ding" introduced in Section 4 is not the correct replacement for "semilattice monomorphism", but the reasoning of Theorem 4.11 is quite general, and should go through with other notions of strong embedding. We conclude that Example 4.10 is a fundamental obstacle to finding a single concept of $\varkappa$-distributivity which will extend both Theorem 1.5 and Theorem 4.1 to arbitrary partially ordered sets. As a final remark, we note that if we restrict ourselves to finite partially ordered sets, all the difficulties just mentioned disappear.

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Axiomatization of the forcing relation with an application to Peano Arithmetic

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Abstract. In the paper we describe those formal properties of forcing which set theoretical forcing and the method of indicators in Peano Arithmetic have in common.

Introduction. We describe a formal similarity between forcing in set theory and the indicator method in Peano Arithmetic.

Let $\boldsymbol{P}$ be a set of forcing conditions, $\langle \mathscr{Y}, \mathcal{O}\rangle$ a topological space in $V^{\boldsymbol{P}}$ which is V-codable, i.e. such that there is a set $\langle 0, \leqslant\rangle$ and an isomorphism φ in $V^{\boldsymbol{P}}$ of $\langle 0, \leqslant\rangle$ and $\langle \mathcal{O}, \subseteq\rangle$ such that the relation "$y \in \varphi(\check{q})$" for a $q \in 0$ is an absolute relation of y and q w.r.t. V and $V^{\boldsymbol{P}}$ (see [1]). We shall always assume that $q_1 q_2 \in \mathcal{O} \,\&\, q_1 \cap q_2 \neq \emptyset \Rightarrow q_1 \cap q_2 \in \mathcal{O}$.

Let $y \in V^{\boldsymbol{P}}$ be an element of $\mathscr{Y}$. Let us identify in $V^{\boldsymbol{P}}$ $\langle 0, \leqslant\rangle$ with $\langle \mathcal{O}, \subseteq\rangle$.

We formulate two systems of axioms characterizing, respectively, the following relations:

$$R(p, q) \text{ defined as } p \Vdash (\underset{\sim}{y} \in \check{q})$$

and

$$R'(p, q) \text{ defined as } p \wedge \|\underset{\sim}{y} \in \check{q}\| \neq \emptyset.$$

Then R satisfies the first system of axioms iff there is a $\underset{\sim}{y} \in V^{\boldsymbol{P}}$ such that R is the relation $p \Vdash (\underset{\sim}{y} \in \check{q})$ and R' satisfies the second system iff there is a $\underset{\sim}{y} \in V^{\boldsymbol{P}}$ such that R' is the relation $p \wedge \|\underset{\sim}{y} \in \check{q}\| \neq \emptyset$. We call relations of the first type *forcing relations* and those of the second type *consistency relations*.

We show that the Kirby–Paris indicator for models of PA defined by means of a game where questions are Gödel numbers of formulas naturally determines a relation satisfying the second group of axioms, i.e., a consistency relation.

We also show that there is a strict correspondence between consistency and forcing relations.

A consistency relation canonically determines a forcing relation and conversely. Thus the Kirby–Paris indicator determines a forcing relation. This explains certain analogies between the forcing and the indicator method.

The forcing relation determined by the indicator is not definable within

PA. The consistency relation in question is definable in PA like the indicator itself. That is why it is more natural in PA to consider this consistency relation or the indicator itself and not the corresponding forcing relation.

On the other hand, forcing in set theory can be defined game-theoretically similarly to the way in which the Kirby–Paris indicator is defined. This shows the similarity of forcing and indicator in the other direction.

The paper is organized as follows:

Section 1 is devoted to forcing in set theory. We first introduce by algebraical axioms the notion of a quasi-forcing relation. An illustration of this notion is forcing in set theory and also a certain relation in recursion theory.

Then we strengthen our axioms and we obtain a characterization of relations of the form $p \Vdash (\underline{y} \in \check{q})$ — we call them forcing relations.

Theorem 1 expresses the equivalence: R is a forcing relation iff there is a $\underline{y} \in V^{\boldsymbol{P}}$ such that R is the relation $p \Vdash (\underline{y} \in \check{q})$.

We finish § 1 by defining forcing in set theory game-theoretically.

In § 2 we introduce the notion of a consistency relation. We show the one-one correspondence between forcing and consistency relations. We define the consistency relation determined by the usual indicator for models of PA. We define the corresponding forcing relation and show that it is not definable in PA. Finally, using our notions, we give a proof, simpler than the original one, of the independence of a sentence analogous to the Kirby–Paris sentence.

Our sentence is very similar to the Kirby–Paris sentence; however, we do not know whether it is equivalent to the Kirby–Paris sentence.

§ 1

Let us introduce the following notion of a quasi-forcing relation.

Definition 1. Let $\boldsymbol{P}$ be a partially ordered set. Let $\langle \mathscr{Y}, \mathscr{O} \rangle$ be a T_2 topological space. Let $R \subseteq \boldsymbol{P} \times \mathscr{O}$ be called a *quasi-forcing relation* iff

(1) $\langle p, q \rangle \in R \,\&\, p' \leqslant p \Rightarrow \langle p', q \rangle \in R,$

$\langle p, q \rangle \in R \,\&\, q' \supseteq q \Rightarrow \langle p, q' \rangle \in R,$

(2) $\langle p, q \rangle \in R \,\&\, \langle p, q' \rangle \in R \Rightarrow q \cap q' \neq \emptyset \,\&\, R(p, q \cap q').$

Example 1. Notice first that the usual forcing relation in set theory provides an example for this definition. Indeed, let M be an inner model. Let $\boldsymbol{P} \in M$ and let $\langle \mathscr{Y}, \mathscr{O} \rangle$ be a topological space such that $\mathscr{O}$ is M-codable (see [1]). Assume that $\mathscr{Y} \subseteq \omega^\omega$. Let $\underline{y}$ be an element of $M^{\boldsymbol{P}}$ denoting a real. Then if we define $R(p, q) \equiv p \Vdash (\underline{y} \in \check{q})$ where q codes an element of $\mathscr{O}$, then R is a quasi-forcing relation.

Let us consider a less trivial example of a quasi-forcing relation.

Example 2. Let $\boldsymbol{P} = \omega^{<\omega}$ with the inverse inclusion. Consider the functions $\{e\}^x \in \omega^\omega$ for a given e and x running over ω^ω. Then there is a recursive relation $R_e \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that for $t \in \omega^{<\omega}$, $t \subseteq \{e\}^x \equiv (\mathrm{E}s)_{\subseteq x} R_e(s, t)$.

Let $\langle \mathscr{Y}, \mathscr{O} \rangle$ be the Baire space. Then if we define $R(p, q)$ as

$$R(p, q) \equiv (\mathrm{E}p')(\mathrm{E}q')\big(p \subseteq p' \,\&\, R_e(p', q') \,\&\, q' \subseteq q\big)$$

(note that we identify p, q with finite sequences determining them) then R is quasi-forcing relation. In other symbols we have

$$R(s, t) \equiv (\mathrm{E}n)(\mathrm{E}t')\big(t \subseteq t' \,\&\, R_e(s_{\restriction n}, t')\big).$$

Let us make a few remarks.

Notice that (1) of Definition 1 corresponds to the Cohen "extension" lemma and (2) to the "consistency lemma". As we shall see, the third Cohen lemma, i.e., the "truth lemma", in a suitably restricted form is a consequence of (1), (2), i.e., of the extension lemma and the consistency lemma, no matter what concrete quasi-forcing relation is considered.

Remark 1. *Let $\langle \mathscr{X}, \mathscr{O} \rangle$ be a topological space with the basis $\mathscr{O}$ induced by $\boldsymbol{P}$, i.e., there is an isomorphism φ of $\langle \boldsymbol{P}, \leqslant \rangle$ and $\langle \mathscr{O}, \subseteq \rangle$. For example, $\langle \mathscr{X}, \mathscr{O} \rangle$ can be the space of ultrafilters in $\boldsymbol{P}$* (see [2]) *or if $\boldsymbol{P} = \omega^{<\omega}$ it can be the Baire space $\langle \omega^\omega, \omega^{<\omega} \rangle$. Let $\langle \mathscr{Y}, \mathscr{O}' \rangle$ be given, $\langle \mathscr{Y}, \mathscr{O}' \rangle$ codable. Let $R \subseteq \boldsymbol{P} \times \mathscr{O}'$ be a quasi-forcing relation. Then there is a function $f\colon \mathscr{X} \to \mathscr{Y}$ such that for $x \in \operatorname{Dom} f$, $q \in \mathscr{O}'$ we have*

(*) $$f(x) \in q \equiv (\mathrm{E}p)\big(x \in p \,\&\, R(p, q)\big)$$

and f is maximal among functions with the property (*).

Before proving this remark let us refer again to set theory. Let M, $\boldsymbol{P}$, $\mathscr{Y}$, $\mathscr{O}'$, $\underline{y}$ be as before. Let $\langle \mathscr{X}, \mathscr{O} \rangle$ be the space of ultrafilters in $\boldsymbol{P}$. Then f is an extension of f' defined as

$$f'(x) = i_x(\underline{y})$$

for x being generic over M, $\boldsymbol{P}$.

Then we have

$$i_x(\underline{y}) \in q = (\mathrm{E}p)\big(x \in p \,\&\, (p \Vdash \underline{y} \in \check{q})\big),$$

which is an instance of the original Cohen truth lemma.

In the case of recursion we have

$$t \subseteq \{e\}^x \equiv (\mathrm{E}s)\big(s \subseteq x \,\&\, R(s, t)\big)$$

and we can define $f(x) = \{e\}^x$.

Proof of Remark 1. Define f as

$$\langle x, y \rangle \in f \equiv (q)_{\mathscr{O}'}\big(y \in q \Leftrightarrow (\mathrm{E}p)(x \in p \,\&\, R(p, q))\big).$$

It suffices to show that f is a function, because $(*)$ and the maximality is immediate by the definition. Indeed, let $\langle x, y\rangle, \langle x, y'\rangle \in f$. Let $y \in q$, $y' \in q'$. Then there are p, p' such that $x \in p \cap p'$ and $R(p, q)$, $R(p', q')$. Let p'' be such that $p'' \subseteq p \cap p'$. Then by (1) of Definition 1 $R(p'', q)$, $R(p'', q')$. Hence by (2) $q \cap q' \neq \emptyset$. Thus $y = y'$ because q, q' were arbitrary and $\langle \mathscr{Y}, \mathscr{O}'\rangle$ is T_2. ■

Remark 2. If R is a quasi-forcing relation then the f satisfying $(*)$ is continuous in $\mathscr{O}$ as a function from $\mathscr{X}$ to $\mathscr{Y}$.

Indeed, this follows directly from $(*)$.

Let us recall here that there is a possible approach to the axiomatization of forcing starting from the "truth lemma" by means of purelly topological notions, mainly the continuity. We did it in [3]. In fact, in [3] we dealt not with quasi-forcing relations but with forcing relations defined below. We give a full topological characterization of the relation $p \Vdash (\underset{\sim}{y} \in \check{q})$.

In § 2 we shall see another example of a quasi-forcing relation.

Now let us limit our definition in such a way as to preserve its algebraical character but to characterize fully the usual forcing relation.

Definition 2. Let $\boldsymbol{P}$, $\mathscr{Y}$, $\mathscr{O}$ be as before. Let $R \subseteq \boldsymbol{P} \times \mathscr{O}$ be a *forcing relation* iff

(1) $\langle p, q\rangle \in R \,\&\, p' \leqslant p \Rightarrow \langle p', q\rangle \in R$,
$\langle p, q\rangle \in R \,\&\, q' \supseteq q \Rightarrow \langle p, q'\rangle \in R$,

(2) $\langle p, q\rangle \in R \,\&\, \langle p, q'\rangle \in R \Rightarrow q \cap q' \neq \emptyset \,\&\, \langle p, q \cap q'\rangle \in R$,

(3) $(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'} R(p'', q) \Rightarrow R(p, q)$,

(4) $(q)_{\mathscr{O}}$ the set $D_q = \{p \in \boldsymbol{P}\colon (\mathrm{E}q')((q' \subseteq q \text{ or } q' \cap q = \emptyset) \,\&\, R(p, q'))\}$ is dense in $\boldsymbol{P}$.

Theorem 1. *Let $\langle \mathscr{Y}, \mathscr{O}'\rangle$ be a topological space V-codable in $V^{\boldsymbol{P}}$. Let $\langle \mathscr{Y}, \mathscr{O}'\rangle$ have the centralization property, i.e., any subfamily $\mathscr{O}'' \subseteq \mathscr{O}'$ such that all intersections of finite subsets of $\mathscr{O}''$ are non-empty, has a non-empty intersection $\bigcap \mathscr{O}''$* (see [3]). *Let $R \subseteq \boldsymbol{P} \times \mathscr{O}'$. Then R is a forcing relation iff there is a $\underset{\sim}{y}$ in $V^{\boldsymbol{P}}$ such that $R(p, q) \equiv (p \Vdash \underset{\sim}{y} \in \check{q})$.*

Proof. We shall first give a general proof of this lemma and then we shall give a simpler proof in the case where $\langle \mathscr{Y}, \mathscr{O}'\rangle$ is the Baire space, because this is the most natural case.

To prove that for a $\underset{\sim}{y}$ from $V^{\boldsymbol{P}}$ the relation $p \Vdash (\underset{\sim}{y} \in \check{q})$ is a forcing relation it is enough to use standard properties of forcing and it is easy.

So let us assume that R is a forcing relation. Let $\langle \mathscr{X}, \mathscr{O}\rangle$ be the space of ultrafilters in $\boldsymbol{P}$. Let f_R be the function from $\mathscr{X}$ to $\mathscr{Y}$ defined for R in Remark 1. For simplicity work in a boolean extension of V in which $(p)_{\boldsymbol{P}}(\mathrm{E}x)$ (x is generic over $V, \boldsymbol{P}$), but we can also work in $V^{\boldsymbol{P}}$. If x is generic over $V, \boldsymbol{P}$ (let us say "$\boldsymbol{P}$-generic") then $x \in \operatorname{Dom} f_R$ — we prove it as the last part of Fact 1 in [3]. Let us recall this proof.

We have
$$\langle x, y\rangle \in f_R \equiv (q)_{\mathscr{O}'}\big(y \in q \Rightarrow (\mathrm{E}p)(x \in p \,\&\, R(p, q))\big).$$
Let x be $\boldsymbol{P}$-generic. Let
$$\mathscr{O}'_x = \{q \in \mathscr{O}'\colon (\mathrm{E}p)(x \in p \,\&\, R(p, q))\}.$$

First we shall show that intersection of finite subsets of $\mathscr{O}'_x$ are non empty. So let $q_1 \ldots q_n \in \mathscr{O}'_x$. There are $p_1 \ldots p_n \in \boldsymbol{P}$ such that $x \in p_1 \cap \ldots \cap p_n$ and $R(p_i, q_i)$. Let p be such that $x \in p \subseteq p_1 \cap \ldots \cap p_n$. By (1), $R(p, q_i)$. By a consecutive application of (2) we infer $q_1 \cap \ldots \cap q_i \neq \emptyset$ and $R(p, q_1 \cap \ldots \cap q_i)$ for $i \leqslant n$. Hence $q_1 \cap \ldots \cap q_n \neq \emptyset$, which is what we wanted to show.

Now by the fact that $\langle \mathscr{Y}, \mathscr{O}'\rangle$ has the centralization property, $\bigcap \mathscr{O}'_x \neq \emptyset$. Let $y \in \bigcap \mathscr{O}'_x$.

We shall show that $\langle x, y\rangle \in f_R$. So let $y \in q \in \mathscr{O}'$. Consider
$$D_q = \{p' \in \boldsymbol{P}\colon (\mathrm{E}q')((q' \subseteq q \vee q' \cap q = \emptyset) \,\&\, R(p', q'))\}.$$
By (4), D_q is dense in $\boldsymbol{P}$.

By the fact that x is $\boldsymbol{P}$-generic, there is a p' such that $x \in p' \,\&\, p' \in D_q$. Let q' be such that $(q' \subseteq q \vee q' \cap q = \emptyset)$ and $R(p', q')$. We have $q' \in \mathscr{O}'_x$ because $x \in p'$ and $R(p', q')$. Thus $y \in q'$. Hence $q' \cap q \neq \emptyset$ because $y \in q \cap q'$. Thus $q' \subseteq q$. Hence by (1), $R(p', q)$. Thus $q \in \mathscr{O}'_x$.

We have shown that if $q \in \mathscr{O}'$ and $y \in q$ then $q \in \mathscr{O}'_x$, i.e., $(\mathrm{E}p)(x \in p \,\&\, R(p, q))$. Thus $\langle x, y\rangle \in f_R$.

Hence follows $x \in \operatorname{Dom} f_R$, which is what we wanted to show.

Define $\underset{\sim}{y}$ to be an element of $V^{\boldsymbol{P}}$ such that $V^{\boldsymbol{P}} \Vdash (f_{\check{R}}(\underset{\sim}{x}) = \underset{\sim}{y})$, where $\underset{\sim}{x}$ is the canonical name for a generic filter. Let us show that $R(p, q) \equiv p \Vdash (\underset{\sim}{y} \in \check{q})$.

We have: for every $\boldsymbol{P}$-generic x, $f_R(x) = i_x(\underset{\sim}{y})$, by the definition of $\underset{\sim}{y}$. Let us show that
$$R(p, q) \equiv (x)(x \ \boldsymbol{P}\text{-generic} \,\&\, x \in p \Rightarrow f_R(x) \in q).$$

This will suffice to prove that R is the relation $p \Vdash (\underset{\sim}{y} \in \check{q})$ in view of the above equality. Indeed, let $R(p, q)$, x be $\boldsymbol{P}$-generic, $x \in p$. Then, by the definition of f_R, $f_R(x) \in q$. Assume conversely that (x) (x $\boldsymbol{P}$-generic & $x \in p \Rightarrow f_R(x) \in q$). Suppose that $\neg R(p, q)$. Let $p' \leqslant p$. By (4) of Definition 2 there is a q' such that $q' \subseteq q$ or $q' \cap q = \emptyset$ and a $p'' \leqslant p'$ such that $R(p'', p')$. Two cases are possible:

(1) $(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}(\mathrm{E}q')_{\subseteq q} R(p'', q')$ or

(2) $(\mathrm{E}p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}(\mathrm{E}q')(q' \cap q = \emptyset \,\&\, R(p'', q'))$.

If (1) then by (1) of Definition 2 we have $(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'} R(p'', q)$ and hence by (3), $R(p, q)$. Contradiction. Thus (2) holds.

We have $R(p'', q')$ for a $p'' \leqslant p$ and a q' disjoint with q. Let x be $\boldsymbol{P}$-

generic, $x\in p''$. Then by the definition of f_R, $f_R(x)\in q'$. But by our assumption, $f_R(x)\in q$ (note that $x\in p$). Contradiction. Hence $R(p,q)$. ∎

Assume now that $\langle \mathcal{Y}, \mathcal{O}\rangle$ is the Baire space $\langle \omega^\omega, \omega^{<\omega}\rangle$. Again, if there is a $\underline{y}$ in $V^{\boldsymbol{P}}$ for which R is the relation $p\Vdash(\underline{y}\in\check{q})$, then evidently R is a forcing relation. Thus assume that we are given a forcing relation $R\subseteq \boldsymbol{P}\times\omega^{<\omega}$. Let $\underline{y}$ be defined as

$$\langle\langle n,m\rangle,p\rangle\in\underline{y}\equiv R(p,\{\langle n,m\rangle\}).$$

Let us show that $R(p,q)\equiv p\Vdash(\underline{y}\in\check{q})$.

Assume $R(p,q)$. Then, by (1) of Definition 2, $(n)_{\mathrm{dom}\,q}R(p,\{\langle n,q(n)\rangle\})$, (we identify finite sequences with neighbourhoods). Thus $(n)_{\mathrm{dom}\,q}(p\Vdash(\underline{y}(n)=q(n)))$ by the definition of $\underline{y}$. Hence $p\Vdash(\underline{y}\in\check{q})$.

Conversely, let $p\Vdash(\underline{y}\in\check{q})$. Let $p'\leqslant p$. By the density of D_q there is a $p''\leqslant p'$ and a q' such that $q'\cap q=\emptyset$ or $q'\subseteq q$ and $R(p'',q')$. By what we showed before, $p''\Vdash(\underline{y}\in\check{q}')$. But hence $q\cap q'\neq\emptyset$ because $p''\Vdash(\underline{y}\in\check{q})$. Hence $q'\subseteq q$. Thus by (1) of Definition 2, $R(p'',q)$. Hence, by the fact that p' was arbitrary and by (3), $R(p,q)$. ∎

Theorem 1 shows that it is easy to characterize the family of relations $\{p\Vdash(\underline{y}\in\check{q})\}_{\underline{y}\in V^{\boldsymbol{P}}}$ by very simple algebraic conditions. We are also able to characterize the whole relation $\{p\Vdash\varphi(\underline{y})\}_{\underline{y}\in V^{\boldsymbol{P}}}$ as a relation of p and φ, by algebraic conditions.

Let M be an inner model (e.g. $M=V$). Let us consider the language $L_{ZF}(M)$, i.e., the language of set theory with elements of M as constants. Let us admit long conjunctions of the form $\bigwedge_{r\in X}\varphi(r)$ where $X\subseteq M$ is a definable class. Let us consider the following M-logic — as axioms we take the usual logical axioms and the axiom "$\neg x\in\emptyset$". As rules of inference we take the usual rules and the following rules:

$$\frac{\bigwedge_{\substack{r''\in M\\ \operatorname{rank} r''<\operatorname{rank} r,r'}}(r''\in r\equiv r''\in r')}{r'=r}$$

$$\frac{\bigwedge_{\substack{r''\in M\\ \operatorname{rank} r''<\operatorname{rank} r}}(r''\neq r'\vee r''\notin r)}{r'\notin r}$$

and the following M-rule:

$$\frac{\bigwedge_{r\in M}\varphi(r)}{(x)\varphi(x)}.$$

Let $\langle \boldsymbol{P},\leqslant\rangle$ be a partially ordered set.

DEFINITION 3. Let $F\subseteq\boldsymbol{P}\times L_{ZF}(M)$ be called a *forcing relation* if:

(1) $F(p,\varphi)\,\&\,p'\leqslant p\Rightarrow F(p',\varphi)$,
$F(p,\varphi)\,\&\vdash_M(\varphi\Rightarrow\psi)\Rightarrow F(p,\psi)$.

(2) It is not true that $F(p,\varphi)\,\&\,F(p,\neg\varphi)$.

(3) $(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}F(p'',\varphi)\Rightarrow F(p,\varphi)$.

(4) For every φ the set $D_\varphi=\{p\in\boldsymbol{P}\colon F(p,\varphi)\vee F(p,\neg\varphi)\}$ is dense in $\boldsymbol{P}$.

(5) $F(p,\varphi)\,\&\,F(p,\psi)\Rightarrow F(p,\varphi\,\&\,\psi)$,
$F(p,\varphi(r))$ for $r\in X\Rightarrow F(p,\bigwedge_{r\in X}\varphi(r))$.

(6) Let us define the following topological space $\langle\mathcal{Y},\mathcal{O}\rangle$: Let

$$\mathcal{Y}=\{0,1\}^M,\qquad q_r^\varepsilon\in\mathcal{O}\equiv q_r^\varepsilon=\{f\in\{0,1\}^M\colon f(r)=\varepsilon\}$$

for a given $r\in M$, $\varepsilon\in\{0,1\}$ (then $\mathcal{O}$ is a subbasis of a topology). Let $R\in M$, $R\subseteq\boldsymbol{P}\times\mathcal{O}$ be a forcing relation in the sense of Definition 2. Then there is a constant $\bar{r}$ such that

$$R(p,q_r^0)\equiv F(p,r\in\bar{r}),$$
$$R(p,q_r^1)\equiv F(p,r\notin\bar{r}).$$

Conversely, if $\bar{r}\in M$ and we define

$$R(p,q_r^0)\equiv F(p,r\in\bar{r})\,\&\operatorname{rank} r<\operatorname{rank}\bar{r},$$
$$R(p,q_r^1)\equiv F(p,r\notin\bar{r})\,\&\operatorname{rank} r<\operatorname{rank}\bar{r},$$

then $R\in M$ under a coding of $\mathcal{O}$.

Remark 3. Notice that axiom (4) corresponds exactly to the famous Cohen definition of forcing the negation. This is the main feature that distinguishes a forcing relation from a quasi-forcing relation ((3) is purely technical). Thus we can infer that this axiom is not the deepest point in the definition of forcing, as has been sometimes thought. Indeed, as we have already seen and as we shall still see in § 2, quasi-forcing relations are quite interesting, i.e., we can do a lot without (4). The reason for axiom (4) is mainly to permit finding "generic" filters in $\boldsymbol{P}$, i.e., filters deciding all the sentences.

Let us prove now that Definition 2 really characterizes forcing in set theory.

Remark 4. *$F\subseteq\boldsymbol{P}\times L_{ZF}(M)$ is a forcing relation iff there is a valuation of constants $v\colon M\to M^{\boldsymbol{P}}$ such that $v(\emptyset)=\emptyset$, $\operatorname{rank} r'<\operatorname{rank} r\Rightarrow\operatorname{rank} v(r')<\operatorname{rank} v(r)$, $F(p,\varphi(r))\equiv p\Vdash\varphi(v(r))$ and v is onto the class of those $\underline{y}\in M^{\boldsymbol{P}}$ which hereditarily have the property $\langle p',\underline{y}'\rangle\in\underline{y}\equiv p\Vdash(\underline{y}'\in\underline{y})$.*

Proof. The "if" part of the remark is standard.

Let us assume that F is a forcing relation and define v.

Let $v(\emptyset)=\emptyset$. Assume that for $r\in R^M_\xi$, $v(r)$ has been defined. Let $r\in R^M_{\xi+1}$. Let $v(r)\subseteq\{v(r')\colon r'\in R^M_\xi\}\times\boldsymbol{P}$ be defined as

$$\langle v(r'),p\rangle\in v(r)\equiv F(p,r'\in r).$$

Assume that $v_{\restriction R_\xi}$ is onto N_ξ where N_ξ is the submodel of M consisting of those $\underline{y}$ which hereditarily have the property:

$$\langle p,\underline{y}'\rangle\in\underline{y}\equiv(p\Vdash\underline{y}'\in\underline{y}).$$

Assume $v_{\restriction R_\xi}\in M$. Let us show that $v_{\restriction R_{\xi+1}}$ is onto $N_{\xi+1}$. Indeed, let $\underline{y}\in N_{\xi+1}$. Then

$$\underline{y}=\{\langle p,\underline{y}'\rangle\colon p\Vdash\underline{y}'\in\underline{y}\},\qquad \underline{y}\subseteq N_\xi\times\boldsymbol{P},\qquad \underline{y}\in M.$$

Let R be defined as

$$R(p,q^0_{r'})\equiv\langle p,v(r')\rangle\in\underline{y},$$
$$R(p,q^1_{r'})\equiv\langle p,v(r')\rangle\notin\underline{y}\quad\text{for}\quad r'\in R_\xi.$$

Then R is a forcing relation and $R\in M$ because $v_{\restriction R_\xi}$, $\underline{y}\in M$. Thus by (6) there is a constant $\bar r$ such that

$$R(p,q^0_{r'})\equiv F(p,r'\in\bar r),$$
$$R(p,q^1_{r'})\equiv F(p,r'\notin\bar r).$$

Let us show that $v(\bar r)=\underline{y}$. Indeed, this follows from the fact that $v_{\restriction R_\xi}$ is onto N_ξ.

Let us show

(1) $$F(p,r'\in r)\equiv p\Vdash v(r')\in v(r),$$

(2) $$F(p,r'\notin r)\equiv p\Vdash v(r')\notin v(r),$$

(3) $$F(p,r'=r)\equiv p\Vdash v(r')=v(r),$$

(4) $$F(p,r'\neq r)\equiv p\Vdash v(r')\neq v(r).$$

Let us first make the following remark: if $F(p,\varphi(r))$ and $F(p,r=r')$ then $F(p,\varphi(r'))$.

Indeed, by (6) we have $F(p,r=r'\,\&\,\varphi(r))$. But

$$\vdash_M(r=r'\,\&\,\varphi(r)\Rightarrow\varphi(r')).$$

Hence by (1) $F(p,\varphi(r'))$.

Now let us assume that (1)–(4) hold for r', r'' from R_ξ. Let $r\in R_{\xi+1}$.

1° If $F(p,r'\in r)$ then $\langle v(r'),p\rangle\in v(r)$ and thus $p\Vdash v(r')\in v(r)$.

If $p\Vdash v(r')\in v(r)$ then

$$(p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}(\mathrm{E}r'')_{R_\xi}\big(p''\Vdash(v(r'')=v(r'))\big)$$

and $\langle v(r''),p\rangle\in v(r)$. Thus $F(p'',r'=r)$ and $F(p'',r''\in r)$ by the inductive assumption. Hence by the initial remark $F(p'',r'\in r)$ and by (3) $F(p,r'=r)$.

2° Let $F(p,r'\notin r)$. Let $p'\leqslant p$. Then $F(p',r'\notin r)$. Hence by the preceding point, which we have just shown, and by (1) p' not $\Vdash v(r')\in v(r)$. Thus $p\Vdash v(r')\notin v(r)$.

Let $p\Vdash v(r')\notin v(r)$. Let $p'\leqslant p$. Then $\neg F(p',r'\in r)$ by 1°. By (4) there is a $p''\leqslant p'$ such that $F(p'',r'\notin r)$. Hence by (3), $F(p,r'\notin r)$.

Thus we have shown 1°, 2° for $r'\in R_\xi$, $r\in R_{\xi+1}$. Now let us add this to our inductive assumption and let $\bar r\in R_{\xi+1}$.

3° Let us prove the implication "⇒".

Let $F(p,\bar r=r)$. Suppose that there is an $r'\in R_\xi$, $p'\leqslant p$ such that $F(p',r'\in\bar r)\,\&\,F(p',r'\notin r)$. Hence, by the initial remark, $F(p',r'\in r)$ because $F(p',r=\bar r)$. But this together with $F(p,r'\notin r)$ contradicts (2). Thus, for every $r'\in R_\xi$, $p'\leqslant p$ $\neg\big(F(p',r'\in\bar r)\,\&\,F(p',r'\notin r)\big)$. Hence, for every $r'\in R_\xi$, $p'\leqslant p$, p' not $\Vdash\big(v(r')\in v(\bar r)\,\&\,v(r')\notin v(r)\big)$ by the inductive assumption. The same holds symmetrically for r interchanged with $\bar r$. Thus $p\Vdash\big(v(r)=v(\bar r)\big)$.

Let $p\Vdash v(r)=v(\bar r)$. Then for every r' such that $\operatorname{rank} r'<\operatorname{rank} r,\bar r$,

$$p\Vdash\Big(\big(v(r')\in v(r)\big)\equiv\big(v(r')\in v(\bar r)\big)\Big).$$

Hence $F\big(p,(r'\in r\equiv r'\in\bar r)\big)$. By (5)

$$F\Big(p,\ \bigwedge\!\!\!\bigwedge_{\substack{r'\\ \operatorname{rank} r'<\operatorname{rank} r}}(r'\in r\equiv r'\in\bar r)\Big).$$

Then, by (1), $F(p,r=\bar r)$.

4° Let $F(p,r\neq\bar r)$. Let $p'\leqslant p$. Then $F(p',r\neq\bar r)$. Then p' not $\Vdash v(r)=v(\bar r)$ by 3°. Thus $p\Vdash v(r)\neq v(\bar r)$.

Let $p\Vdash v(r)\neq v(\bar r)$. Let $p'\leqslant p$. Then $\neg F(p',r=\bar r)$ by (3). By (4) there is a $p''\leqslant p'$ such that $F(p'',r\neq\bar r)$. Thus by (3) $F(p,r\neq\bar r)$.

It remains to prove (1), (2) under the assumption r, $r'\in R_{\xi+1}$.

(2) Let $F(p,r'\notin r)$. Suppose that there is an r'', $p'\leqslant p$ such that $\operatorname{rank} r''<\operatorname{rank} r$ and $p'\Vdash v(r'')=v(r')\,\&\,v(r'')\in v(r)$. Thus $p'\Vdash v(r')\in v(r)$. By the inductive assumption $F(p',r'\in r)$. But $F(p',r'\notin r)$ by (1). This contradicts (2). Hence, for every r'' with rank less than rank r and for every $p'\leqslant p$, p' not $\Vdash v(r'')=v(r)\,\&\,v(r'')\in v(r)$. Thus $p\Vdash v(r')\notin v(r)$.

Let $p\Vdash v(r')\notin v(r)$. Then, for every r'' such that $\operatorname{rank} r''<\operatorname{rank} r$, $p\Vdash v(r'')\neq v(r')\vee v(r'')\notin v(r)$. Thus $F\big(p,(r''\neq r'\vee r''\notin r)\big)$ for every suitable r''. By (5)

$$F\Big(p,\ \bigwedge\!\!\!\bigwedge_{\substack{r''\\ \operatorname{rank} r''<\operatorname{rank} r}}(r''\neq r'\vee r''\notin r)\Big).$$

By (1), $F(p,r'\notin r)$.

Thus we have proved the remark for atomic formulas and their negations.

Assume now that the remark is true for φ, ψ. Let θ be of the form $\varphi\,\&\,\psi$.

Let $F(p, \theta)$. Then by (1), $F(p, \varphi)$ and $F(p, \psi)$. Hence, by the inductive assumption, $p \Vdash \varphi$ and $p \Vdash \psi$ and thus $p \Vdash \theta$.

Let $p \Vdash \theta$. Then $p \Vdash \varphi$ and $p \Vdash \psi$. Hence, by the inductive assumption, $F(p, \varphi)$ and $F(p, \psi)$. Thus by (5) $F(p, \theta)$.

Let θ be of the form $(x)\varphi(x)$. Let $F(p, \theta)$. Then by (1), $F(p, \varphi(r))$ for an r. Hence, by the inductive assumption, $p \Vdash \varphi(v(r))$ and thus $p \Vdash \theta$.

Let $p \Vdash \theta$. Then $p \Vdash \varphi(v(r))$ for every r. Hence, by the inductive assumption, $F(p, \varphi(r))$ and by (1), (5), $F(p, \theta)$.

Let θ be of the form $\neg\varphi$. Let $F(p, \theta)$. Suppose that p not $\Vdash \theta$. Then there is a $p' \leqslant p$ such that $p' \Vdash \varphi$. By the inductive assumption, $F(p', \varphi)$. But $F(p', \neg\varphi)$. Contradiction. Thus $p \Vdash \theta$.

Let $p \Vdash \theta$. Suppose that $\neg F(p, \theta)$. Then by (4) there is a $p' \leqslant p$ such that $F(p', \varphi)$. By what we have just proved, $p' \Vdash \varphi$. Contradiction. Thus $F(p, \theta)$. ∎

Let us close this section with a piece of discussion.

Remark 4 shows that the axioms (1)–(6) fully characterize the forcing relation in set theory. In other words, if F satisfies (1)–(6) then this is the forcing relation defined by Cohen. Let us consider the meaning of (1)–(6).

Suppose we are looking for a relation $F \subseteq \boldsymbol{P} \times L_{\mathrm{ZF}}(M)$ which should say the following: if we have the information p about the generic set G, then it gives the information q about the generic model. Then (1) says: if we have more information about G then we have not less information about $M[G]$, and if we want less information about $M[G]$ then the former information about G suffices.

(2) says that we do not obtain contradictory information.

(3) is rather technical but it is not important to prove the "$\Leftarrow$" part of the equivalence $p \Vdash \varphi \equiv F(p, \varphi)$.

(5) says that we can join information.

(6) says that there are enough constants.

Axiom (4) has a different character – it permits easy finding of sets of conditions deciding every sentence (i.e., generic).

Remark 4 shows that if we put natural requirements about giving information on a relation F then it is determined uniquely. To be precise, it is also determined by the M-logic that we have defined but our requirements put on the M-logic are also natural set-theoretical requirements.

Example 3. Let M be an inner model. Let $\mathscr{P}^M(\boldsymbol{P}) \approx \omega$, $\boldsymbol{P} \in M$. Let $L_{\mathrm{ZF}}(M)$ be defined as before. Let $L_\xi(M)$ be the subset of $L_{\mathrm{ZF}}(M)$ consisting of sentences with constants of rank $< \xi$. Let us define the following game: $G_\xi(p)$. Let I ask questions and II – answers.

The possible questions are of the form "p" for $p' \in \boldsymbol{P}$ or "$\varphi(r)$" for $\varphi(r) \in L_\xi(M)$. Player II answers "yes" or "no". Moreover, (a) If φ is of the form $(\mathrm{E}x)\psi(x)$ then II shows an r. (b) If φ is of the form $r' \in r$ then II shows r'' such that $\operatorname{rank} r'' < \operatorname{rank} r$; (c) If φ is of the form $r' \neq r$ then II shows r'' such that $\operatorname{rank} r'' < \operatorname{rank} r', r$.

Player II fails if he has given the following answers:

(1) a. To the question "$\varphi(r)$" he has said "yes" and φ is of the form $(\mathrm{E}x)\psi(x)$ and has shown r, and he has said "no" to the question $\psi(r)$.

b. He has said "yes" to the question "$r' \in r$" and has shown r'', and he has said "no" to the question "$r'' \in r \,\&\, r' = r''$".

c. He has said "yes" to the question "$r' \neq r$" and has shown r'', and he has said "no" to the question "$(r'' \in r \,\&\, r'' \notin r') \vee (r'' \in r' \,\&\, r'' \notin r)$".

(2) He has said "yes" to "$\varphi(r)$" and "no" to "$\psi(r)$" where $\vdash_M (\varphi \Rightarrow \psi)$.

(3) He has said "yes" to "$\varphi(r)$" and to "$\neg\varphi(r)$".

(4) He has said "yes" to "$\varphi(r) \,\&\, \psi(r')$" but "no" to "$\varphi(r)$" or "$\psi(r')$". Similar conditions concern other connectives.

(5) $r \subseteq \boldsymbol{P} \times \mathscr{C}$ is a forcing relation, $r \in R_\xi$ and $\langle p', q^0_{r'} \rangle \in r$, and he has said "no" to "$r' \notin r$" and "yes" to "p'" or $\langle p', q^1_{r'} \rangle \in r$ and he has said "no" to "$r' \notin r$" and "yes" to "p'".

(6) He has said "no" to "p".

(7) He has said "yes" to "p''" and "no" to "p'" where $p'' \leqslant p'$.

(8) He has said "yes" to "p''" and to "p'" where p', p'' are incompatible.

Let II win if he has not given wrong answers of the form (1)–(8) and if he has answered every question "p'", "$\varphi(r)$" for $\varphi(r) \in L_\xi(M)$.

Let $F(p, \varphi(r))$ be defined as $(p')_{\leqslant p}$ (there is a winning strategy for II in $G_\xi(p')$ saying "yes" to the question "$\varphi(r)$" where $\xi = \operatorname{rank} r$).

Let us show that F is a forcing relation. Notice that (1), (5), (6) are obvious. Let us show (4), which is the most difficult. We show it by induction w.r.t. the length of $\varphi(r)$ and the rank of r. Assume that for $r, r' \in R_\eta$ (4) holds for φ of the form $r' \in r$ or $r' = r$.

a. Let $r' \in R_\eta$, $r \in R_{\eta+1}$ and let φ be of the form $r' \in r$. Let $p \in \boldsymbol{P}$. Let $r \subseteq \boldsymbol{P} \times \mathscr{C}$ be a forcing relation. Then there is a $p' \leqslant p$ such that $\langle p', q^0_{r'} \rangle \in r$ or $\langle p', q^1_{r'} \rangle \in r$. Then we have either "every strategy winning for II in $G_\xi(p')$ says "yes" to $r' \in r$" or "every strategy winning for II in $G_\xi(p')$ says "yes" to "$r' \notin r''$". Thus $F(p', r' \in r)$ or $F(p', r' \notin r)$.

b. Let $r' \in R_{\eta+1}$, $r \in R_{\eta+1}$ and φ be of the form $r' \neq r$. Let $p \in \boldsymbol{P}$.

Assume that $(r'')_{R_\eta}$ (for every strategy σ winning for II in $G_\xi(p)$, σ says "yes" to "$(r'' \in r' \equiv r'' \in r)$"). Then every winning strategy for II in $G_\xi(p)$ says "yes" to "$r' = r$". Hence $F(p, r' = r)$.

Assume that $(\mathrm{E}r'')_{R_\eta}$ $(\mathrm{E}\sigma)$ (σ is a winning strategy for II in $G_\xi(p)$ and σ says "no" to "$(r'' \in r' \equiv r'' \in r)$").

Take σ. Then σ says "yes" to $r'' \in r'$, $r'' \notin r$ or σ says "yes" to "$(r'' \notin r'$, $r'' \in r)$". Assume the first case. Then we have $(\mathrm{E}p')_{\leqslant p} \langle p', q^0_{r''} \rangle \in r' \,\&\, \langle p', q^1_{r''} \rangle \in r$.

Then every strategy τ winning for II in $G_\xi(p')$ says "yes" to "$r''\in r'$, $r''\notin r$". Thus $F(p', r'=r)$. Treat the second case similarly.

Notice that we have shown more than (4), namely the following:

(*) $(\mathrm{E}p')_{\leqslant p}$ (every strategy σ winning for II in $G_\xi(p')$ says "yes" to "φ")

or

$(\mathrm{E}p')_{\leqslant p}$ (every strategy σ winning for II in $G_\xi(p')$ says "yes" to "$\neg\varphi$").

We show (*) similarly for other φ.

As a corollary, from (*) we obtain

(**) $F(p, \varphi) \equiv (p')_{\leqslant p}(\mathrm{E}p'')_{\leqslant p'}$ (every strategy winning for II in $G_\xi(p'')$ says "yes" to φ).

Indeed, let $F(p, \varphi)$. Let $p' \leqslant p$. Then $(\mathrm{E}p'')_{\leqslant p'}$ (every strategy winning for II in $G_\xi(p'')$ says "yes" to "φ") or $(\mathrm{E}p'')_{\leqslant p'}$ (every strategy winning for II in $G_\xi(p'')$ says "no" to "φ").

The second eventuality is impossible because $F(p', \varphi)$. Thus the first holds.

The implication $\Leftarrow$ is obvious.

From (**) we easily obtain properties (2), (3) of Definition 3.

Thus F is "the" forcing relation.

§ 2

Now we shall introduce a new notion.

Definition 4. Having a quasi-forcing relation R we can define another relation, connected with R, R^*, which will be called a *consistency relation.* Let $\langle p, q\rangle \in R^* \equiv (\mathrm{E}p')_{\leqslant p} R(p', q)$.

Example 4. Let $\boldsymbol{P}$, R, $\underline{y}$ be as in Example 1, § 1. Let $R'(p, q) \equiv (p$ is compatible with $\|\underline{y}\in\check{q}\|)$. Then $R' = R^*$.

Example 5. Let $R(p, q)$ be the quasi-forcing relation of Example 2 of § 1 for a given e, $R \subseteq \omega^{<\omega}\times\omega^{<\omega}$. Then R^* is the relation:

$$R^*(s, t) \equiv (\mathrm{E}s')(s' \supseteq s \,\&\, t \subseteq \{e\}^{s'}).$$

Remark 5. *Let $R \subseteq \boldsymbol{P}\times\mathcal{O}'$ be a quasi-forcing relation. Let $\langle \mathcal{X}, \mathcal{O}\rangle$ be as in Remark* 1. *For $q \in \mathcal{O}'$, let $\mathcal{X}_q^R$ be defined as*

$$x \in \mathcal{X}_q^R \equiv (\mathrm{E}p)\big(x\in p \,\&\, R(p, q)\big) \vee (\mathrm{E}p)(\mathrm{E}q')\big(q\cap q' = \emptyset \,\&\, x\in p \,\&\, R(p, q')\big).$$

Then R^* has the following properties:

(1) $R(p, q) \,\&\, p \leqslant p' \,\&\, q \leqslant q' \Rightarrow R^*(p', q')$,

(2) $[p\cap\mathcal{X}_q^R \neq \emptyset \,\&\, (p')_{\leqslant p}(p'\cap\mathcal{X}_q^R \neq \emptyset \Rightarrow R^*(p', q)) \,\&\, R^*(p, q')]$
$\Rightarrow q\cap q' \neq \emptyset \,\&\, R^*(p, q\cap q')$.

Remark 6. $\mathcal{X}_q^R$ is the set of those $x \in \mathcal{X}$ that "decide" about q. In Example 4 it is the set of those x that intersect an appropriate D_q of Definition 2 and in Example 5 it is $\{x\colon \{e\}^x$ is defined at $\operatorname{dom} q\}$.

If R is a forcing relation then $\mathcal{X}_q^R$ is dense in $\mathcal{X}$ in $\mathcal{O}$. In this case (2) is equivalent to $[(p')_{\leqslant p} R^*(p', q) \,\&\, R(p, q')] \Rightarrow q\cap q' = \emptyset$, and $(p')_{\leqslant p} R^*(p', q) \equiv R(p, q)$. Thus (2) follows directly from (2) of Definition 2.

Proof of Remark 5. (1) is evident.

(2) Assume the hypothesis of (2). Let $p' \leqslant p$ be such that $R(p', q')$ (such a p' exists because $R^*(p, q')$). Then $p'\cap\mathcal{X}_q^R \neq \emptyset$. Hence $R^*(p', q)$. Then $(\mathrm{E}p'')_{\leqslant p'} R(p'', q)$. But $R(p'', q')$ because $R(p', q')$. Thus $q\cap q' \neq \emptyset \,\&\, R^*(p, q\cap q')$. Contradiction. ∎

Definition 5. Let $R' \subseteq \boldsymbol{P}\times\mathcal{O}'$ be called a *consistency relation* iff:

(1) $R'(p, q) \,\&\, p \leqslant p' \,\&\, q \leqslant q' \Rightarrow R'(p', q')$.

(2) Let for $q \in \mathcal{O}'$, $\mathcal{X}_q^{R'}$ be defined as

$$x \in \mathcal{X}_q^{R'} \equiv (p)\Big(x\in p \Rightarrow \big(R'(p, q) \vee (\mathrm{E}q')(q'\cap q = \emptyset \,\&\, R'(p, q'))\big)\Big).$$

We assume

$$[p\cap\mathcal{X}_q^{R'} \neq \emptyset \,\&\, (p')_{\leqslant p}(p'\cap\mathcal{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', q)) \,\&\, R'(p, q')]$$
$$\Rightarrow q\cap q' \neq \emptyset \,\&\, R'(p, q\cap q').$$

Remark 7. The above definition is an attempt to introduce the notion of a consistency relation axiomatically. It can be proved that if R' is of the form R^* for a quasi-forcing relation R then (1), (2) hold although $\mathcal{X}_q^R$ defined for R can be different from $\mathcal{X}_q^{R'}$ — the proof of (1), (2) is just a repetition of the proof of Remark 5 with $\mathcal{X}_q^{R'}$ in place of $\mathcal{X}_q^R$.

Remark 8. *For $q \in \mathcal{O}'$, let $-q \in \mathcal{O}'$. Let $R' \subseteq \boldsymbol{P}\times\mathcal{O}'$ be a consistency relation. Then there is a $\boldsymbol{P} \subseteq \boldsymbol{P}$ and a relation $\boldsymbol{R} \subseteq \boldsymbol{P}\times\mathcal{O}'$ such that, for p such that $\emptyset \neq p\cap\mathcal{X}_q^{R'}$, $R^*(p, q) \equiv R'(p, q)$.*

Moreover, $\mathcal{X}_q^R \subseteq \mathcal{X}_q^{R'}$ and $R_{\restriction \boldsymbol{P}'\times\mathcal{O}'}$ is a quasi-forcing relation for a $\boldsymbol{P}' \subseteq \boldsymbol{P}$ and $R_{\restriction \boldsymbol{P}''\times\mathcal{O}'}$ is a forcing relation for a $\boldsymbol{P}'' \subseteq \boldsymbol{P}'$.

Proof. Define R as

$$R(p, q) \equiv \big[(p')_{\leqslant p}(p'\cap\mathcal{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', q)) \,\&\, p\cap\mathcal{X}_q^{R'} \neq \emptyset\big].$$

Let $\mathcal{X}' = \bigcap_{q\in\mathcal{O}'} \mathcal{X}_q^{R'}$. Let $\mathcal{X}'' = \bigcap_{q\in\mathcal{O}'} \mathcal{X}_q^{R}$. Let $\boldsymbol{P}' = \{p\colon p\cap\mathcal{X}' \neq \emptyset\}$. Let $\boldsymbol{P}'' = \{p\colon p\cap\mathcal{X}'' \neq \emptyset\}$.

Then R is a quasi-forcing relation in $\boldsymbol{P}'$, a forcing relation in $\boldsymbol{P}''$.

Let us show $R^*(p, q) \Rightarrow R'(p, q)$.

Indeed, we have

$$R^*(p, q) \equiv (\mathrm{E}p')_{\leqslant p} R(p', q) \equiv (\mathrm{E}p')_{\leqslant p}\big(p'\cap\mathcal{X}_q^{R'} \neq \emptyset \,\&$$
$$\&\, (p'')_{\leqslant p'}(p''\cap\mathcal{X}_q^{R'} \neq \emptyset \Rightarrow R'(p'', q))\big).$$

Hence for p' itself we have $R'(p', q)$. By (1) of Definition 5 we infer $R'(p, q)$.

Assume now $R'(p,q) \& p \cap \mathscr{X}_q^{R'} \neq \emptyset$. Let $\bar{q} = -q$. Suppose that

$$(p')_{\leqslant p}(p' \cap \mathscr{X}_q^{R'} \neq \emptyset) \Rightarrow (\mathrm{E}p'')_{\leqslant p'}(p'' \cap \mathscr{X}_q^{R'} \neq \emptyset \& R'(p'', \bar{q})).$$

Then, by (1) of Definition 5, $(p')_{\leqslant p}(p' \cap \mathscr{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', \bar{q}))$. Hence, by (2) of Definition 5, $q \cap \bar{q} \neq \emptyset$. Contradiction. Thus

$$(\mathrm{E}p')_{\leqslant p}(p' \cap \mathscr{X}_q^{R'} \neq \emptyset \& (p'')_{\leqslant p'}(p'' \cap \mathscr{X}_q^{R'} \neq \emptyset \Rightarrow R'(p'', q))).$$

Hence $(\mathrm{E}p')_{\leqslant p} R'(p', q)$. Thus $R^*(p,q)$.

Let us show that $\mathscr{X}_q^R \subseteq \mathscr{X}_q^{R'}$. Indeed, this follows directly from the fact that $R^*(p,q) \Rightarrow R'(p,q)$ for every $p \in \boldsymbol{P}$. ∎

Remark 9. Let us remark here that it can happen that $\mathscr{X}_q^R \neq \mathscr{X}_q^{R^*}$ even if R is a forcing relation. Indeed, let $\boldsymbol{P} = 2^{<\omega}$, $\langle \mathscr{Y}, \mathscr{O}' \rangle = \langle 2^\omega, 2^{<\omega} \rangle$. Let $\underline{y} \in V^{\boldsymbol{P}}$ be defined as

$$\underline{y} = \{\langle s \langle n, \varepsilon \rangle \rangle \colon [(\varepsilon = 1 \& 2 | \min\{m \in \operatorname{dom} s\colon s(m) = 0\}) \vee$$
$$\vee (\varepsilon = 0 \& 2 \nmid \min\{m \in \operatorname{dom} s\colon s'(m) = 0\})] \& (\mathrm{E}m)_{\operatorname{dom} s}(s(m) = 0)\}.$$

Let $R(s,t) \equiv s \Vdash \check{t} \subseteq \underline{y}$ where $\Vdash$ denotes the weak Cohen forcing. Then R is a forcing relation. Let $x \in 2^\omega$ be such that $x \equiv 1$. Let $t = \langle 1 \rangle$. Then $(s)(s \subseteq x \Rightarrow (\mathrm{E}s')(s' \supseteq s \& s' \Vdash \check{t} \subseteq \underline{y}))$. Thus $(s)(s \subseteq x \Rightarrow R^*(s,t))$. Hence $x \in \mathscr{X}_t^{R^*}$. But

$$(s)(s \subseteq x \Rightarrow (\mathrm{E}s')(s' \supseteq s \& s' \Vdash \check{t} \subseteq \underline{y})).$$

Indeed, if $s \subseteq x$, let s' be such that $s' \supseteq s \& 2 \nmid \min\{m \in \operatorname{dom} s'\colon s'(m) = 0\}$ (the first 0 is at an odd place). Hence $(s)(s \subseteq x \Rightarrow s \text{ not} \Vdash \check{t} \subseteq \underline{y})$. Thus $x \notin \mathscr{X}_t^R$.

Remark 10. If $\mathscr{X}_q^R = \mathscr{X}_q^{R'}$ and it is closed in $\langle \mathscr{X}, \mathscr{O} \rangle$, then $\mathscr{X}_q^R = \mathscr{X}_q^{R^*}$. Indeed, we have $\mathscr{X}_q^{R^*} = \mathscr{X}_q^R$ directly from the definition.

Remark 11. If $\mathscr{X}_q^R = \mathscr{X}_q^{R'}$ then for $x \in \mathscr{X}$, $q \in \mathscr{O}'$ we have

$$(\mathrm{E}p)(x \in p \& R(p,q)) \equiv (p')(x \in p' \Rightarrow (\mathrm{E}p'')_{\leqslant p'}(x \in p'' \& R'(p'', q))).$$

The proof is straightforward.

Let us introduce one more definition.

Definition 6. Let $R' \subseteq \boldsymbol{P} \times \mathscr{O}'$ be called a *weak consistency relation* iff

(1) $R'(p,q) \& p \leqslant p' \Rightarrow R'(p', q')$,

(2′) if $p \cap \mathscr{X}_q^{R'} \neq \emptyset \& (p')_{\leqslant p}(p' \cap \mathscr{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', q)) \&$ $\& (p')_{\leqslant p}(p' \cap \mathscr{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', q'))$, then $q \cap q' \neq \emptyset \& R'(p, q \cap q')$.

Notice that (2′) is weaker than (2) of Definition 5. However, (2′) suffices to show that the relation R defined in Remark 8 is a quasi-forcing relation when restricted to $\boldsymbol{P}' \times \mathscr{O}'$ and that for p in $\boldsymbol{P}$ we have $R^*(p,q) \Rightarrow R'(p,q)$. Thus we have the following

Remark 12. If $R' \subseteq \boldsymbol{P} \times \mathscr{O}'$ is a weak consistency relation, then there is a relation $R \subseteq \boldsymbol{P} \times \mathscr{O}'$ such that $R^* \subseteq R'$. Moreover, $\mathscr{X}_q^R \subseteq \mathscr{X}_q^{R'}$ and $R_{\upharpoonright \boldsymbol{P}' \times \mathscr{O}'}$ is a quasi-forcing relation for a $\boldsymbol{P}' \subseteq \boldsymbol{P}$.

Notice that Remarks 10, 11 are still true if instead of R we put a weak consistency relation R' and R is defined for R' as in Remark 8.

Now we are ready to show a weak consistency relation in Peano Arithmetic PA.

Example 6. Consider the language of arithmetic with relational symbols $+$, $\cdot$, $=$, $<$ and constants 0, 1.

Let M be a countable non-standard model for PA. Let us define $\boldsymbol{P}$. Let $p \in \boldsymbol{P}$ if p is a pair (a,b) of elements of M such that $a < b$. Let $(a,b) \geqslant (a', b')$ if $a \leqslant a' < b' \leqslant b$. Let us define $\langle \mathscr{X}, \mathscr{O} \rangle$. Let $\mathscr{X}$ be the collection of all initial segments of M. Let $\mathscr{O}$ be identified with $\boldsymbol{P}$ and let $I \in (a,b)$ if $a \in I$ and $b \notin I$.

Let us define $\langle \mathscr{Y}, \mathscr{O}' \rangle$. Let $\mathscr{Y} = \mathscr{X}$ and let $q \in \mathscr{O}'$ if there is a sentence φ which is Σ_1 or Π_1 such that $q = \{I\colon I \models \varphi\}$.

Assume that we are given an enumeration $\ulcorner \varphi \urcorner$ of sentences that are Σ_1 or Π_1.

For $a, b, c \in M$, let $G_c(a,b)$ be the Paris game of length c.

Let $p = (a,b)$ and let q be determined by φ. Let us define $R'(p,q)$ $\equiv$ there is a winning strategy for player II in $G_{\ulcorner \varphi \urcorner + 3}(a,b)$ such that he always says "yes" to the question "φ".

We shall show that R' is a weak consistency relation.

First let us show the following

Remark 13. *Let p, q be as above. If $R'(p,q)$ then there is an $I \in \mathscr{X}$ such that $I \in p$ and $I \models \varphi$.*

Proof. If φ is of the form $(\mathrm{E}x)\psi(x)$ where ψ is bounded, then let player I ask "φ". Then player II, following his strategy, answers "yes" and he has to show an x that $x < b$ and $\psi(x)$. Let $I = \{y\colon y \leqslant x\} = \leqslant_x$ if $x > a$ and $I = \leqslant_a$ if $x \leqslant a$.

If φ is of the form $(x)\psi(x)$ then let player I ask "φ". Player II answers "yes". Now if player I asks "$x \in I$?" for any $x \leqslant a$ then player II has to answer "yes" because if he answers "no" and the next question is "$a \in I$?" then he fails. Hence for every $x \leqslant a$ we have $\psi(x)$. Let $I = \leqslant_a$. ∎

Remark 14. *R' is a weak consistency relation.*

Proof. Let us show (1) of Definition 6. Let $a \leqslant a' < b' \leqslant b$ and let $R'((a', b'), q)$. Then $R'((a,b), q)$ because player II can follow the same strategy in the appropriate game $G_{\ulcorner \varphi \urcorner + 3}(a,b)$ as in $G_{\ulcorner \varphi \urcorner + 3}(a', b')$.

Let us show (2′) of Definition 6. Notice first that if $I \in \mathscr{X}_q^{R'}$ then I has no biggest element. Indeed, suppose that $I \in \mathscr{X}_q^{R'}$ and $I = \leqslant_c$. Then $I \in (c, c+1)$. By the definition of $\mathscr{X}_q^{R'}$ we have $R'((c, c+1), q)$ or $R'((c, c+1), q')$ for a q' disjoint with q. Hence there is a winning strategy for player II in the game $G_3((c,c+1), q)$ or in $G_3((c, c+1), q')$. Let player I ask the question "$c \in I$?". Then player II has to answer "yes". Now let player I ask "$c+1 \in I$?". Then player II has no answer. Thus for no q can there be a winning strategy for player II in $G_3((c, c+1), q)$. Thus I has no biggest element.

Assume now $p \cap \mathcal{X}_q^{R'} \neq \emptyset$ and

$$(p')_{\leqslant p}(p' \cap \mathcal{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', q)) \& (p')_{\leqslant p}(p' \cap \mathcal{X}_q^{R'} \neq \emptyset \Rightarrow R'(p', q')).$$

Let $I \in p$ be such that $I \in \mathcal{X}_q^{R'}$. Let q be determined by φ. Let $I \in (a, b)$. Then $(a, b) \cap \mathcal{X}_q^{R'} \neq \emptyset$. Thus $R'((a, b), q)$. Thus as in the proof of Remark 13 we can show that if φ is of the form $(\mathrm{E}x)\psi(x)$ then there is such an x less than b and if φ is of the form $(x)\psi(x)$ then for every x less than or equal to a, $\psi(x)$. Let φ be of the form $(\mathrm{E}x)\psi(x)$. Then for every b such that $b \notin I$ there is an x such that $x < b$ and $\psi(x)$. Using underspill (note that I is not definable in M because it has no biggest element) we have an x in I such that $\psi(x)$. Thus $I \models \varphi$. If φ is of the form $(x)\psi(x)$ then for every a in I we have $\psi(a)$. Thus $I \models \varphi$. Let q' be determined by φ'. As before, we show that $I \models \varphi'$. Thus $q \cap q' \neq \emptyset$. Similarly we show $R'(p, q \cap q')$. ■

Let us show a few properties of the relation R' that we have defined and of the relation R definable by R' as in Remark 11.

First consider the properties that resemble forcing in set-theory.

Remark 15. $\mathcal{X}_q^R = \mathcal{X}_q^{R'}$.

Proof. Let $x \in \mathcal{X}_q^{R'}$. Suppose that $x \notin \mathcal{X}_q^R$. Then

$$(p)(x \in p \Rightarrow \neg R(p, q) \& \neg R(p, \bar{q})).$$

Thus

$$(p)\big(x \in p \Rightarrow (\mathrm{E}p')_{\leqslant p}(p' \cap \mathcal{X}_q^{R'} \neq \emptyset \& \neg R'(p', q))\big).$$

But

$$p' \cap \mathcal{X}_q^{R'} \neq \emptyset \& (\mathrm{E}p'')_{\leqslant p'}(\mathrm{E}q')(R'(p', q') \& q' \cap q = \emptyset).$$

Thus

$$(p)\big(x \in p \Rightarrow (\mathrm{E}p'')_{\leqslant p}(\mathrm{E}q')(R'(p', q') \& q' \cap q = \emptyset)\big).$$

Hence by Remark 14

$$(p)(x \in p \Rightarrow (\mathrm{E}J)(\mathrm{E}\varphi')(J \in p \& J \models \varphi'))$$

where φ' is such that $\{I\colon I \models \varphi'\} \cap \{I\colon I \models \varphi\} = \emptyset$ where q is determined by φ. But $I \models \varphi' \Rightarrow I \models \neg\varphi$ because if $I \models \varphi'$ then I not $\models \varphi$. Thus

$$(p)(x \in p \Rightarrow (\mathrm{E}I)(I \in p \& I \models \neg\varphi)).$$

Using "underspill" if $\neg\varphi$ is Σ_1 or directly if $\neg\varphi$ is Π_1, we infer that $x \models \neg\varphi$. Similarly we show that $x \models \varphi$. Contradiction. Thus $x \in \mathcal{X}_q^R$. ■

Remark 16. *Let* $I \in \mathcal{X}_q^R$. *Then*

$$(\mathrm{E}p)(I \in p \& R(p, q)) \equiv I \models \varphi$$

where q *is determined by* φ.

Thus we have the truth lemma, in other words, $f_R = \mathrm{id}$.

Proof. Assume $(\mathrm{E}p)(R(p, q) \& I \in p)$. Then, as in Remark 14, we show that $I \models \varphi$. Let $I \models \varphi$. Suppose that $(p)\neg R(p, q)$. Then, as in the proof of Remark 15, we show that $I \models \neg\varphi$. Hence $(\mathrm{E}p)(I \in p \& R(p, q))$. ■

Corollary 1. *R is not* PA-*definable.*

Proof. Indeed, suppose it is PA-definable. Let $R(p, q)$. Let $p' \leqslant p$ be minimal such that $R(p', q)$. Let $I \in p' \cap \mathcal{X}_q^R$, $I \models \varphi$. Let $p' = (a, b)$. Then $I \in (a+1, b)$. By Remark 16, $R((a+1, b), q)$. This contradicts the minimality of p'. ■

Remark 17. *Let* $p \cap \mathcal{X}_q^R \neq \emptyset$. *We have*

$$R(p, q) \equiv (x)_{\mathcal{X}_q^R}(x \in p \Rightarrow x \models \varphi),$$

where q *is determined by* φ.

Proof. $R(p, q)$ implies $(x)_{\mathcal{X}_q^R}(x \in p \Rightarrow x \models \varphi)$ by Remark 16. Thus assume that $(x)_{\mathcal{X}_q^R}(x \in p \Rightarrow x \models \varphi)$. Suppose that $\neg R(p, q)$. But $p \cap \mathcal{X}_q^R \neq \emptyset$. Let $x \in p \cap \mathcal{X}_q^R$. We have $x \models \neg\varphi$ by the fact that $x \in \mathcal{X}_q^R$ and by Remark 16. Contradiction. Thus $R(p, q)$. ■

Remark 18.

(1) $$p \cap \mathcal{X}' \neq \emptyset \Rightarrow R(p, q) \equiv (x)(x \in p \Rightarrow x \models \varphi).$$

(2) $$I \in \mathcal{X}' \Rightarrow (\mathrm{E}p)(R(p, q) \& I \in p) \equiv I \models \varphi.$$

Indeed, we prove (1) exactly as Remark 16 and (2) follows from Remark 16.

The above remark shows that elements of $\mathcal{X}'$ play the role of generic filters in set theory. Let us mention that models of PA are in $\mathcal{X}'$ and hence it follows that Remark 18 holds also if we replace $\mathcal{X}'$ by the family of models of PA. Thus models of PA also play the role of generic filters.

Remark 19. *$R_{\restriction \boldsymbol{P}'}$ satisfies* (4) *of Definition* 2 *of a forcing relation.*

Proof. Let us show that for every q the set

$$D_q = \{p \in \boldsymbol{P}'\colon R(p, q) \vee (\mathrm{E}q')(q' \cap q = \emptyset \& R(p, q'))\}$$

is dense in $\boldsymbol{P}'$.

Indeed, let $p \in \boldsymbol{P}'$. Let $I \in p \cap \mathcal{X}'$. Let q be determined by φ. If $I \models \varphi$ then, by Remark 15, $(\mathrm{E}p')(I \in p' \& R(p', q))$. Let p'' be such that $I \in p''$, $p'' \leqslant p', p$. Then $p'' \leqslant p$, $p'' \in D_q$, $p'' \in \boldsymbol{P}'$. If $I \models \neg\varphi$, then $(\mathrm{E}p'')_{\leqslant p}(p'' \in \boldsymbol{P}' \& R(p'', \bar{q}))$. Thus $p'' \in D_q$.

All we have said up to now about R' and R, except the facts that if $x \in \mathcal{X}_q^R$ then x has no biggest element and that R is not PA-definable, and except the observations following Remark 17, remains true if we define $G_c(a, b)$ in a slightly different way. If player I plays "$(\mathrm{E}x)\theta(x)$" then in the case where player II answers "yes" he is obliged to show an x such that $\theta(x)$

not necessarily the least such x. Player II wins if he has not given contradictory answers and answers contradicting the fact that he describes an initial segment such that a is in it and b is not (we drop the requirements about successor, addition and multiplication). This game would even be more natural for our considerations so far, e.g., we would be able to prove $R'(p, q) \equiv (\mathrm{E}x)_p (x \models \varphi)$ where q is determined by φ. However, we would not be able to obtain certain models of PA, which is important in our next considerations.

Let us now show a few properties of R' connected with the theory PA.

Remark 20. *Let $I \models \mathrm{PA}$, $I \in p$. Then for every q, $R'(p, q)$ or $R'(p, \bar{q})$. Thus $I \in \mathscr{X}'$. Moreover, if $p \in \boldsymbol{P}'$ then $(\mathrm{E}J)(J \in p \,\&\, J \models \mathrm{PA})$. Thus $\mathscr{X}'$ and $\{I\colon I \models \mathrm{PA}\}$ are symbiotic.*

Proof. Let q be determined by φ. Then $\bar{q}$ is determined by $\neg\varphi$. If $I \models \varphi$ then we define the strategy for player II in $G_{\ulcorner\varphi\urcorner+3}(p)$ so that player II tells the truth about I. This is a winning strategy. If $I \models \neg\varphi$ then the reasoning is analogous. If $p \in \boldsymbol{P}'$ then for every q, $R'(p, q) \vee R'(p, \bar{q})$. Hence $(\mathrm{E}J)_p (J \models \mathrm{PA})$. ∎

Remark 21. *Let $I \in \mathscr{X}^R_q$. Then $(\mathrm{E}p)(I \in p \,\&\, R(p, q))$, i.e.,*

$$I \models (a)(\mathrm{E}b)\big(R'(a', b, \ulcorner\varphi\urcorner) \,\&\, b > a\big)$$

where q is determined by φ (we identify q with $\ulcorner\varphi\urcorner$).

Proof. Let $(\mathrm{E}p)(I \in p \,\&\, R(p, q))$. By Remark 11 and Remark 15 we infer

$$(p')\big(I \in p' \Rightarrow (\mathrm{E}p'')_{\leqslant p'} (I \in p'' \,\&\, R'(p'', q))\big).$$

Replacing elements of $\boldsymbol{P}$ by pairs and using "underspill", we infer

$$I \models (a)(\mathrm{E}b)\big((R'(a', b), \ulcorner\varphi\urcorner) \,\&\, b > a\big).$$

To prove "if" we use "overspill". ∎

Corollary 2. *The Paris sentence $(x)(\mathrm{E}a, b)\big(a > x \,\&\, Y(a, b) > x\big)$ is valid in N (the set of standard integers); Y is the Paris indicator.*

Proof. We have $N \models \mathrm{PA}$, thus $N \in \mathscr{X}'$. Hence

$$N \models \varphi \equiv (\mathrm{E}p)\big(R(p, q) \,\&\, N \in p\big) \equiv (p')\big(N \in p' \Rightarrow R'(p', q)\big)$$

where q is determined by φ. We have $N \models \varphi$ or $N \models \neg\varphi$ for every φ. Thus for every q

$$(p')\big(N \in p' \Rightarrow R'(p', q) \vee R'(p', \bar{q})\big).$$

By Remark 21 hence follows

$$N \models (\mathrm{E}a, b)\big(a > \ulcorner\varphi\urcorner \,\&\, (R'(a, b), \ulcorner\varphi\urcorner) \vee (\dot{R}'(a, b), \ulcorner\neg\varphi\urcorner)\big)$$

for every φ. But

$$R\big((a, b), \ulcorner\varphi\urcorner\big) \vee R'\big((a, b), \ulcorner\neg\varphi\urcorner\big) \Rightarrow Y(a, b) > \ulcorner\varphi\urcorner.$$

Hence

$$N \models (x)(\mathrm{E}b)\big(b > x \,\&\, Y(x, b) > x\big). \;∎$$

We shall use our technique also to prove the independence of the Paris sentence; however, this will be only a translation of the Paris proof. Next we shall show the independence of a sentence very close to the Paris sentence. We do not know whether the two sentences are equivalent.

Assume that the enumeration $\ulcorner\varphi\urcorner$ of Σ_1 and Π_1 sentences is considered within M. Then the Gödel numbers $\ulcorner\varphi\urcorner$ correspond both to standard and non-standard sentences. We can extend our definition of $R'(p, q)$ onto $\boldsymbol{P} \times M$ as follows:

let $R'(p, x) \equiv x$ is a number of a Σ_1 or a Π_1 sentence and there is a winning strategy for player II in $G_{x+3}(p)$ answering "yes" to the question x.

It is also reasonable to extend the definition of R onto $\boldsymbol{P} \times M$.

If x is non-standard then let us define $R(p, x)$:

$$\begin{aligned} R(p, x) &\equiv \big[p \cap \mathscr{X}' \neq \emptyset \,\&\, (p')_{\leqslant p} (p \cap \mathscr{X}' \neq \emptyset \Rightarrow R'(p', x))\big] \\ &\equiv \big[p \in \boldsymbol{P}' \,\&\, (p')_{\boldsymbol{P}'} (p' \leqslant p \Rightarrow p' \cap \mathscr{X}' \neq \emptyset)\big]. \end{aligned}$$

Here $\mathscr{X}'$ plays the role of $\mathscr{X}^{R'}_q$. Indeed, it is reasonable to consider points $I \in \mathscr{X}$ that are "generic" w.r.t. standard sentences less than x and for a non-standard x this means all standard sentences.

Let us show that the sentence

$$(x)(a)(\mathrm{E}b)\big(b > a \,\&\, R'((a, b), x)\big) \vee (a)(\mathrm{E}b)\big(b > a \,\&\, R'((a, b), \bar{x})\big)$$

where $\bar{x}$ is the number of the negation of the sentence with the number x, is independent of PA.

Our sentence says that arbitrarily large intervals (a, b) decide about the sentence x. In particular, for arbitrarily large (a, b) the Paris game has length at least x. Hence our sentence implies the Paris sentence. However, it is not clear whether the converse implication holds. Indeed, if the Paris game at (a, b) has length greater than x, then it means that there is a strategy for the second player answering all sufficiently small sequences of questions; if a sequence ends with "x", then this strategy answers "yes" or "no", but the answer may depend on the sequence of questions. It is not necessarily true that for every sentence ending with "x" the strategy answers "yes" or for every such sequence it answers "no", and this is what is stated in our sentence.

Remark 22. *If x is standard then both*

$$(a)(\mathrm{E}b)\big(b > x,\, a \,\&\, R'((a, b), x)\big) \vee (a)(\mathrm{E}b)\big(b > x,\, a \,\&\, R'((a, b), \bar{x})\big)$$

where $x = \ulcorner\varphi\urcorner$ and $\bar{x} = \ulcorner\neg\varphi\urcorner$; and

$$(a)(\mathrm{E}b)\big(b > x,\, a \,\&\, Y(a, b) > x\big)$$

are provable sentences of PA, *and thus are equivalent. Indeed, if* $I \models$ PA *and* $I \in p$ *then* $R'(p, x)$ *if* $I \models \varphi$ *and* $R'(p, \bar{x})$ *if* $I \models \neg \varphi$. *Then, using underspill, we show that* I *satisfies both sentences in question. By the fact that every model of* PA *is an initial segment of another model we infer that our sentences are provable in* PA.

If x *is non-standard we no longer know whether the sentence*

$$\text{“}(a)(\mathrm{E}b)\big(b > x,\ a \,\&\, R'((a, b), x)\big) \vee (a)(\mathrm{E}b)\big(b > x,\ a \,\&\, R'((a, b), \bar{x})\big)\text{”}$$

is equivalent to $(a)(\mathrm{E}b)\big(b > x,\ a \,\&\, Y(a, b) > x\big)$. *If it is not equivalent, it is another independent sentence.*

Proof. We shall transform the Kirby–Paris proof. For $p \in \boldsymbol{P}$, let $p = (a, b)$, $(p)_0 = a$, $(p)_1 = b$. Assume that every a is a number of a Σ_1 or Π_1 sentence. We shall show the following: the set $D = \{p \in \boldsymbol{P}'\colon Y(p) < (p)_0\}$ is dense in $\boldsymbol{P}'$ (Y is the Kirby–Paris indicator).

Indeed, let $p \in \boldsymbol{P}'$. We can assume that $(p)_0$ is non-standard. Indeed, suppose that $(p)_0$ is standard. In [4] it is shown (Proposition 4.3) that if $p \in \boldsymbol{P}'$ then there are infinitely many I such that $I \in p$, $I \models$ PA. Take a non-standard I such that $I \in p$. Let $a \in I$, $a > N$. Let $p' = \big(a, (p)_1\big)$. Then $p' \leqslant p$, $p' \in \boldsymbol{P}'$, $(p')_0 > N$.

If $Y(p) < (p)_0$ then $p \in D$. Thus assume that $Y(p) \geqslant (p)_0$. Let $p' \leqslant p$ be minimal such that $Y(p') \geqslant (p')_0$. Then $p' \in \boldsymbol{P}'$ because $Y(p') > N$. Thus $\bar{p} = \big((p')_0 + 1, (p')_1\big) \in \boldsymbol{P}'$. By the minimality of p', $Y(\bar{p}) < (\bar{p})_0$. Hence it follows that D is dense.

Let $I \in \mathscr{X}$ be such that $(\mathrm{E}p)_D(I \in p)$. Let $(p)_0 = x$. Then $x \in I$. Suppose that $I \models (a)(\mathrm{E}b)\big(Y(a, b) \geqslant x\big)$. Then, by "overspill", $(\mathrm{E}p')\big(p' \leqslant p \,\&\, Y(p') \geqslant x \,\&\, I \in p'\big)$. But $Y(p) \geqslant Y(p')$. Thus $Y(p) \geqslant x$ contradicting the fact that $p \in D$. Hence

$$I \models \neg (x)(a)(\mathrm{E}b)\big(Y(a, b) > x\big).$$

Thus

$$I \models \neg (x)\big[(a)(\mathrm{E}b)\big(b > a, x \,\&\, R'((a, b), x)\big) \vee (a)(\mathrm{E}b)\big(b > a, x \,\&\, R'((a, b), \bar{x})\big)\big]. \ \blacksquare$$

COROLLARY 3. *The negation of the Paris sentence is distinguished from the Paris sentence by the fact that it is satisfied in all segments that are in a certain sense "generic"* (intersect D).

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