then $\{N \in Y\colon \forall a > \omega \exists I < a\, I$ codes $N\}$ is the complement of a set of first category in the Cantor set Y.

5.3. What is the exact distribution of the values of the functions F_n? In particular, is the type $\{x > F_n(n)\colon n \in \omega\} \cup \{C_n(x)\colon n \in \omega\}$ consistent?

5.4. For $X \subseteq M \models \mathrm{PA}$ we define the closure of X in the usual way: $b \in \mathrm{cl}(X)$ iff, for each $g \in \mathrm{Aut}(M)$, if $\forall x \in X\, g(x) = x$ then $g(b) = b$.

Conjecture. There exist two consistent extensions Δ_1, Δ_2 of the type Γ_0 (cf. the proof of Theorem 3.2) such that, for each countable and recursively saturated $M \models \mathrm{PA}$, if b_1 realizes Δ_1 in M then $\mathrm{cl}(M[b]) =$ the Skolem closure of $(M[b] \cup \{b\})$ and if b_2 realizes Δ_2 in M then $\mathrm{cl}(M[b]) \supsetneqq$ the Skolem closure of $(M[b] \cup \{b\})$.

5.5. Conjecture (Smorynski [11]). Let $M \models \mathrm{PA}$ be countable and recursively saturated, and let $b_1, b_2 \in M$. If $(M, M[b_1])$ is elementary equivalent to $(M, M[b_2])$ then $M(b_1)$ is isomorphic with $M(b_2)$.

The author would like to thank Roman Kossak for fruitful discussions.

References

[1] L. Kirby, *Initial segment of models of Arithmetic*, Doctoral thesis, Manchester 1977.

[2] — and J. Paris, *Initial segments of models of Peano axioms*, in: Set theory and hierarchy theory V, Springer Lecture Notes 619 (1977), pp. 211–226.

[3] — K. Mc Aloon and R. Murawski, *Indicators, recursive saturation and expandability*, Fund. Math. 114 (1981), pp. 127–139.

[4] H. Kotlarski, *On elementary cuts in models of arithmetic*, Fund Math. 115 (1983), pp. 27–31.

[5] — *Elementary cuts in models of arithmetic*, abstract, to appear in Bull. Acad. Polon. Sci.

[6] S. Krajewski, *Non-standard satisfaction classes*, in: Set theory and hierarchy theory, Springer Lecture Notes 537 (1976), pp. 125–144.

[7] H. Lessan, *Models of arithmetic*, Doctoral thesis, Manchester 1978.

[8] J. Paris, *Some independence results for Peano Arithmetic*, J. Symb. Logic. 43 (4) (1978), pp. 725–732.

[9] J. Schlipf, *Toward model theory through recursive saturation*, J. Symb. Logic 43 (2) (1978), pp. 183–203.

[10] C. Smorynski, *Recursively saturated non-standard models of arithmetic*, to appear in J. Symb. Logic. 46 (1981), pp. 259–286.

[11] — *Elementary extensions of recursively saturated models of arithmetic*, Notre Dame J. Formal Logic 22, pp. 193–203.

Accepté par la Rédaction le 3.8.1981

On locally contractive fixed-point mappings

by

Ljubomir Ćirić (Beograd)

Abstract. Let (M, d) be a metric space and T a selfmapping on M. Suppose that for each $u \in M$ there exists a sphere $S(u, r(u))$ such that $x, y \in S(u, r(u))$ with $x \neq y$ implies $d(Tx, Ty) < d(x, y)$ and $Tx, Ty \in S(v, r(v))$ for some $v \in M$. Furthermore, suppose that $\{T^n x\}$ contains a convergent subsequence for some $x \in M$. Under these assumptions our main result states that the set of fixed or periodic points of T is non-void. This generalizes one result of M. Edelstein for ε-contractive mappings. A fixed point theorem for corresponding mappings on Hausdorff uniform spaces is stated also.

Introduction. Let (M, d) be a metric space and T a selfmapping on M. A mapping T is said to be *locally contractive on* M if for each $u \in M$ there exists a sphere $S(u, r(u)) = \{x\colon d(u, x) < r(u)\}$, $r(u) > 0$, such that $d(Tx, Ty) < d(x, y)$ holds for all $x, y \in S(u, r(u))$ with $x \neq y$. If there exists $\varepsilon > 0$ such that $r(u) \geq \varepsilon$ for all $u \in M$, then T is called *ε-contractive*. M. Edelstein in [3] proved that if $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$, then an ε-contractive mapping has fixed or periodic points. On compact spaces locally contractive mappings are ε-contractive, and therefore have fixed or periodic points. However, M. Edelstein in [3] and S. Naimpally in [4] have constructed examples which show that if M is not compact, then locally contractive mappings may be without fixed or periodic points, even though $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$.

Our aim is to present a subclass of locally contractive mappings which need not be ε-contractive, but still have fixed or periodic points in the case that $\{T^n x\}$ contrains a convergent subsequence for some $x \in M$.

Definition. A mapping T of a metric space M into itself is said to be *well locally contractive* if for each $u \in M$ there exists $S(u, r(u))$ such that $x, y \in S(u, r(u))$ with $x \neq y$ implies

$$d(Tx, Ty) < d(x, y) \quad \text{and} \quad Tx, Ty \in S(v, r(v))$$

for some $v \in M$.

1. Now we shall prove the following result.

Theorem 1. *Let T be a well locally contractive selfmapping on a metric* *space M. If $\{T^n x\}$, $x \in M$, contains a convergent subsequence $\{T^{n_i}x\}$ then $u = \lim_{i\to\infty} T^{n_i}x$ is a periodic point of T.*

Proof. Let x in M be such that $\lim T^{n_i}x = u$ for some $u \in M$. We may suppose that $T^r x \neq T^s x$ for all positive integers r and s, since otherwise the theorem follows immediately. Choose fixed positive integers p and k such that

$$T^p x,\ T^{p+k}x \in S(u, r(u)) \tag{1}$$

with $d(T^p x, T^{p+k}x) < \frac{1}{3}r(u)$. We shall show that $T^k u = u$.

As T is well locally contractive, (1) implies that for $n = p+1$ we have

$$d(T^n x, T^{n+k}x) = d(TT^p x, TT^{p+k}x) < d(T^p x, T^{p+k}x)$$

and $T^{p+1}x$, $T^{p+1+k}x \in S(v, r(v))$ for some $v \in M$. This again implies that

$$d(TT^{p+1}x, TT^{p+1+k}x) \leqslant d(T^{p+1}x, T^{p+1+k}x)$$

and $T^{p+2}x$, $T^{p+2+k}x \in S(w, r(w))$ for some $w \in M$, and so

$$d(T^n x, T^{n+k}x) < d(T^p x, T^{p+k}x) \quad \text{for} \quad n = p+2.$$

If we proceed in this manner we conclude that (1) implies

$$d(T^n x, T^{n+k}x) < d(T^p x, T^{p+k}x) \quad \text{for all } n > p. \tag{2}$$

Since a locally contractive mapping is continuous and $\lim_{i\to\infty} T^{n_i}x = u$, it follows from (2) that

$$d(u, T^k u) \leqslant d(T^p x, T^{p+k}x).$$

This and $d(T^p x, T^{p+k}x) < \frac{1}{3}r(u)$ imply $T^k u \in S(u, \frac{1}{3}r(u))$.

Assume now that $T^k u \neq u$. Then

$$u,\ T^k u \in S(u, \tfrac{1}{3}r(u))$$

implies, similarly as (1) implies (2), that

$$d(T^n u, T^{n+k}u) < d(u, T^k u) \quad \text{for all } n. \tag{3}$$

Now we shall show that

$$T^n x \in S(u, \tfrac{1}{3}r(u)) \text{ with } n > p \text{ implies } T^{n+k}x,\ T^{n+2k}x \in S(u, r(u)). \tag{4}$$

Let $d(u, T^n x) < \frac{1}{3}r(u)$ and $n > p$. Then by (2) and by triangle inequality we have

$$d(u, T^{n+k}x) \leqslant d(u, T^n x) + d(T^n x, T^{n+k}x) < \tfrac{1}{3}r(u) + d(T^p x, T^{p+k}x),$$

$$\begin{aligned} d(u, T^{n+2k}x) &\leqslant d(u, T^{n+k}x) + d(T^{(n+k)}x, T^{(n+k)+k}x) \\ &< d(u, T^{n+k}x) + d(T^p x, T^{p+k}x). \end{aligned}$$

Hence, as $d(T^p x, T^{p+k}x) < \frac{1}{3}r(u)$, we obtain that

$$d(u, T^{n+k}x) < \tfrac{2}{3}r(u);\ d(u, T^{n+2k}x) < r(u).$$

Thus we have proved (4).

Now, as $\lim_{i\to\infty} T^{n_i}x = u$, we may suppose that each n_i is chosen such that $n_i > p$ and

$$T^{n_i}x \in S(u, \tfrac{1}{3}r(u)).$$

Then, by (4), for any fixed n_i we have

$$T^{n_i+k}x,\ T^{n_i+2k}x \in S(u, r(u)).$$

This implies, similarly as (1) implies (2), that

$$d(T^n x, T^{n+k}x) < d(T^{n_i+k}x, T^{(n_i+k)+k}x) \quad \text{for all } n > n_i + k.$$

Hence, as $d(u, T^k u)$ is a cluster point of $\{d(T^n x, T^{n+k}x)\}_{n=0}^{\infty}$,

$$d(u, T^k u) \leqslant d(T^{n_i+k}x, T^{n_i+2k}x).$$

But this and $\lim_{i\to\infty} T^{n_i}x = u$ imply the following relation:

$$d(u, T^k u) \leqslant \lim_{i\to\infty} d(T^{n_i+k}x, T^{n_i+k}x) = d(T^k u, T^{2k}u)$$

which contradicts (3) for $n = k$. Therefore, our assumption that $T^k u \neq u$ — is not correct and so u is a fixed (in the case $k = 1$) or periodic point of T. The proof is complete.

It is clear that each ε-contractive mapping is also well contractive. So we have

COROLLARY 1 (Edelstein [3, Th. 2]). *Let M be a metric space, T an ε-contractive selfmapping on M and let $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$. Then u is a periodic point.*

The following example shows that our Theorem 1 do in fact improve the Edelstein's result.

EXAMPLE. Let $M_1 = \{(x, 1/x)\colon x \geqslant 1\}$, $M_2 = \{(x, -1/x)\colon x \geqslant 1\}$ and put $M = M_1 \cup M_2$. Define on $M \subset R^2$ a mapping T by

$$T\left(x, \frac{1}{x}\right) = \left(\frac{x+1}{2}, -\frac{2}{x+1}\right) \quad \text{and} \quad T\left(x, -\frac{1}{x}\right) = \left(\frac{x+1}{2}, \frac{2}{x+1}\right).$$

If for any $u = (x, 1/x)$ and $v = (x, -1/x)$ we take, for instans, $r(u) = r(v) = 1/x$, then it is easy to see that T is well locally contractive. Since for each $z \in M$ the sequence $\{T^n z\}$ contains the convergent sub-sequence $\{T^{2n}z\}$, we may apply our Theorem and $u = (1, 1)$ and $Tu = (1, -1)$ are periodic points of T.

However, Edelstein's theorem is not applicable, as T is not ε-contractive for any $\varepsilon > 0$. Indeed, for any $\varepsilon > 0$ there are points $x = (a, 1/a)$ and $y = (a, -1/a)$ with $a > 2/\varepsilon$ (and $a \neq 1$) such that

$$d(Tx, Ty) > d(x, y),$$

although $d(x, y) = 2/a < \varepsilon$.

Note that T is not also the Bailey's mapping (6) of [1, p. 101], since for the above chosen x and y in M the relations

$$d(x, y) > d(Tx, Ty) > d(T^2x, T^2y) > \ldots > d(T^nx, T^ny) > \ldots$$

hold, although $d(x, y) < \varepsilon$.

2. Now we are going to present sufficient conditions for the existence of a fixed point of well locally contractive mappings.

We remember that a metric space M is said to be *convex* provided x and y in M implies there exists z in M such that $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$.

THEOREM 2. *If T is a well locally contractive selfmapping on a complete convex metric space M and if for some x in M a subsequence of $\{T^nx\}$ is contained in a compact subset of M then T has a unique fixed point.*

Proof. By Theorem 1 there exists some u in M such that $T^ku = u$ for some $k \geq 1$. Assume that $Tu \neq u$. Since M is a complete convex space there exists a metric interval $[u, Tu]$ in M. Therefore, there exists $\varepsilon > 0$ such that x, y in $[u, Tu]$ and $d(x, y) < \varepsilon$ imply that x, y in $S(z, r(z))$ for some z in $[u, Tu]$. Now let

$$u = x_0, x_1, \ldots, x_m = Tu$$

be a finite sequence of elements in $[u, Tu]$ such that

$$d(u, Tu) = \sum_{j=1}^{m} d(x_{j-1}, x_j) \quad \text{and} \quad d(x_{j-1}, x_j) < \varepsilon \ (j = 1, 2, \ldots, m).$$

Since $x_{j-1}, x_j \in S(z, r(z))$ for some $z \in [u, Tu]$ and T is well locally contractive, we have

$$d(T^kx_{j-1}, T^kx_j) < d(x_{j-1}, x_j) \quad (j = 1, 2, \ldots, m).$$

Hence, as $T^ku = u$, we obtain

$$d(u, Tu) = d(T^ku, T^kTu) \leq \sum_{j=1}^{m} d(T^kx_{j-1}, T^kx_j)$$
$$< \sum_{j=1}^{m} d(x_{j-1}, x_j) = d(u, Tu),$$

which is absurd. Therefore, $Tu = u$. The uniqueness of a fixed point follows by the same arguments. This completes the proof.

Note that for ε-chainable metric spaces M Edelstein [3] proved the following result:

THEOREM A. *Let M be an ε-chainable metric space, T an ε-contractive selfmapping of M and let $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$. If u has a compact spherical neighborhood $K[u, r]$ of radius $r \geq \varepsilon$ then u is a unique fixed point.*

For well locally contractive mappings a similar result is not valid. Indeed, in our example the space M is ε-chainable for any $\varepsilon > 0$ and for each $z = (x, 1/x)$ in M the sequence $\{T^nz\}$ contains a convergent subsequence. Also $u = (1, 1)$ (and $v = (1, -1)$) has a compact neighborhood $K[u, r]$ for any $r < +\infty$, but T is a fixed point free.

THEOREM 3. *Let T be well locally contractive on M. If for some $x \in M$ a certain subsequence of $\{T^nx\}$ is contained in a compact subset of M and*

$$\inf_{n>0} d(T^nx, T^{n+1}x) = 0, \tag{5}$$

then the set of fixed points of T is non-void.

Proof. By (5) we may choose p such that $d(T^px, T^{p+1}x) < \frac{1}{3}r(u)$ and following the arguments given in the proof of Theorem 1 (for $k = 1$) we obtain $Tu = u$.

COROLLARY 2. *Let T be well locally contractive on M. If $\lim_{i\to\infty} T^{n_i}x = u \in M$ for some $x \in M$ and T is asymptotically regular at x (i.e. $\lim_{i\to\infty} d(T^nx, T^{n+1}x) = 0$), then the set of fixed points of T is non-void.*

Note that in Theorem 3 a well locally contractive mapping T can not be replaced by a mapping T which satisfies the following conditions: T is continuous and

$$d(T^{n_k}x, T^{n_k}y) < d(x, y) \tag{6}$$

holds for some subsequence $\{n_k\}$ of positive integers (and $x \neq y$), even if $S(u, r(u)) = M$ for all $u \in M$. To show this, we shall use Bailey's counter example X from [1], pages 103 and 105. If Bailey's X we replace by $M = X\setminus\{(0, 0)\}$, then for every $x \in M$ with $x \neq (1/i, 0)$, $i \in I$, the sequence $\{T^nx\}$ contains a convergent subsequence in M and T satisfies (5) on M. The condition (6) is also satisfied, as Bailey's condition (5) of [1] implies (6). However, T has not fixed neither periodic points in M.

3. Now we shall show how fixed-point theorems of § 2 can be extended to Hausdorff topological spaces, whose topology can be generated by the family of pseudometrics.

Let X be a nonempty set and $\mathscr{P}$ a nonempty family of pseudo-metrics on X such that the collection $\{S_p(x, r)\colon x \in X, 0 < r < +\infty\}$ forms a base for a Hausdorff topology for X, where $S_p(x, r)$ is the open sphere of p-radius r about x (see Kelley [4]). It is known that a set X has the structure which can be such described iff X with a certain uniformity is a Hausdorff uniform space, or iff X with a certain topology is a Hausdorff completely regular space.

Note also that a sequence $\{x_n\}$ in X converges to u iff it converges to u in p-topology for all $p \in \mathscr{P}$, or equivalently, the numbers $p(x_n, u)$ converges to zero for all $p \in \mathscr{P}$. For $x, y \in X$, $x \neq y$ iff there exists some $p \in \mathscr{P}$ such that $p(x, y) > 0$.

Now we shall present an extended form of Theorem 3.

THEOREM 4. *Let X be a Hausdorff topological space and $\mathscr{P}$ a family of pseudo-metrics which generate the topology on X. Let $T\colon X \to X$ be a mapping such that for each $u \in M$ and $p \in \mathscr{P}$ there exists an open sphere $S_p(u, r_p(u))$ such that $x, y \in S_p(u, r_p(u))$ with $p(x, y) > 0$ implies*

$$p(Tx, Ty) < p(x, y) \quad \text{and} \quad Tx, Ty \in S_p(v, r_p(v))$$

for some $v \in X$. If $\lim_{i\to\infty} T^{n_i}x \in M$ for some $x \in M$ and

$$\inf_{n>0} p(T^n x, T^{n+1}x) = 0 \tag{7}$$

holds for every $p \in \mathscr{P}$, then the set of fixed point of T is non-void.

Proof. Let p be any member of $\mathscr{P}$. If in the proof of Theorem 1 we replace $d(x, y)$ by $p(x, y)$ and $r(u)$ by $r_p(u)$, then by (7) we may choose a positive integer m such that $p(T^m x, T^{m+1}x) < \frac{1}{3}r_p(u)$ and (as $\lim_{i\to\infty} T^{n_i}x = u$) $T^m x, T^{m+1}x \in S_p(u, r_p(u))$.

Following arguments given in the proof of Theorem 1 we obtain that $p(Tu, u) = 0$. Since $p \in \mathscr{P}$ was arbitrary, it follows that $p(Tu, u) = 0$ for all $p \in \mathscr{P}$. Therefore, $Tu = u$ and the proof is complete.

References

[1] D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. 41 (1966), pp. 101–106. MR 32 # 6434.

[2] Lj. Ćirić, *Fixed and periodic points for a class of contractive operators*, Publ. Inst. Math. 18 (32) (1975), pp. 57–69. MR 55 # 4121.

[3] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962), pp. 74–79. MR 24 # 2936.

[4] J. L. Kelley, *General Topology*, Van Nostrand, Princeton, N. J., 1955. MR 16 # 1136.

[5] S. A. Naimpally, *A note on contractive mappings*, Indag. Math. 26 (1964), pp. 275–279. MR 29 # 590.

[6] E. Rakotch, *A note on α-locally contractive mappings*, Bull. Res. Council Israel, F 10 F (1962), pp. 188-191. MR 26 # 4319.

[7] — *On ε-contractive mappings*, Bull. Res. Council Israel, F 10 F (1961), pp. 53–58. MR 32 # 4685.

[8] B. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. 226 (1977), pp. 257–290. MR 55 # 6406.

Accepté par la Rédaction le 7.12.1981

Weak-chainability of tree-like continua and the combinatorial properties of mappings

by

T. A. Moebes (Houston, Tex.)

Abstract. In 1951, R. H. Bing mentioned the question of the existence of atriodic tree-like continua which are not chainable. In 1972, W. T. Ingram constructed an example of an atriodic tree-like continuum with positive span which is not chainable. A. Lelek introduced the notion of weak chainability and characterized it by the property of being a continuous image of a chainable continuum. A. Lelek introduced the concept of span and proved chainable continua have span zero. The question of Ingram's example of 1972 mentioned above being weakly chainable was mentioned by W. T. Ingram in 1976.

We present a theorem in this paper that gives sufficient conditions for a continuum expressed in terms of inverse expansions in finite trees not to be weakly chainable. Since Ingram's example given in 1972 was obtained as an inverse limit on simple triods, our theorem is applied to show that this example is not weakly chainable. The argument given is not span dependent but does, however, depend upon the combinatorial properties of the bonding maps of the inverse system in question.

1. Introduction. In 1972, W. T. Ingram [1] constructed an example of an atriodic tree-like continuum with positive span. This example in [2] answered the question mentioned by R. H. Bing [1] of the existence of atriodic tree-like continua which are not chainable. A. Lelek [4] introduced the notion of weak chainability and characterized it by the property of being a continuous image of a chainable continuum. A. Lelek [5] introduced the concept of span and proved chainable continua have span zero (p. 210). The question of the continuum given in [2] being weakly chainable was mentioned by W. T. Ingram in [3]. In this paper we give a theorem that gives sufficient conditions for a continuum expressed in terms of inverse expansions in finite trees not to be weakly chainable. Since the continuum given in [2] was obtained as an inverse limit on simple triods, our theorem is applied to show that the example given in [2] is not weakly chainable. The argument given is not span dependent. The argument does, however, depend upon the combinatorial properties of the bonding maps of the inverse system in question. The bonding maps between consecutive factor spaces do not necessarily have to be identical for the main theorem in this paper to apply.

This paper makes use of the results given in 1963 by J. Mioduszewski [6] for a compact metric space to be a continuous image of another one expressed in terms of inverse expansions in polyhedra. This paper also makes