$C$ is an approximation of $\dot{U}$ by $E$ over $X$. This completes our proof of Lemma R. ■

Now, we give a proof of Theorem A. Suppose that $A$ is an $\boldsymbol{R}$-positive formula in $L$, $X$ is a finite set of $\boldsymbol{R}$-atomic formulas, $U$ is a finite set of uniqueness conditions of $\boldsymbol{R}$, and $E$ is a finite set of existence conditions of $\boldsymbol{R}$ The "if-part" of Theorem A is obvious because that the formula $\bigwedge X \wedge \bigwedge U \wedge \bigwedge E . \supset B$ is provable in $L$, for every approximation $B$ of $U$ by $E$ over $X$. Also, if $E$ consists of simple existence conditions only, for any approximation $B$ of $U$ by $E$ over $X$, we can find a simple approximation $B'$ of $U$ by $E$ over $X$ such that $B' \supset B$ is provable in $L$. So, it is sufficient to prove the "only-if-part" of Theorem A.

Assume that the formula $\bigwedge X \wedge \bigwedge U \wedge \bigwedge E . \supset A$ is provable in $L$. Then, the sequent $X, U, E \to A$ is provable in $L$. By Lemma K, the sequent $X, U, E \to A^{=}$ is provable in $L_1$. Then, this sequent is also provable in $L_2$ by Lemma L. Since $A^{=}$ is $\boldsymbol{R}$-positive, the sequent $X \to A^{=}$ is an $\boldsymbol{R}$-sequent. By Lemma P, this sequent is provable in $L_3(U, E)$. By Lemma Q, the sequent $X \to A^{=}$ is provable in $L_4(U, E)$. By Lemma R, there is an approximation $B$ of $U$ by $E$ over $X$ such that the sequent $B \to A^{=}$ is provable in $L$. Since the sequent $A^{=} \to A$ is provable in $L$, the sequent $B \to A$ is provable in $L$. Hence, the formula $B \supset A$ is provable in $L$. This completes our proof of Theorem A.

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# On the homotopical classification of DJ-mappings of infinitely dimensional spheres

by

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**Abstract.** This paper contains some results which concern the DJ-homotopical classification of DJ-mappings of the sphere in an infinitely dimensional Hilbert space into itself.

In connection with the appearance of the definition of the category in which a sphere in the infinitely dimensional Banach space is not contractible, we have to consider the homotopical classification of transformations of such a sphere into itself within this category. This problem has been presented by B. Nowak [5]. It will be a certain simplification to notice that the set of homotopy classes is a group, as it is in the finite-dimensional case. This paper is an attempt to present certain numerical invariants of the homotopy classes. However, we will not be able to prove that there is a one-to-one correspondence between them.

We will first define objects and morphisms of DJ-category.

DEFINITION. A pair $(X, (X_n)_{n\in N})$ where $X$ is a metric space and $(X_n)_{n\in N}$ is an increasing sequence of its subspaces such that

$$X = \overline{\bigcup_{n\in N} X_n} \tag{1}$$

is called a *metric space with filtration.*

DEFINITION. Let $(X, (X_n)_{n\in N})$ and $(Y, (Y_n)_{n\in N})$ be two metric spaces with filtration and $d_Y$- the distance in $Y$. A uniformly continuous transformation $f\colon X \to Y$ such that

$$\lim_{n\to\infty} \sup_{x\in X_n} d_Y(f(x), Y_n) = 0 \tag{2}$$

is called a DJ-*mapping*. If condition (2) is replaced by:

(3) there is $n_0 \in N$ such that for $n \geqslant n_0$: $f(X_n) \subset Y_n$,

then $f$ is called an FJ-*mapping.*

FJ-mappings are a particular case of DJ-mappings.

We will consider an infinitely dimensional Hilbert space $H$ with a filtration containing finite-dimensional linear subspaces $(H_n)_{n\in N}$. A filtration induced on $S_H = \{x\in H\colon \|x\| = 1\}$ makes the unit sphere a metric space with filtration by finite-dimensional spheres. We have a canonical abelian group structure in the set of classes of DJ-homotopically equivalent DJ-mappings of $S_H$ into $S_H$. This group is marked $[S_H, S_H]$. In paper [5] B. Nowak has proved that this group is nontrivial in the general case of a Banach space.

The theory of DJ-mappings of infinitely dimensional spheres is a kind of generalization of the classical Leray–Schauder theory [3] and of the theory of approximatively-proper mappings by Browder and Petryshyn [1], [6]. Neither the set of Leray–Schauder mappings nor the set of approximatively-proper mappings is closed with regard to the algebraic operations and the operation of taking inverse transformation, while DJ-mappings have these properties. We can define a degree of DJ-mapping and show [5] that homotopical classification by degree of DJ-mapping is better than by the Browder–Petryshyn degree and that there exist DJ-mappings which are not approximatively-proper.

Now we will prove two theorems which reduce the study of the group $[S_H, S_H]$ to the study of FJ-mappings and FJ-homotopies between them.

THEOREM 1 (FJ-approximation). *The set of all FJ-mappings of a metric space with filtration $(X, (X_n)_{n\in N})$ into a Hilbert space with filtration by finite-dimensional linear subspaces $(H, (H_n)_{n\in N})$ is dense in the set of all bounded DJ-mappings of $X$ into $H$ in the uniform convergence topology. We get the same result when we replace $(H, (H_n)_{n\in N})$ by $(S_H, (S_H \cap H_n)_{n\in N})$.*

Proof. Let $f\colon (X, (X_n)_{n\in N}) \to (H, (H_n)_{n\in N})$ be a bounded DJ-mapping. For every $\varepsilon > 0$ we will find an FJ-mapping $f_\varepsilon\colon (X, (X_n)_{n\in N}) \to (H, (H_n)_{n\in N})$ such that $\sup_{x\in X} \|f(x)-f_\varepsilon(x)\| \leqslant \varepsilon$.

From the definition of a DJ-mapping, there is an $n_0 \in N$ such that for $n \geqslant n_0$ and $x \in X_n$

$$d_H(f(x), H_n) \leqslant \tfrac{1}{2}\varepsilon. \tag{4}$$

We will show that there is a $\delta > 0$ such that for $n \geqslant n_0$ and $x \in X$

$$d_X(x, X_n) \leqslant \delta \Rightarrow d_H(f(x), H_n) \leqslant \varepsilon. \tag{5}$$

In fact, since $f$ is uniformly continuous, there is a $\delta > 0$ such that $d_X(x, y) \leqslant 2\delta$ implies $d_H(f(x), f(y)) \leqslant \frac{1}{2}\varepsilon$. If $n \geqslant n_0$, $x$ is a point of $X$ and $d_X(x, X_n) \leqslant \delta$, then there exists a $y \in X_n$ such that $d_X(x, y) \leqslant 2\delta$. Hence $d_H(f(x), f(y)) \leqslant \frac{1}{2}\varepsilon$ and from (4) $d_H(f(y), H_n) \leqslant \frac{1}{2}\varepsilon$. So $d_H(f(x), H_n) \leqslant \varepsilon$. We have shown (5).

We define a family of functions $\alpha_k\colon X \to [0, 1]$ for $k \in N$:

$$\alpha_k(x) = \begin{cases} 1 & \text{when } k \leqslant n_0, \\ \min[1, \delta^{-1} d_X(x, X_{k-1})] & \text{when } k > n_0. \end{cases} \tag{6}$$



The functions $\alpha_k$ satisfy the Lipschitz condition

$$|\alpha_k(x) - \alpha_k(y)| \leqslant \delta^{-1} d_X(x, y) \tag{7}$$

for $x, y \in X$ and $k \in N$, and have the following property:

$$x \in X_n \Rightarrow \bigwedge_{k > \max(n, n_0)} \alpha_k(x) = 0. \tag{8}$$

We will write $P_n$ for the orthogonal projection onto $H_n$, $n \in N$ and $P_{-1} = 0$. For all $x \in X$ we have the formula

$$f(x) = \sum_{k=0}^{\infty} (P_k - P_{k-1}) f(x). \tag{9}$$

Now we can define the mapping $f_\varepsilon\colon X \to H$:

$$f_\varepsilon(x) = \sum_{k=0}^{\infty} \alpha_k(x)(P_k - P_{k-1}) f(x). \tag{10}$$

This series is convergent because its terms are orthogonal to each other,

$$\|\alpha_k(x)(P_k - P_{k-1}) f(x)\|^2 \leqslant \|(P_k - P_{k-1}) f(x)\|^2$$

and the series $\sum_{k=0}^{\infty} \|(P_k - P_{k-1}) f(x)\|^2$ is convergent to $\|f(x)\|^2$.

We will show that $f_\varepsilon$ is uniformly continuous. Using the Minkowski inequality and (7), we get:

$$\begin{aligned} \|f_\varepsilon(x) - f_\varepsilon(y)\| &\leqslant \Big(\sum_{k=0}^{\infty} |\alpha_k(x) - \alpha_k(y)|^2 \|(P_k - P_{k-1}) f(x)\|^2\Big)^{1/2} + \\ &\quad + \Big(\sum_{k=0}^{\infty} |\alpha_k(y)|^2 \|(P_k - P_{k-1})(f(x) - f(y))\|^2\Big)^{1/2} \\ &\leqslant \delta^{-1} \|f(x)\| d_X(x, y) + \|f(x) - f(y)\| \quad \text{for} \quad x, y \in X. \end{aligned}$$

Since $f$ is bounded, $f_\varepsilon$ is uniformly continuous.

If $n \geqslant n_0$ and $x \in X_n$, then by (8) $f_\varepsilon(x) \in H_n$. Hence $f_\varepsilon$ is the FJ-mapping.

Now we will prove that $f_\varepsilon$ is the FJ-approximation of the mapping $f$. Let $x$ be a point of $X$. A sequence $(d_X(x, X_n))_{n\in N}$ is decreasing and convergent to 0, therefore there is an $n' \in N$ such that $d_X(x, X_{n'}) \leqslant \delta$ and $d_X(x, X_{n'-1}) > \delta$. We get an inequality:

$$\begin{aligned} \|f(x) - f_\varepsilon(x)\| &= \Big(\sum_{k=0}^{\infty} |1 - \alpha_k(x)|^2 \|(P_k - P_{k-1}) f(x)\|^2\Big)^{1/2} \\ &= \Big(\sum_{k=n'+1}^{\infty} |1 - \alpha_k(x)|^2 \|(P_k - P_{k-1}) f(x)\|^2\Big)^{1/2} \\ &\leqslant \Big(\sum_{k=n'+1}^{\infty} \|(P_k - P_{k-1}) f(x)\|^2\Big)^{1/2}. \end{aligned}$$

The last expression is equal to $d_H(f(x), H_n)$ and from (5) we have $\|f(x)-f_\varepsilon(x)\| \leqslant \varepsilon$.

Now let $F\colon X \to S_H$ be a certain DJ-mapping. Since $S_H \subset H$, by the first part of the theorem for $\varepsilon > 0$ there is an FJ-mapping $f_\varepsilon\colon X \to H$ such that $\sup_{x\in X}\|F(x)-f_\varepsilon(x)\| \leqslant \varepsilon$. For $\varepsilon < \frac{1}{2}$ we can define an FJ-mapping $F_\varepsilon\colon X \to S_H$

(11) $$F_\varepsilon(x) = \|f_\varepsilon(x)\|^{-1} f_\varepsilon(x).$$

For each $x \in X$ we have

$$\|F(x)-F_\varepsilon(x)\| \leqslant \|F(x)-f_\varepsilon(x)\| + \big\|f_\varepsilon(x)-\|f_\varepsilon(x)\|^{-1} f_\varepsilon(x)\big\| \leqslant 2\varepsilon.$$

We have proved the second part of our theorem.

This theorem and its proof can be written without changes for spaces $l^p$ with filtration generated by vectors of the standard basis. The problem whether the FJ-approximation theorem is true for all Banach spaces is open.

THEOREM 2. *If $f, g\colon S_H \to S_H$ are DJ-homotopically equivalent FJ-mappings, then there is an FJ-homotopy between them.*

Proof. Let $h$ be a DJ-homotopy between $f$ and $g$. Due to Theorem 1 there is an FJ-mapping $h^*\colon X\times[0, 1] \to S_H$ such that $\|h^*-h\| \leqslant 1$. We define $G\colon X\times[0, 1] \to H$ by the formulas

(12) $$G(x, t) = \begin{cases} (1-3t)\, f(x) + 3t h^*(x, 0) & \text{for } t\in[0, \frac{1}{3}),\\ h^*(x, 3t-1) & \text{for } t\in[\frac{1}{3}, \frac{2}{3}),\\ (3t-2)\, g(x) + (3-3t)\, h^*(x, 1) & \text{for } t\in[\frac{2}{3}, 1].\end{cases}$$

It is clear that $G$ is an FJ-homotopy between $f$ and $g$ and that $\|G(x, t)\| \geqslant \frac{1}{2}$ for all $x \in S_H$, $t\in[0, 1]$. Therefore, $(x, t) \to G(x, t)/\|G(x, t)\|$ also has the above properties and takes values in $S_H$.

The following theorem is proved in [5]: if $f, g\colon S_H \to S_H$ are FJ-mappings and are DJ-homotopically equivalent, then from a certain $n_0 \in N$ we have

(13) $$\deg f|_{S_H \cap H_n} = \deg g|_{S_H \cap H_n}.$$

We will find a better characterization of DJ-homotopy classes and, by constructing an example, show that the converse is false.

We will first prove two lemmas about the extension of homotopy.

LEMMA 1. *We suppose that $f\colon (S_n, S_{n-k}) \to (S_n, S_{n-k})$ where $k \geqslant 2$, $f \sim 0$ and $f|_{S_{n-k}} \sim 0$. Then there exists a homotopy $h^*\colon f \sim 0$ such that $h^*(S_{n-k}\times [0, 1]) \subset S_{n-k}$.*

Proof. All the sets in question are included in $R^{n+1}$:

$$B_m = \{x\in R^{n+1}\colon x_i = 0,\ i > m;\ \|x\| \leqslant 1\},$$
$$B_m^+ = \{x\in B_m\colon x_m \geqslant 0\},$$
$$S_m = \{x\in R^{n+1}\colon x_i = 0,\ i > m+1;\ \|x\| = 1\}$$



and

$$S_m^+ = \{x\in S_m\colon x_{m+1} \geqslant 0\}.$$

Since we have a homotopy $f|_{S_{n-k}} \sim 0$, we can define a transformation $F\colon B_{n-k+1} \cup S_n \to S_n$ such that $F|_{S_n} = f$ and $F(B_{n-k+1}) \subset S_{n-k}$. We will prove that $F$ has an extension $F^*\colon B_{n+1} \to S_n$. To show this, it is sufficient by Borsuk's Homotopy Extension Theorem to verify that $F$ is homotopic to a map $G\colon B_{n-k+1} \cup S_n \to S_n$ which is extendable to $B_{n+1}$. To this end take any extension $\bar{f}\colon B_{n+1} \to S_n$ of $f$ and let $G = \bar{f}\,|\,B_{n-k+1} \cup S_n$. To see that $F$ and $G$ are homotopic, as maps to $S_n$, it is sufficient to show that their restrictions to $B_{n-k+1}$ are homotopic modulo $S_{n-k}$. This, however, is a direct consequence of the fact that $\pi_{n-k+1}(S_n) = 0$.

Let $f\colon S_n \to S_n$ be a continuous mapping. We will write $\deg^+ f|_{\mathring{S}_n^+}$ for the local degree of $f|_{\mathring{S}_n^+}\colon \mathring{S}_n^+ \to S_n$ at points $y \in \mathring{S}_n^+$ where $\mathring{S}_n^+ = \{x\in S_n\colon x_{n+1} > 0\}$. This number does not depend on the choice of the point $y$ [2]. Similarly, we define $\mathring{S}_n^-$, $\deg^- f|_{\mathring{S}_n^+}$ as the local degree of the same mapping at points $y\in \mathring{S}_n^-$, and $\deg^+ f|_{\mathring{S}_n^-}$, $\deg^- f|_{\mathring{S}_n^-}$. We have the obvious relations:

(16) $$\deg f = \deg^+ f|_{\mathring{S}_n^+} + \deg^+ f|_{\mathring{S}_n^-} = \deg^- f|_{\mathring{S}_n^+} + \deg^- f_{\mathring{S}_n^-}.$$

LEMMA 2. *Let $f\colon (S_n, S_{n-1}) \to (S_n, S_{n-1})$ be a continuous mapping of pairs, $f \sim 0$ and $f|_{S_{n-1}} \sim 0$. There is a homotopy $h^*\colon f \sim 0$ such that $h^*(S_{n-1}\times [0; 1]) \subset S_{n-1}$ if and only if*

(17) $$\deg^+ f|_{\mathring{S}_n^+} = \deg^+ f|_{\mathring{S}_n^-} = 0$$

*or equivalently $\deg^- f|_{\mathring{S}_n^+} = \deg^- f|_{\mathring{S}_n^-} = 0$. If such a homotopy exists, then each homotopy $f|_{S_{n-1}} \sim 0$ has an extension to $f \sim 0$.*

Proof. Suppose that $h^*\colon f \sim 0$ and $h^*(S_{n-1}\times[0, 1]) \subset S_{n-1}$. Then we can define a continuous transformation $F\colon B_{n+1} \to S_n$ such that $F|_{S_n} = f$ and $F(B_n) \subset S_{n-1}$. Hence $\deg F|_{\mathring{S}_n^+ \cup B_n} = \deg F|_{\mathring{S}_n^- \cup B_n} = 0$ and $\deg F|_{\mathring{S}_n^+ \cup B_n} = \deg_y F|_{\mathring{S}_n^+}$, $\deg F|_{\mathring{S}_n^- \cup B_n} = \deg_y F|_{\mathring{S}_n^-}$ where $y \notin S_{n-1}$. But the degrees $\deg_y F|_{\mathring{S}_n^+}$ and $\deg_y F|_{\mathring{S}_n^-}$ are equal to $\deg^+ f|_{\mathring{S}_n^+}$ and $\deg^+ f|_{\mathring{S}_n^-}$, respectively, or $\deg^- f|_{\mathring{S}_n^+}$ and $\deg^- f|_{\mathring{S}_n^-}$ if $y \in \mathring{S}_n^-$. Hence condition (17) is necessary.

Suppose now that condition (17) is satisfied. Having a homotopy $h\colon f|_{S_{n-1}} \sim 0$ and the homeomorphism $(S_{n-1}\times[0, 1]/(S_{n-1}\times\{1\}) \cong B_n$, we construct the mapping $G\colon S_n \cup B_n \to S_n$, $G|_{S_n} = f$, $G(B_n) \subset S_{n-1}$. The existence of a homotopy $h^*$ is equivalent to the existence of an extension of $G$ to $B_{n+1}$. Let us consider the mappings $G|_{S_n^+ \cup B_n}$ and $G|_{S_n^- \cup B_n}$. Since their domains are homeomorphic with $S_n$, this extension exists if the degrees of these mappings vanish. But this is the consequence of the assumption.

Both lemmas imply the following:

If $f\colon S_H \to S_H$ is an FJ-mapping which is DJ-homotopically trivial, then

$$(18)\qquad \begin{aligned} &\deg f|_{S_H\cap H_n} = 0, && \text{if } \dim H_n - \dim H_{n-1} > 1,\\ &\deg^+ f|_{S_H\cap \mathring{H}_n^+} = \deg^+ f|_{S_H\cap \mathring{H}_n^-} = 0 && \text{in the remaining cases.}\end{aligned}$$

We have chosen an orthonormal basis $\{e_n\colon n\in N\}$ in $H$ such that $H_n = \operatorname{Lin}\{e_k\colon k\leqslant k_n\}$ and we have marked $\mathring{H}_n^+ = \{x\in H_n\colon (x, e_{k_n}) > 0\}$, $\mathring{H}_n^- = \{x\in H_n\colon (x, e_{k_n}) < 0\}$. Due to the group structure in $[S_H, S_H]$ and Theorems 1, 2, we get:

THEOREM 3. *There is a homomorphism $[S_H, S_H]$ into the group $\mathop{\mathrm{X}}_{n=1}^{\infty} G_n / \bigoplus_{n=1}^{\infty} G_n$ where*

$$(19)\qquad G_n = \begin{cases} Z, & \text{if } \dim H_n - \dim H_{n-1} > 1,\\ Z\oplus Z, & \text{if } \dim H_n - \dim H_{n-1} = 1.\end{cases}$$

This homomorphism is not canonical in the general case. There are two equivalent ways to choose it: the first if we use the pairs $\deg^+$, and the second if we take $\deg^-$. If, for each $n\in N$, $\dim H_n - \dim H_{n-1} > 1$, this homomorphism is canonical.

EXAMPLE. Now we will construct an FJ-mapping $f\colon S_H\to S_H$ such that, for every $n\in N$, $\deg f|_{S_H\cap H_n} = 0$ but $f$ is not DJ-homotopically trivial.

Let $H = c^2$, $H_n = \operatorname{Lin}\{e_k\colon k\leqslant n\}$ where $\{e_k\colon k\in N\}$ is the standard basis in $c^2$, and $0<\varepsilon<\frac{1}{2}$. For $\varepsilon$-neighbourhoods of elements of the basis and vectors opposite to them: $V_n^+ = K(e_n, \varepsilon)\cap S_H$, $V_n^- = K(-e_n, \varepsilon)\cap S_H$, $n\geqslant 1$, we define transformations $T_n^+\colon V_n^- \to H$, $T_n^-\colon V_n^-\to H$ by the formulas

$$(20)\qquad \begin{aligned} T_n^+(x_0, x_1, \ldots, x_n, \ldots) &= 2\varepsilon^{-1}(4-\varepsilon^2)^{-1/2}(x_0, x_1, \ldots, x_{n-1}, x_{n+1}, \ldots),\\ T_n^-(x_0, x_1, \ldots, x_n, \ldots) &= 2\varepsilon^{-1}(4-\varepsilon^2)^{-1/2}(-x_0, x_1, \ldots, x_{n-1}, x_{n+1}, \ldots).\end{aligned}$$

Then we define $F\colon B_H\to H$ where $B_H$ is the unit ball in $H$:

$$(21)\qquad F(x_1, x_2, \ldots, x_n, \ldots) = \begin{cases} (2\|x\|-1, 2x_1, \ldots, 2x_n, \ldots), & \text{if } \|x\|\leqslant \frac{1}{2},\\ (2\|x\|-1, 2\|x\|^{-1}x_1(1-\|x\|), \ldots \\ \quad \ldots, 2\|x\|^{-1}x_n(1-\|x\|), \ldots) & \text{if } \|x\| > \frac{1}{2}\end{cases}$$

and $R(y) = \|y\|^{-1}y$ for $y\in H$. Now we get $f$ by defining on pieces:

$$(22)\qquad f|_{V_n^+} = R\circ F\circ T_n^+,\quad f|V_n^- = R\circ F\circ T_n^-,\quad f|_{S_H\setminus\bigcup_{n=1}^{\infty}(V_n^+\cup V_n^-)} = e_0.$$

We can see that $\|T_n^+(x)\|\leqslant 1$. Therefore, on the images $T_n^+(V_n^+)$ and $T_n^-(V_n^-)$ the norm $\|F(x)\|$ is not less than $\frac{1}{2}$. Moreover, for $x\in \operatorname{Fr}(V_n^+)$ we have $\|x-e_n\| = \varepsilon$, i.e.

$$\sum_{k\neq n} x_k^2 + (1-x_n)^2 = \varepsilon^2;$$



then $\|T_n^+(x)\| = 1$ and $f(x) = e_0$. In the same way we show that $x\in\operatorname{Fr}(V_n^-)$ implies $f(x) = e_0$. Hence $f$ is well defined and continuous. It is easy to check that $f$ is also uniformly continuous and, for each $n\geqslant 1$, $f(S_H\cap H_n)\subset H_n$. Therefore $f$ is an FJ-mapping.

We can find degrees and local degrees using the fact that all these mappings are of $C^\infty$-class [4]. We will obtain:

$$\deg f|_{S_H\cap H_n} = \deg_{-e_0} f|_{S_H\cap H_n} = \sum_{i=1}^{n}(+1) + \sum_{i=1}^{n}(-1) = 0,$$

$$\deg^+ f|_{S_H\cap \mathring{H}_n^+} = \deg_{e_n} f|_{S_H\cap \mathring{H}_n^+} = \sum_{i=1}^{n}(+1) + \sum_{i=1}^{n-1}(-1) = +1,$$

$$\deg^+ f|_{S_H\cap \mathring{H}_n^-} = \deg_{e_n} f|_{S_H\cap \mathring{H}_n^-} = \sum_{i=1}^{n-1}(+1) + \sum_{i=1}^{n}(-1) = -1.$$

All degrees $\deg f|_{S_H\cap H_n}$ vanish but $f$ is not DJ-homotopically trivial.

It is important for the applications that the described homomorphism of $[S_H, S_H]$ into $\mathop{\mathrm{X}}_{n=1}^{\infty} G_n/\bigoplus_{n=1}^{\infty} G_n$ should be an injection. For a mapping $f$ which has local degrees equal to 0 from Lemmas 1 and 2, there is a homotopy $f|_{S_H\cap \bigcup_{n=1}^{\infty} H_n} \sim 0$ which is continuous on each sphere $S_n$.

The problem whether it is possible to choose a homotopy which is uniformly continuous, i.e. whether the homomorphism is injective, is still unsolved.

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