^{n-1} perpendicular to $(n-2)$-dimensional plane K_i bounding in R^{n-1} the half-space $R^{n-1}\cap B_0(u_i)$ and directed towards the latter half-space. Since v_1, u_i and u_i' are perpendicular to K_i, they lie in a 2-dimensional plane. Moreover, v_1 and u_i' are perpendicular and the angle $\sphericalangle(u_i, u_i')$ is acute. Hence there exist λ_{i1} and positive γ_i such that $u_i = \lambda_{i1}v_1+\gamma_i u_i'$. From $\operatorname*{Lim}_{i\to\infty} B_0(u_i) = B_0(v_1, \ldots, v_k)$ we conclude $\lim_{i\to\infty} u_i = v_1$. Therefore $\lim_{i\to\infty}\lambda_{i1} = 1$, $\lim_{i\to\infty}\gamma_i = 0$ and almost all λ_{i1} are positive. Since almost all sets $R^{n-1}\cap B_0(u_i)$ are open half-spaces of R^{n-1}, we can apply (omitting a finite number of them) the inductive hypothesis. We infer that almost all u_i' are positive combinations $u_i' = \lambda_{i2}'v_2+\ldots+\lambda_{ik}'v_k$ and (when $k\geqslant 3$) that $\lim_{i\to\infty}(\lambda_{i,j+1}'/\lambda_{ij}') = 0$ for $j = 2, \ldots, k-1$. Put $\lambda_{ij} = \gamma_i\lambda_{ij}'$ if λ_{ij}' is defined. Therefore almost all u_i are positive combinations $u_i = \lambda_{i1}v_1+\gamma_i u_i' = \lambda_{i1}v_1+\ldots+\lambda_{ik}v_k$ and $\lim_{i\to\infty}(\lambda_{i,j+1}/\lambda_{ij}) = 0$ for $j = 2, \ldots, k-1$. Since $v_1, \ldots, v_k$ are unit and perpendicular and since u_i' is unit, $|\lambda_{i2}'|\leqslant 1$. Hence $\lim_{i\to\infty}(\lambda_{i2}/\lambda_{i1}) = \lim_{i\to\infty}(\gamma_i\lambda_{i2}'/\lambda_{i1}) = 0$.

2. Let almost all u_i be positive combinations $u_i = \lambda_{i1}v_1+\ldots+\lambda_{ik}v_k$ and let $\lim_{i\to\infty}(\lambda_{i,j+1}/\lambda_{ij}) = 0$ for $j = 1, \ldots, k-1$.

If $k = 1$, then $u_i = \lambda_{i1}v_1$, where $\lambda_{i1} = 1$, and consequently, $B_0(u_i) = B_0(v_1)$ for almost all numbers $i = 1, 2, \ldots$ Hence $\operatorname*{Lim}_{i\to\infty} B_0(u_i) = B_0(v_1)$.

Let $2\leqslant k\leqslant n$. Since u_i is a unit vector and $v_1, \ldots, v_k$ are unit perpendicular vectors, $\lambda_{i1}^2+\ldots+\lambda_{ik}^2 = 1$. Moreover, since $\lim_{i\to\infty}(\lambda_{i,j+1}/\lambda_{ij}) = 0$ for $j = 1, \ldots, k-1$, we have $\lim_{i\to\infty}\lambda_{ij} = 0$ for $j = 2, \ldots, k$ and $\lim_{i\to\infty}\lambda_{i1} = 1$. Therefore $\lim_{i\to\infty} u_i = v_1$. Hence

$$(*)\qquad B_0(v_1)\subset \operatorname*{Lim\,inf}_{i\to\infty} B_0(u_i)\subset \operatorname*{Lim\,sup}_{i\to\infty} B_0(u_i)\subset P_0(v_1).$$

Let R^{n-1} denote the hyperplane bounding the half-space $B_0(v_1)$. Since almost all combinations $\lambda_{i1}v_1+\ldots+\lambda_{ik}v_k$ are positive, almost all combinations $\lambda_{i2}v_2+\ldots+\lambda_{ik}v_k$ are also positive. Hence almost all sets $R^{n-1}\cap B_0(u_i)$ are open half-planes of the plane R^{n-1}. Since the vector $\lambda_{i1}v_1+\ldots+\lambda_{ik}v_k = u_i$ is perpendicular to the hyperplane bounding $B_0(u_i)$ and since it is directed towards the side of $B_0(u_i)$, the vector $\lambda_{i2}v_2+\ldots+\lambda_{ik}v_k$ is perpendicular to the $(n-2)$-dimensional plane K_i bounding the half-plane $R^{n-1}\cap B_0(u_i)$ and directed towards it. Hence the vector $u_i' = \alpha\lambda_{i2}v_2+\ldots+\alpha\lambda_{ik}v_k$, where $\alpha = (\lambda_{i2}^2+\ldots+\lambda_{ik}^2)^{-1}$, is unit, perpendicular and directed towards the side of $R^{n-1}\cap B_0(u_i)$. This is true for almost all numbers $i = 1, 2, \ldots$ Obviously, almost all the combinations above are positive and $\lim_{i\to\infty}(\alpha\lambda_{i,j+1}/\alpha\lambda_{ij}) = 0$ for $j = 2, \ldots, k-1$. Hence, from the inductive hypothesis we conclude that the set $\operatorname*{Lim}_{i\to\infty}(B_0(u_i)\cap R^{n-1})$ is equal (in the notation of R^{n-1}) to $B_0(v_2, \ldots, v_k)\subset R^{n-1}$. This and $(*)$ imply

$$\operatorname*{Lim\,inf}_{i\to\infty} B_0(u_i) = B_0(v_1, \ldots, v_k) = \operatorname*{Lim\,sup}_{i\to\infty} B_0(u_i).$$

Therefore the limit $\operatorname*{Lim}_{i\to\infty} B_0(u_i)$ exists and equals $B_0(v_1, \ldots, v_k)$.

The proof is complete.

It would be interesting to discuss decompositions of R^n onto $m > 2$ disjoint convex subsets.

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Reçu par la Rédaction le 10. 3. 1980
Accepté par la Rédaction le 21. 9. 1981