Approximation theory of uniqueness conditions by existence conditions

by

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Abstract. Let L be a first order classical predicate calculus with equality LK, or a first order intuitionistic predicate calculus with equality LJ. Suppose that $\boldsymbol{R}$ is a set of predicate symbols in L. Then, $\boldsymbol{R}$-free ($\boldsymbol{R}$-positive) formulas are formulas which have no (no negative) occurrences of predicate symbols in $\boldsymbol{R}$. Uniqueness conditions of $\boldsymbol{R}$ are sentences of the form: $\forall \bar{x}_1 \ldots \forall \bar{x}_p (R_1(\bar{x}_1) \wedge \ldots \wedge R_p(\bar{x}_p). \supset A(\bar{x}_1, \ldots, \bar{x}_p))$, where $R_1, \ldots, R_p \in \boldsymbol{R}$ and A is $\boldsymbol{R}$-free. Existence conditions of $\boldsymbol{R}$ are sentences of the form:

$$\forall \bar{x}_0 \forall \bar{x}_1 \ldots \forall \bar{x}_p \big(R_1(\bar{x}_1) \wedge \ldots \wedge R_p(\bar{x}_p) \wedge A_1(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_p) \wedge \ldots \wedge A_q(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_p). \supset \exists \bar{y}\big(Q_1(\bar{s}_1(\bar{x}_0, \ldots, \bar{x}_p, \bar{y})) \vee \ldots \vee Q_r(\bar{s}_r(\bar{x}_0, \ldots, \bar{x}_p, \bar{y}))\big)\big),$$

where $r \geqslant 1$, $R_1, \ldots, R_p, Q_1, \ldots, Q_r \in \boldsymbol{R}$, and $A_1, \ldots, A_q$ are $\boldsymbol{R}$-free. Suppose that X is a finite set of formulas of the form $P(\bar{t})$, $P \in \boldsymbol{R}$, and U is a uniqueness condition of $\boldsymbol{R}$ of the form above. Then, $U[X]$ is the formula $\bigwedge\limits_{R_1(\bar{s}_1) \in X} \ldots \bigwedge\limits_{R_p(\bar{s}_p) \in X} A(\bar{s}_1, \ldots, \bar{s}_p)$.

Suppose that $\boldsymbol{U}$ is a finite set of uniqueness conditions of $\boldsymbol{R}$ and $\boldsymbol{E}$ is a set of existence conditions of $\boldsymbol{R}$. Then, the set of approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X, denoted by $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$, is defined by the following (1), (2), (3), and (4):

(1) $\bigwedge X \wedge \bigwedge\limits_{A \in \boldsymbol{U}} A[X] \in \mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$.

(2) If $A, B \in \mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$, then $A \wedge B \in \mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$.

(3) If $B_j(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_p, \bar{b}) \in \mathrm{Ap}\big(\boldsymbol{U}, \boldsymbol{E}, X \cup \{Q_j(\bar{s}_j(\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_p, \bar{b}))\}\big)$, $j = 1, \ldots, r$ and

$$\forall \bar{x}_0 \forall \bar{x}_1 \ldots \forall \bar{x}_p \Big(\bigwedge_{i=1}^{p} R_i(\bar{x}_i) \wedge \bigwedge_{k=1}^{q} A_k(\bar{x}_0, \ldots, \bar{x}_p). \supset \exists \bar{y} \bigvee_{j=1}^{r} Q_j(\bar{s}_j(\bar{x}_0, \ldots, \bar{x}_p, \bar{y}))\Big) \in \boldsymbol{E}$$

then

$$\bigwedge_{R_1(\bar{u}_1) \in X} \ldots \bigwedge_{R_p(\bar{u}_p) \in X} \forall \bar{x}_0 \Big(\bigwedge_{k=1}^{q} A_k(\bar{x}_0, \bar{u}_1, \ldots, \bar{u}_p). \supset \exists \bar{y} \bigvee_{j=1}^{r} B_j(\bar{x}_0, \bar{u}_1, \ldots, \bar{u}_p, \bar{y})\Big) \in \mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X).$$

(4) Every element in $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$ is obtained from (1), (2), (3) above only.

Our main theorem is:

APPROXIMATION THEOREM. *Suppose that A is an $\boldsymbol{R}$-positive formula in L. Then, the formula $\bigwedge X \wedge \bigwedge \boldsymbol{U} \wedge \bigwedge \boldsymbol{E}. \supset A$ is provable in L if and only if $B \supset A$ is provable in L for some approximation B of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X.*

This theorem and its variations have many applications.

In this paper, we shall introduce a new syntactical theory, named "approximation theory of uniqueness conditions by existence conditions", which gives us new proofs of "elimination theorems of uniqueness conditions" in Motohashi [6], [7], Minc's theorem on Skolem functions in the intuitionistic predicate calculus in Smorynski [10], [11], Barwise's approximation theorem on Henkin quantifiers in Barwise [1], and a new approximation theorem on Vaught sentences (cf. Harnik–Makkai [2]). Let L be a first order classical predicate calculus with equality LK or a first order intuitionistic predicate calculus with equality LJ. Now, we fix a set $\boldsymbol{R}$ of predicate symbols. Then, a formula A in L is said to be *$\boldsymbol{R}$-free* (*$\boldsymbol{R}$-positive*, *$\boldsymbol{R}$-negative*) if A has no (no negative, no positive) occurrences of predicate symbols in $\boldsymbol{R}$. $\boldsymbol{R}$-atomic formulas are formulas of the form $P(\bar t)$, where $P\in\boldsymbol{R}$ and $\bar t$ is a sequence of terms, and normal $\boldsymbol{R}$-atomic formulas are $\boldsymbol{R}$-atomic formulas of the form $P(\bar x)$, where $\bar x=\langle x_1, x_2, \dots, x_n\rangle$ is a sequence of distinct free variables.

Uniqueness conditions of $\boldsymbol{R}$ are sentences of the form

$$\forall\bar x_1 \dots \forall\bar x_p(R_1(\bar x_1)\wedge\dots\wedge R_p(\bar x_p).\supset A(\bar x_1, \dots, \bar x_p)),$$

where $R_1(\bar x_1), \dots, R_p(\bar x_p)$ are normal $\boldsymbol{R}$-atomic formulas and A is an $\boldsymbol{R}$-free formula in L.

$$\forall x_1\forall x_2\forall y_1\forall y_2(R(x_1, y_1)\wedge R(x_2, y_2).\supset(x_1=x_2\supset y_1=y_2))$$

is a typical example of uniqueness conditions of $\{R\}$, where R is a binary predicate symbol.

Existence conditions of $\boldsymbol{R}$ are sentences of the form

$$\forall\bar x_0\forall\bar x_1\dots\forall\bar x_p(R_1(\bar x_1)\wedge\dots\wedge R_p(\bar x_p)\wedge A_1(\bar x_0, \bar x_1, \dots, \bar x_p)\wedge\dots$$
$$\wedge A_q(\bar x_0, \bar x_1, \dots, \bar x_p).\supset\exists\bar y(Q_1(\bar s_1(\bar x_0, \dots, \bar x_p, \bar y))\vee\dots\vee Q_r(\bar s_r(\bar x_0, \dots, \bar x_p, \bar y)))),$$

where $r\geqslant 1$, $R_1(\bar x_1), \dots, R_p(\bar x_p)$ are normal $\boldsymbol{R}$-atomic formulas, $A_1, \dots, A_q$ are all $\boldsymbol{R}$-free formulas, and $Q_1(\bar s_1(\bar x, \bar y)), \dots, Q_r(\bar s_r(\bar x, \bar y))$ are all $\boldsymbol{R}$-atomic formulas which have at least one occurrence of free variables in $\bar y$ (this implies that $\bar y$ is not the empty sequence).

Simple existence conditions of $\boldsymbol{R}$ are existence conditions of $\boldsymbol{R}$ such that $p=q=0$ in the above form. $\forall x\exists yR(x, y)$ is a typical example of simple existence conditions of $\{R\}$. Suppose that X is a finite set of $\boldsymbol{R}$-atomic formulas. For each $R\in\boldsymbol{R}$, let $X(R)$ be the set of n-tuples $\langle t_1, \dots, t_n\rangle$ of terms such that $R(t_1, \dots, t_n)\in X$, where n is the number of arguments of R. Suppose that U is a uniqueness condition of $\boldsymbol{R}$ of the form

$$\forall\bar x_1\dots\forall\bar x_p(R_1(\bar x_1)\wedge\dots\wedge R_p(\bar x_p).\supset A(\bar x_1, \dots, \bar x_p)),$$

then $U[X]$ is the formula

$$\bigwedge_{\bar s_1\in X(R_1)}\dots\bigwedge_{\bar s_p\in X(R_p)} A(\bar s_1, \dots, \bar s_p).$$

Suppose that $\boldsymbol{U}$ is a finite set of uniqueness conditions of $\boldsymbol{R}$ and $\boldsymbol{E}$ is a finite set of existence conditions of $\boldsymbol{R}$. Then, the set of approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X, denoted by $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$, is defined by the following (1), (2), (3), and (4):

(1) $\bigwedge X\wedge\bigwedge_{A\in\boldsymbol{U}} A[X]\in\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$,

(2) If $A, B\in\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$, then $A\wedge B\in\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$.

(3) If $B_j(\bar a_0, \bar a_1, \dots, \bar a_p, \bar b)\in\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)\cup\{Q_j(\bar s_j(\bar a_0, \bar a_1, \dots, \bar a_p, \bar b))\}$, $j=1, \dots, r$ and

$$\forall\bar x_0\forall\bar x_1\dots\forall\bar x_p(\bigwedge_{i=1}^{p} R_i(\bar x_i)\wedge\bigwedge_{k=1}^{q} A_k(\bar x_0, \dots, \bar x_p).$$
$$\supset\exists\bar y\bigvee_{j=1}^{r} Q_j(\bar s_j(\bar x_0, \dots, \bar x_p, \bar y)))\in\boldsymbol{E},$$

then

$$\bigwedge_{\bar u_1\in X(R_1)}\dots\bigwedge_{\bar u_p\in X(R_p)}\forall\bar x_0(\bigwedge_{k=1}^{q} A_k(\bar x_0, \bar u_1, \dots, \bar u_p).$$
$$\supset\exists\bar y\bigvee_{j=1}^{r} B_j(\bar x_0, \bar u_1, \dots, \bar u_p, \bar y))\in\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X).$$

(4) Every element in $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$ is obtained from (1), (2), (3) above only.

Moreover, if $\boldsymbol{E}$ consists of simple existence conditions only, then the set of kth approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X, denoted by $\mathrm{Ap}^k(\boldsymbol{U}, \boldsymbol{E}, X)$, is defined by the following $(1)^0$ and $(2)^{k+1}$:

$(1)^0$ $$\mathrm{Ap}^0(\boldsymbol{U}, \boldsymbol{E}, X)=\{\bigwedge X\wedge\bigwedge_{A\in\boldsymbol{U}} A[X]\}.$$

$(2)^{k+1}$ $$\mathrm{Ap}^{k+1}(\boldsymbol{U}, \boldsymbol{E}, X)$$
$$=\{\forall\bar x\exists\bar y\bigvee_{j=1}^{r} A_j(\bar x, \bar y) \mid A_j(\bar a, \bar b)\in\mathrm{Ap}^k(\boldsymbol{U}, \boldsymbol{E}, X)\cup\{Q_j(\bar s_j(\bar a, \bar b))\},$$
$$j=1, 2, \dots, r, \text{ and } \forall\bar x\exists\bar y\bigvee_{j=1}^{r} Q_j(\bar s_j(\bar x, \bar y))\in\boldsymbol{E}\}.$$

Note that any formula in $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, X)$ or $\mathrm{Ap}^k(\boldsymbol{U}, \boldsymbol{E}, X)$ is $\boldsymbol{R}$-positive and has no occurrences of free variables which do not occur in any formula in X. Let $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E})=\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E}, \emptyset)$ and $\mathrm{Ap}^k(\boldsymbol{U}, \boldsymbol{E})=\mathrm{Ap}^k(\boldsymbol{U}, \boldsymbol{E}, \emptyset)$. Then, sentences in $\mathrm{Ap}(\boldsymbol{U}, \boldsymbol{E})$ or $\mathrm{Ap}^k(\boldsymbol{U}, \boldsymbol{E})$ are called approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ or simple approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$, respectively. Our main theorem is:

THEOREM A (Approximation theorem). *Suppose that A is an **R**-positive formula in L. Then, the formula $\bigwedge X \wedge \bigwedge \boldsymbol{U} \wedge \bigwedge \boldsymbol{E} . \supset A$ is provable in L if and only if $B \supset A$ is provable in L for some approximation of **U** by **E** over X. In particular, if **E** consists of simple existence conditions only, we can take a simple approximation of **U** by **E** over X in the above statement.*

Usually we use Theorem A in case that X is the empty set. We have many variations of Theorem A, which will be explained in § 1 below and many applications of Theorem A, which will be given in § 2 below. In § 3 below, we shall give a syntactical proof of Theorem A.

§ 1. Variations. In this section, we shall explain some variations of Theorem A in the introduction of this paper.

1. Uniqueness conditions. We can obtain a similar approximation theorem even if we admit the following type of sentences as uniqueness conditions of $\boldsymbol{R}$.

$$\forall \bar{x}(R_1(\bar{t}_1(\bar{x})) \wedge \ldots \wedge R_p(\bar{t}_p(\bar{x})) . \supset A(\bar{x})),$$

where $R_1, \ldots, R_p \in \boldsymbol{R}$ and A is an $\boldsymbol{R}$-free formula in L. In fact, this type of uniqueness conditions of $\boldsymbol{R}$ can be replaced by the following uniqueness conditions of $\boldsymbol{R}$:

$$\forall \bar{x}_1 \ldots \forall \bar{x}_p (R_1(\bar{x}_1) \wedge \ldots \wedge R_p(\bar{x}_p) . \supset \forall \bar{x}(\bar{x}_1 = \bar{t}_1(\bar{x}) \wedge \ldots \wedge \bar{x}_p = \bar{t}_p(\bar{x}) . \supset A(\bar{x})))$$

where

$$\bar{x}_i = \bar{t}_i(\bar{x}) \quad \text{is} \quad x_{i,1} = t_{i,1}(\bar{x}) \wedge x_{i,2} = t_{i,2}(\bar{x}) \wedge \ldots$$

2. Existence conditions. We can obtain a similar approximation theorem even if we admit the following type of sentences as existences conditions of $\boldsymbol{R}$.

$$\forall \bar{x}(R_1(\bar{t}_1(\bar{x})) \wedge \ldots \wedge R_p(\bar{t}_p(\bar{x})) \wedge A_1(\bar{x}) \wedge \ldots \wedge A_q(\bar{x}) . \supset \exists \bar{y}(Q_1(\bar{s}_1(\bar{x}, \bar{y})) \vee \ldots \vee Q_r(\bar{s}_r(\bar{x}, \bar{y})))),$$

where $r \geqslant 1$, $R_1 \in \boldsymbol{R}, \ldots, R_p \in \boldsymbol{R}$, $Q_1 \in \boldsymbol{R}, \ldots, Q_r \in \boldsymbol{R}$, $A_1, \ldots, A_q$ are all $\boldsymbol{R}$-free formulas and $Q_1(\bar{s}_1(\bar{x}, \bar{y})), \ldots, Q_r(\bar{s}_r(\bar{x}, \bar{y}))$ have at least one occurrence of one free variable in $\bar{y}$.

In fact, this type of existence conditions of $\boldsymbol{R}$ can be replaced by the following existence conditions of $\boldsymbol{R}$:

$$\forall \bar{x} \forall \bar{x}_1 \ldots \forall \bar{x}_p(R_1(\bar{x}_1) \wedge \ldots \wedge R_p(\bar{x}_p) \wedge A_1(\bar{x}) \wedge \ldots \wedge A_q(\bar{x}) \wedge \wedge \bar{x}_1 = \bar{t}_1(\bar{x}) \wedge \ldots \wedge \bar{x}_p = \bar{t}_p(\bar{x}) . \supset \exists \bar{y}(Q_1(\bar{s}_1(\bar{x}, \bar{y})) \vee \ldots \vee Q_r(\bar{s}_r(\bar{x}, \bar{y})))).$$

3. $\boldsymbol{R}$-free approximations. The definition of the set of $\boldsymbol{R}$-free approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X is obtained from that of approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X by deleting $\bigwedge X$ in (1). Similarly, $\boldsymbol{R}$-free simple approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X are obtained. Then, we have the following theorem.

THEOREM B ($\boldsymbol{R}$-free approximation theorem). *Suppose that A is an **R**-free formula in L. Then, the formula $\bigwedge X \wedge \bigwedge \boldsymbol{U} \wedge \bigwedge \boldsymbol{E} . \supset A$ is provable in L if and only if $B \supset A$ is provable in L for some **R**-free approximation B of **U** by **E** over X. In particular, if **E** consists of simple existence conditions of **R**, then we can take a simple **R**-free approximation B of **U** by **E** over X in the above statement.*

This theorem is an immediate consequence of Theorem A and the following lemma.

POSITIVE LEMMA. *Suppose that P, Q are n-ary predicate symbols, and $B(P)$ is a P-positive (P-negative) formula. Then, the formula*

$$\forall \bar{x}(P(\bar{x}) \supset Q(\bar{x})) \wedge B(P) . \supset B(Q) \qquad (\forall \bar{x}(P(\bar{x}) \supset Q(\bar{x})) \wedge B(Q) . \supset B(P))$$

is provable in L, where $B(Q)$ is obtained from B by replacing some occurrences of P of the form $P(\bar{t})$ by $Q(\bar{t})$.

4. Parametrical versions. Let $\bar{c} = \langle c_1, \ldots, c_N \rangle$ be a fixed sequence of distinct free variables of length N, and $\boldsymbol{R}$ a set of predicate symbols such that each predicate symbol in $\boldsymbol{R}$ has the number of arguments, $n+N$, where $n > 0$. $\bar{c}$-formulas are formulas which have no free variables except those in $\bar{c}$. $\boldsymbol{R}$-atomic formulas with parameters $\bar{c}$ are $\boldsymbol{R}$-atomic formulas of the form $R(t_1, \ldots, t_n, \bar{c})$, and normal $\boldsymbol{R}$-atomic formulas with parameters $\bar{c}$ are normal $\boldsymbol{R}$-atomic formulas of the form $R(a_1, \ldots, a_n, \bar{t})$. By replacing "$\boldsymbol{R}$-atomic formulas" and "normal $\boldsymbol{R}$-atomic formulas" by "$\boldsymbol{R}$-atomic formulas with parameters $\bar{c}$" and "normal $\boldsymbol{R}$-atomic formulas with parameters $\bar{c}$" respectively, in the definitions of uniqueness conditions of $\boldsymbol{R}$, existence conditions of $\boldsymbol{R}$, and approximations of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X, we have parametrical versions of these notions. For example, uniqueness conditions of $\boldsymbol{R}$ with parameters $\bar{c}$ are $\bar{c}$-formulas in L of the form:

$$\forall \bar{x}_1 \forall \bar{x}_2 \ldots \forall \bar{x}_p(\bigwedge_{i=1}^{p} R_i(\bar{x}_i, \bar{c}) . \supset A(\bar{x}_1, \ldots, \bar{x}_p, \bar{c})),$$

where $R_i(\bar{x}_i, \bar{c})$, $i = 1, \ldots, p$, are normal $\boldsymbol{R}$-atomic formulas with parameters $\bar{c}$ and $A(\bar{x}_1, \ldots, \bar{x}_p, \bar{c})$ is an $\boldsymbol{R}$-free formula. Suppose that $U(\bar{c})$ is a finite set of uniqueness conditions of $\boldsymbol{R}$ with parameters $\bar{c}$, $\boldsymbol{E}(\bar{c})$ is a finite set of existence conditions of $\boldsymbol{R}$ with parameters $\bar{c}$, and $X(\bar{c})$ is a finite set of R-atomic formulas with parameters $\bar{c}$.

THEOREM C (Parametrical approximation theorem). *Suppose that $A(\bar{c})$ is an **R**-free, $\bar{c}$-formula and B is an **R**-positive formula in L. Then, the formula*

$$\forall \bar{x}(A(\bar{x}) \supset . \bigwedge X(\bar{x}) \wedge \bigwedge \boldsymbol{U}(\bar{x}) \wedge \bigwedge \boldsymbol{E}(\bar{x})) \supset B$$

is provable in L if and only if the formula $\forall \bar{x}(A(\bar{x}) \supset C(\bar{x})) \supset B$ is provable in L for some approximation $C(\bar{c})$ of $\boldsymbol{U}(\bar{c})$ by $\boldsymbol{E}(\bar{c})$ over $X(\bar{c})$ with parameters $\bar{c}$.

In particular, if E consists of simple existence conditions with parameters $\bar{c}$ only, then we can take a simple approximation $C(\bar{c})$ of $U(\bar{c})$ by $E(\bar{c})$ over $X(\bar{c})$ in the above statement.

As a corollary of Theorem C, we have the following fact (cf. Theorem 0.1 in [2]).

Let P be an N-ary predicate symbol which does not belong to R. Suppose that $A(P)$ is a $\{P\}$-negative and R-positive sentence in L. For each $\bar{c}$-formula $B(\bar{c})$, let $A(B)$ be the formula obtained from $A(P)$ by replacing every occurrence $P(\bar{t})$ of P in $A(P)$ by $B(\bar{t})$.

COROLLARY D. *$A(\bigwedge U \wedge \bigwedge E)$ is provable in L if and only if $A(B)$ is provable in L for some approximation $B(\bar{c})$ of $U(\bar{c})$ by $E(\bar{c})$, with parameters $\bar{c}$.*

Proof. We can assume that the predicate symbol P occurs in none of formulas in $U(\bar{c}) \cup E(\bar{c})$. Then,

$A(\bigwedge U \wedge \bigwedge E)$ is provable in L

$\Leftrightarrow \forall \bar{x}(P(\bar{x}) \supset . \bigwedge U(\bar{x}) \wedge \bigwedge E(\bar{x})) \supset A(P)$ is provable in L

(by Positive Lemma)

$\Leftrightarrow \forall \bar{x}(P(\bar{x}) \supset B(\bar{x})) \supset A(P)$ is provable in L for some approximation $B(\bar{c})$ of $U(\bar{c})$ by $E(\bar{c})$ with parameters $\bar{c}$

(by Theorem C)

$\Leftrightarrow A(B)$ is provable in L for some approximation $B(\bar{c})$ of $U(\bar{c})$ by $E(\bar{c})$ with parameters $\bar{c}$

(by Positive Lemma). ∎

5. Infinitary versions. We can generalize almost all the notions mentioned in the introduction of this paper into the infinitary logic $L_{\omega_1\omega}$ (cf. Keisler [3]), by replacing "formulas in L" by "formulas in $L_{\omega_1\omega}$". But some natural modifications are necessary. For example, (2) in the definition of $\mathrm{Ap}(U, E, X)$ should be replaced by the following $(2)^{\infty}$:

$(2)^{\infty}$ If K is a countable set of approximations of U by E over X, then $\bigwedge K$ is an approximation of U by E over X.

Also, the definition of kth approximations can be extended to αth approximations ($\alpha < \omega_1$) by adding the following $(2)^{\sigma}$, where σ is a limit countable ordinal number

$$(2)^{\sigma} \qquad \mathrm{Ap}^{\sigma}(U, E, X) = \{\bigwedge_{\alpha<\sigma} A_{\alpha} |\ A_{\alpha} \in \mathrm{Ap}^{\alpha}(U, E, X),\ \alpha < \sigma\}.$$

§ 2. Applications. In this section, we shall show some applications of our approximation theorem.

1. Axiomatization. Theorem B shows us that the set of R-free approximations of U by E is an axiomatization of the set of first order formulas which

are provable from the second order sentence $\exists R(\bigwedge U \wedge \bigwedge E)$. As for the classical predicate calculus, this fact means that we always obtain an axiomatization of the set of first order formulas which are provable from an arbitrary given second order sentence of the form $\exists R(A(R))$, where $A(R)$ is a sentence in L, because there is a finite set R' of new predicate symbols, a finite set U of uniqueness conditions of $R \cup R'$, and a finite set E of simple existence conditions of $R \cup R'$ such that $\exists R(A(R))$ and $\exists R \cup R'(\bigwedge U \wedge \bigwedge E)$ are equivalent (see Motohashi [8] for details).

2. Elimination theorems of uniqueness conditions. Let R be an $(n+1)$-ary predicate symbol in L. By $\mathrm{Un}\, R$, we denote the sentence

$$\forall \bar{x} \forall \bar{y} \forall x \forall y (R(\bar{x}, x) \wedge R(\bar{y}, y) . \supset (\bar{x} = \bar{y} \supset x = y)),$$

where $\bar{x} = \bar{y}$ is the formula $x_1 = y_1 \wedge \ldots \wedge x_n = y_n$. Clearly, $\mathrm{Un}R$ is a uniqueness condition of $\{R\}$. Let $\mathrm{Ex}\, R$ be the sentence: $\forall \bar{x} \exists y R(\bar{x}, y)$, clearly, $\mathrm{Ex}\, R$ is a simple existence condition of $\{R\}$. Let $\mathrm{Ex}^k R$ be the sentence

$$\forall \bar{x}_1 \exists y_1 \forall \bar{x}_2 \exists y_2 \ldots \forall \bar{x}_k \exists y_k (\bigwedge_{i=1}^{k} R(\bar{x}_i, y_i) \wedge \bigwedge_{i,j=1}^{k} (\bar{x}_i = \bar{x}_j \supset y_i = y_j)).$$

Then, $\mathrm{Ap}^k(\{\mathrm{Un}\, R\}, \{\mathrm{Ex}\, R\}) = \{\mathrm{Ex}^k R\}$. So, by Theorem A we have:

THEOREM E (Theorem B in [7]). *For any $\{R\}$-positive formula A, the formula $\mathrm{Ex}\, R \wedge \mathrm{Un}\, R . \supset A$ is provable in L if and only if the formula $\mathrm{Ex}^k R \supset A$ is provable in L for some $k < \omega$.*

Since sentences $\mathrm{Ex}\, R \supset \mathrm{Ex}^k R$ $(k = 0, 1, 2, \ldots)$ are all provable in LK, we have:

COROLLARY F (Theorem I in [6]). *For any $\{R\}$-positive formula A, $\mathrm{Ex}\, R \wedge \mathrm{Un}\, R . \supset A$ is provable in LK if and only if $\mathrm{Ex}\, R \supset A$ is provable in LK.*

Similarly, Theorem II, Theorem III in [6], and Main Theorem in [7] are proved by Theorem A above. This fact shows us that Elimination theorems of uniqueness conditions are immediate consequences of our approximation theorem of uniqueness conditions.

3. A generalization of Skolem's Theorem and Minc's Theorem. Suppose that $R_1, \ldots, R_N$ are distinct predicate symbols in L such that the number of arguments of R_i is $n_i + 1$ for each $i = 1, \ldots, N$. For each $i = 1, \ldots, N$, let $\mathrm{Un}\, R_i$ be the uniqueness condition

$$\forall \bar{x}_i \forall \bar{y}_i \forall x \forall y (R_i(\bar{x}_i, x) \wedge R_i(\bar{y}_i, y) . \supset (\bar{x}_i = \bar{y}_i \supset x = y))$$

and $\mathrm{Ex}\, R_i$ be the simple existence condition $\forall \bar{x}_i \exists y R_i(\bar{x}_i, y)$. Also we consider the following uniqueness condition of $\{R_1, \ldots, R_N\}$, which will be denoted by $\mathrm{Un}(R_1, \ldots, R_N; A)$.

$$\forall \bar{x}_1 \forall y_1 \forall \bar{x}_2 \forall y_2 \ldots \forall \bar{x}_N \forall y_N (\bigwedge_{i=1}^{N} R_i(\bar{x}_i, y_i) . \supset A(\bar{x}_1, y_1, \ldots, \bar{x}_N, y_N)).$$

By $A^{=}[k]$ and $A[k]$, we shall denote the following sentences, respectively;

$$\forall \bar{x}_1^1 \exists y_1^1 \forall \bar{x}_2^1 \exists y_2^1 \ldots \forall \bar{x}_N^1 \exists y_N^1 \forall \bar{x}_1^2 \exists y_1^2 \ldots \forall \bar{x}_N^2 \exists y_N^2 \ldots \forall \bar{x}_1^k \exists y_1^k \ldots \forall \bar{x}_N^k \exists y_N^k$$
$$\Big[\bigwedge_{1 \leqslant i_1,\ldots,i_N \leqslant k} A(\bar{x}_1^{i_1}, y_1^{i_1}, \ldots, \bar{x}_N^{i_N}, y_N^{i_N}) \wedge \bigwedge_{1 \leqslant i,j \leqslant k} \bigwedge_{1 \leqslant s \leqslant k} (\bar{x}_s^i = \bar{x}_s^j \supset y_s^i = y_s^j)\Big]$$

and

$$\forall \bar{x}_1^1 \exists y_1^1 \ldots \forall \bar{x}_N^1 \exists y_N^1 \ldots \forall \bar{x}_1^k \exists y_1^k \ldots$$
$$\ldots \forall \bar{x}_N^k \exists y_N^k \Big[\bigwedge_{1 \leqslant i_1,\ldots,i_N \leqslant k} A(\bar{x}_1^{i_1}, y_1^{i_1}, \ldots, \bar{x}_N^{i_N}, y_N^{i_N})\Big].$$

Then $A^{=}(k)$, $k = 1, 2, \ldots$, are all simple approximations of $\{\mathrm{Un}\,R_1, \ldots, \mathrm{Un}\,R_N, \mathrm{Un}(R_1, \ldots, R_N;\, A)\}$ by $\{\mathrm{Ex}\,R_1, \ldots, \mathrm{Ex}\,R_N\}$, and $A(k)$, $k = 1, 2, \ldots$, are all simple approximations of $\{\mathrm{Un}(R_1, \ldots, R_N;\, A)\}$ by $\{\mathrm{Ex}\,R_1, \ldots, \mathrm{Ex}\,R_N\}$, respectively. Moreover, we can easily see that for any simple approximation B of $\{\mathrm{Un}\,R_1, \ldots, \mathrm{Un}\,R_n, \mathrm{Un}(R_1, \ldots, R_N;\, A)\}$ (or $\{\mathrm{Un}(R_1, \ldots, R_N;\, A)\}$) by $\{\mathrm{Ex}\,R_1, \ldots, \mathrm{Ex}\,R_N\}$, there is a k such that $A^{=}(k) \supset B$ (or $A(k) \supset B$) is provable in L. From these facts and Theorem B, we have:

COROLLARY G. *For any $\{R_1, \ldots, R_N\}$-free formula B in L, the formula*

$$\bigwedge_{i=1}^{N} (\mathrm{Ex}\,R_i \wedge \mathrm{Un}\,R_i) \wedge \mathrm{Un}(R_1, \ldots, R_N;\, A) \cdot \supset B$$
$$\Big(\text{or } \bigwedge_{i=1}^{N} (\mathrm{Ex}\,R_i) \wedge \mathrm{Un}(R_1, \ldots, R_N;\, A) . \supset B\Big)$$

is provable in L if and only if the formula $A^{=}(k) \supset B$ (or $A(k) \supset B$) is provable in L for some k.

Let $f_1, \ldots, f_N$ be distinct function symbols in L such that none of them occur in A and the number of arguments of f_i is n_i for each $i = 1, \ldots, N$. By replacing every occurrence of formulas of the form $R_i(\bar{t}, s)$ by "$f_i(\bar{t}) = s$" in Corollary G, we have

COROLLARY H. *For any formula B in L, which has no occurrences of $f_1, \ldots, f_N$, the formula*

$$\forall \bar{x}_1 \ldots \forall \bar{x}_N A(\bar{x}_1, f_1(\bar{x}_1), \ldots, \bar{x}_N, f_N(\bar{x}_N)) \supset B$$

is provable in L if and only if the formula $A^{=}(k) \supset B$ is provable in L for some k.

By Elimination theorem of uniqueness conditions and Corollary G, we have

COROLLARY I. *For any formula B in L, which has no occurrences of $f_1, \ldots, f_N$, the formula*

$$\forall \bar{x}_1 \ldots \forall \bar{x}_N A(\bar{x}_1, f_1(\bar{x}_1), \ldots, \bar{x}_N, f_N(\bar{x}_N)) \supset B$$

is provable in LK if and only if the formula $A(k) \supset B$ is provable in LK for some k.

If $N = 1$, then Corollary H implies Minc's theorem on Skolem functions in LJ (cf. [10], [11]), and Corollary I implies Skolem's theorem on choice functions (cf. Leisenring [4]).

4. Henkin quantifiers and Barwise's approximations. Suppose that $A(\bar{a}_1, a_1, \bar{a}_2, a_2, \ldots, \bar{a}_N, a_N)$ is a formula in L. Then, a Henkin formula H, obtained from A by applying the Henkin quantifier

$$\begin{pmatrix} \forall \bar{x}_1 \exists y_1 \\ \forall \bar{x}_2 \exists y_2 \\ \cdots\cdots \\ \forall \bar{x}_N \exists y_N \end{pmatrix}$$

is the formula

$$\begin{pmatrix} \forall \bar{x}_1 \exists y_1 \\ \forall \bar{x}_2 \exists y_2 \\ \cdots\cdots\cdots \\ \forall \bar{x}_N \exists y_N \end{pmatrix} A(\bar{x}_1, y_1, \bar{x}_2, y_2, \ldots, \bar{x}_N, y_N).$$

For the sake of simplicity, we assume that H is a sentence, i.e. H has no free variables. In [1], J. Barwise defined the kth approximation $H[k]$ of H by the following: $H[1]$ is the sentence,

$$\forall \bar{x}_1^1 \ldots \forall \bar{x}_N^1 \exists y_1^1 \ldots \exists y_N^1 A(\bar{x}_1^1, y_1^1, \bar{x}_2^1, y_2^1, \ldots, \bar{x}_N^1, y_N^1),$$

$H[2]$ is the sentence,

$$\forall \bar{x}_1^1 \ldots \forall \bar{x}_N^1 \exists y_1^1 \ldots \exists y_N^1 \forall \bar{x}_1^2 \ldots \forall \bar{x}_N^2 \exists y_1^2 \ldots \exists y_N^2$$
$$\Big(A(\bar{x}_1^1, y_1^1, \ldots, \bar{x}_N^1, y_N^1) \wedge A(\bar{x}_1^2, y_1^2, \ldots, \bar{x}_N^2, y_N^2) \wedge \bigwedge_{j=1}^{N} (\bar{x}_j^1 = \bar{x}_j^2 \supset y_j^1 = y_j^2)\Big),$$

$H[k]$ is the sentence,

$$\forall \bar{x}_1^1 \ldots \forall \bar{x}_N^1 \exists y_1^1 \ldots \exists y_N^1 \ldots \forall \bar{x}_1^k \ldots \forall \bar{x}_N^k \exists y_1^k \ldots \exists y_N^k$$
$$\Big(\bigwedge_{i=1}^{k} A(\bar{x}_1^i, y_1^i, \ldots, \bar{x}_N^i, y_N^i) \wedge \bigwedge_{1 \leqslant i,s \leqslant k} \bigwedge_{1 \leqslant j \leqslant N} (\bar{x}_j^i = \bar{x}_j^s \supset y_j^i = y_j^s)\Big).$$

Then, Barwise obtained the following theorem by using resplendent models.

THEOREM (Barwise). *For any first order sentence B, the sentence $H \supset B$ is valid if and only if the sentence $H[k] \supset B$ is provable in LK for some $k < \omega$.*

For each $i = 1, \ldots, N$, let n_i be the length of $\bar{a}_i$. Let R be a new $(n_1 + \ldots + n_N + N)$-ary predicate symbol, $\mathrm{Ex}^*\,R$ the sentence

$$\forall \bar{x}_1 \forall \bar{x}_2 \ldots \forall \bar{x}_N \exists y_1 \exists y_2 \ldots \exists y_N R(\bar{x}_1, y_1, \ldots, \bar{x}_N, y_N),$$

$\mathrm{Un}^* R$ the sentence

$$\forall \bar{x}_1 \forall \bar{x}'_1 \forall y_1 \forall y'_1 \ldots \forall \bar{x}_N \forall \bar{x}'_N \forall y_N \forall y'_N [R(\bar{x}_1, y_1, \ldots, \bar{x}_N, y_N) \wedge \wedge R(\bar{x}'_1, y'_1, \ldots, \bar{x}'_N, y'_N). \supset (\bigwedge_{i=1}^{N} (\bar{x}_i = \bar{x}'_i \supset y_i = y'_i))]$$

and $\mathrm{Un}(R:A)$ the sentence

$$\forall \bar{x}_1 \forall y_1 \ldots \forall \bar{x}_N \forall y_N (R(\bar{x}_1, y_1, \ldots, \bar{x}_N, y_N) \supset A(\bar{x}_1, y_1, \ldots, \bar{x}_N, y_N)).$$

Then, $\mathrm{Un}^* R$, $\mathrm{Un}(R:A)$ are uniqueness conditions of $\{R\}$ and $\mathrm{Ex}^* R$ is a simple existence condition of $\{R\}$. Also, the set of simple $\{R\}$-free approximations of $\{\mathrm{Un}^* R, \mathrm{Un}(R:A)\}$ by $\{\mathrm{Ex}^* R\}$ is exactly the set $\{H[k] \mid k = 1, 2, \ldots\}$. On the other hand, it is obvious that $H \supset B$ is valid if and only if $\mathrm{Un}^* R \wedge \mathrm{Un}(R:A) \wedge \mathrm{Ex}^* R. \supset B$ is provable in LK, for any $\{R\}$-free sentence B. Therefore, Theorem B clearly implies Barwise's approximation theorem (cf. [9] for details).

5. Vaught sentences. A Vaught formula $V(\bar{z})$ is a formula of the form

$$\forall x_0 \bigwedge_{k_0 \in \omega} \exists y_0 \bigvee_{l_0 \in \omega} \forall x_1 \bigwedge_{k_1 \in \omega} \exists y_1 \bigvee_{l_1 \in \omega} \ldots \ldots (\bigwedge_{n<\omega} A^{\langle k_0, l_0, \ldots, k_{n-1}, l_{n-1} \rangle}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, \bar{z})),$$

where $A^{\langle k_0, l_0, \ldots, k_{n-1}, l_{n-1} \rangle}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}, \bar{z})$ is a formula in $L_{\omega_1 \omega}$ for each finite sequence $\langle k_0, l_0, \ldots, k_{n-1}, l_{n-1} \rangle$ of natural numbers of length $2n$ (see Harnik–Makkai [2] for details).

For the sake of simplicity, we assume that V is a sentence, i.e. the length of $\bar{z}$ is 0. If readers want to treat the case that $\bar{z}$ is not the empty sequence, please use "parametrical approximation theorems" instead of "approximation theorems" in the following of this section. For each $\alpha < \omega_1$ and each finite sequence $\sigma = \langle k_0, l_0, \ldots, k_{n-1}, l_{n-1} \rangle$ of natural numbers of length $2n$, let $V^\sigma_\alpha(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$ be the formula in $L_{\omega_1 \omega}$ defined by:

$$V^\sigma_0(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) = \bigwedge_{j \leq n} A^{\sigma \restriction 2j}(x_0, y_0, \ldots, x_{j-1}, y_{j-1}),$$

$$V^\sigma_{\alpha+1}(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) = \forall x_n \bigwedge_{k_n \in \omega} \exists y_n \bigvee_{l_n \in \omega} V^{\sigma \frown \langle k_n, l_n \rangle}_\alpha (x_0, y_0, \ldots, x_n, y_n),$$

$$V^\sigma_\alpha(x_0, y_0, \ldots, x_{n-1}, y_{n-1}) = \bigwedge_{\beta < \alpha} V^\sigma_\beta(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$$

if α is a limit ordinal, where $\sigma \restriction 2j$ is the sequence $\langle k_0, l_0, \ldots, k_{j-1} l_{j-1} \rangle$ and $\sigma \frown \langle k_n, l_n \rangle$ is the sequence $\langle k_0, l_0, \ldots, k_{n-1}, l_{n-1}, k_n, l_n \rangle$. Let $V_\alpha = V^\emptyset_\alpha$, where $\emptyset$ is the empty sequence. In Vaught [13], he proved the following approximation theorem.

THEOREM (Vaught). *Suppose that B is a sentence in $L_{\omega_1 \omega}$. Then $V \supset B$ is valid if and only if $V_\alpha \supset B$ is provable in $L_{\omega_1 \omega}$ for some $\alpha < \omega_1$.*

Now, we shall show another approximation theorem of V by using one of our approximation theorems. Let us introduce a new $(n+1)$-ary predicate symbol R^σ for each sequence σ of natural numbers of length $2n$. Let $\boldsymbol{R} = \{R^\sigma \mid \sigma \in \omega^{2n}, n < \omega\}$.

Let U^σ be the sentence

$$\forall x_0 \forall y_0 \ldots \forall x_{n-1} \forall y_{n-1} (\bigwedge_{j \leq n} R^{\sigma \restriction 2j}(x_0, x_1, \ldots, x_{j-1}, y_{j-1}). \supset \bigwedge_{j \leq n} A^{\sigma \restriction 2j}(x_0, y_0, \ldots, x_{j-1}, y_{j-1})),$$

and $E^{\sigma,k}$ the sentence

$$\forall x_0 \forall x_1 \ldots \forall x_n \exists y_n \bigvee_{l \in \omega} R^{\sigma \frown \langle k, l \rangle}(x_0, x_1, \ldots, x_n, y_n),$$

for each $\sigma \in \omega^{2n}$, $k < \omega$.

Let $\boldsymbol{U} = \{U^\sigma \mid \sigma \in \omega^{2n}, n < \omega\}$ and $\boldsymbol{E} = \{E^{\sigma,k} \mid \sigma \in \omega^{2n}, k < \omega, n < \omega\}$. Then, the sentence $V \supset B$ is valid if and only if the sentence $\bigwedge \boldsymbol{U} \wedge \bigwedge \boldsymbol{E} . \supset B$ is provable in $L_{\omega_1 \omega}$, for each $\boldsymbol{R}$-free sentence B in $L_{\omega_1 \omega}$. Hence, we have

THEOREM J. *Suppose that B is a sentence in $L_{\omega_1 \omega}$. Then, $V \supset B$ is valid if and only if $C \supset B$ is provable in $L_{\omega_1 \omega}$, for some $\boldsymbol{R}$-free approximation C of $\boldsymbol{U}$ by $\boldsymbol{E}$.*

Also, by Theorem C, we have an analogous result of Theorem 0.1* in [2].

6. Homomorphism. Let $\mathscr{A}$ be a countable L-structure and $\boldsymbol{R} = \{R_a \mid a \in |\mathscr{A}|\}$ be a set of new binary predicate symbols such that each R_a does not belong to L and R_a and $R_{a'}$ are distinct from each other if $a \neq a'$. Let

$$\boldsymbol{U} = \{\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (R_{a_1}(x_1, y_1) \wedge \ldots \wedge R_{a_n}(x_n, y_n). \supset A(x_1, \ldots, x_n)) \mid \mathscr{A} \models A[a_1, \ldots, a_n], A \text{ is a positive formula in } L \text{ and } a_1, \ldots, a_n \in |\mathscr{A}|\},$$

and

$$\boldsymbol{E} = \{\forall x \exists y \bigvee_{a \in |\mathscr{A}|} R_a(x, y), \forall y \exists x R_a(x, y) \mid a \in |\mathscr{A}|\}.$$

Then, $\boldsymbol{U}$ is a countable set of uniqueness conditions of $\boldsymbol{R}$, and $\boldsymbol{E}$ is a countable set of simple existence conditions of $\boldsymbol{R}$ in the logic $L_{\omega_1 \omega}$. Moreover, $\mathscr{B}$ is a model of $\exists \boldsymbol{R}(\bigwedge \boldsymbol{U} \wedge \bigwedge \boldsymbol{E})$ if and only if $\mathscr{B}$ is a homomorphic image of $\mathscr{A}$, for any L-structure $\mathscr{B}$. Therefore, we have: for any sentence A in $L_{\omega_1 \omega}$,

A holds in any homomorphic image of $\mathscr{A}$
$\Leftrightarrow \bigwedge \boldsymbol{U} \wedge \bigwedge \boldsymbol{E} . \supset A$ is provable in $L_{\omega_1 \omega}$

$\Leftrightarrow B \supset A$ is provable in $L_{\omega_1 \omega}$ for some $\boldsymbol{R}$-free simple approximation B of $\boldsymbol{U}$ by $\boldsymbol{E}$.

Since every simple $\boldsymbol{R}$-free approximation of $\boldsymbol{U}$ by $\boldsymbol{E}$ is positive, this fact implies Theorem 1.1 in [2].

§ 3. A proof. In this section, we shall give a proof of Theorem A. At first, we have to introduce some auxiliary systems.

Let $\boldsymbol{R}$ be a fixed set of predicate symbols. We assume that the logic L is formulated in the usual Gentzen style with a slight modification that every sequent in L is a pair (Γ, Θ), denoted by $\Gamma \to \Theta$, of finite sets of formulas. If $L = LJ$, then we assume that Θ has at most one formula, (cf. Takeuti [12]). $\boldsymbol{R}$-equality axiom sequents are sequents of the form

$$t_1 = s_1, \ \ldots, \ t_n = s_n, \ R(t_1, \ldots, t_n) \to R(s_1, \ldots, s_n) \qquad (R \in \boldsymbol{R}).$$

Let L_1 be the system obtained from L by deleting every $\boldsymbol{R}$-equality axiom sequent. For each formula A in L, let $A^=$ be the formula obtained from A by replacing every occurrence $R(\bar{t})$ of $\boldsymbol{R}$-atomic subformulas by $\exists \bar{v}(\bar{v} = \bar{t} \wedge R(\bar{v}))$.

LEMMA K. *Suppose that A is a formula in L, X is a finite set of $\boldsymbol{R}$-atomic formulas, $\boldsymbol{U}$ is a finite set of uniqueness conditions of $\boldsymbol{R}$, and $\boldsymbol{E}$ is a finite set of existence conditions of $\boldsymbol{R}$. If the sequent $X, \boldsymbol{U}, \boldsymbol{E} \to A$ is provable in L, then the sequent $X, \boldsymbol{U}, \boldsymbol{E} \to A^=$ is provable in L_1.*

Proof. For each set K of formulas in L, let $K^= = \{B^= | \ B \in K\}$. Then we can easily see that if a sequent $\Gamma \to \Theta$ is provable in L, then the sequent $\Gamma^= \to \Theta^=$ is provable in L_1. Assume that the sequent $X, \boldsymbol{U}, \boldsymbol{E} \to A$ is provable in L. Then, the sequent $X^=, \boldsymbol{U}^=, \boldsymbol{E}^= \to A^=$ is provable in L_1. But, the sequent $B \to B^=$ is provable in L_1 for each formula B in $X \cup \boldsymbol{U} \cup \boldsymbol{E}$. Hence, we have that the sequent $X, \boldsymbol{U}, \boldsymbol{E} \to A^=$ is provable in L_1. This completes our proof of Lemma K. ∎

Let L_2 be the system obtained from L_1 by deleting every cut-rule whose cut-formula has at least one occurrence of predicate symbols in $\boldsymbol{R}$. Then, by the usual method of cut-elimination, we have

LEMMA L. *If a sequent $\Gamma \to \Theta$ is provable in L_1, then this sequent is provable in L_2.*

Suppose that B is a uniqueness condition of $\boldsymbol{R}$ of the form

$$\forall \bar{x}_1 \ldots \forall \bar{x}_p (\bigwedge_{i=1}^{p} R_i(\bar{x}_i) . \supset A(\bar{x}_1, \ldots, \bar{x}_p)).$$

Then, (B)-rules are inference rules of the form

$$(B) \frac{\Gamma \to \Theta_1, R_1(\bar{t}_1); \ \Gamma \to \Theta_2, R_2(\bar{t}_2); \ldots; \Gamma \to \Theta_p, R_p(\bar{t}_p); \ A(\bar{t}_1, \ldots, \bar{t}_p), \Gamma \to \Theta}{\Gamma \to \Theta_1, \ldots, \Theta_p, \Theta}$$

and $(B)^*$-rules are inference rules of the form

$$(B^*) \frac{A(\bar{t}_1, \ldots, \bar{t}_p), R_1(\bar{t}_1), \ldots, R_p(\bar{t}_p), \Gamma \to \Theta}{R_1(\bar{t}_1), \ldots, R_p(\bar{t}_p), \Gamma \to \Theta}.$$

Suppose that B is an existence condition of $\boldsymbol{R}$ of the form

$$\forall \bar{x}_0 \forall \bar{x}_1 \ldots \forall \bar{x}_p (\bigwedge_{i=1}^{p} R_i(\bar{x}_i) \wedge \bigwedge_{j=1}^{q} A_j(\bar{x}_0, \ldots, \bar{x}_p) . \supset \exists \bar{y} \bigvee_{k=1}^{r} Q_k(\bar{s}_k(\bar{x}_0, \ldots, \bar{x}_p, \bar{y}))).$$

Then, (B)-rules are inference rules of the form

$$(B) \frac{\Gamma \to \Theta_1, R_1(\bar{t}_1); \ldots; \Gamma \to \Theta_p, R_p(\bar{t}_p); \ \Gamma \to \Theta_{p+1}, \bigwedge_{j=1}^{q} A_j(\bar{x}_0, \ldots, \bar{x}_p); \quad \dfrac{Q_1(\bar{s}_1(\bar{t}_0, \ldots, \bar{t}_p, \bar{a})), \Gamma \to \Theta; \ \ldots; \ Q_r(\bar{s}_r(\bar{t}_0, \ldots, \bar{t}_p, \bar{a})), \Gamma \to \Theta}{\Gamma \to \Theta_1, \ldots, \Theta_{p+1}, \Theta}}{}$$

where $\bar{a}$ is a sequence of distinct free variables which do not occur in $\bar{t}_0, \ldots, \bar{t}_p, \bar{s}_1, \ldots, \bar{s}_r, A_1, \ldots, A_q, \Gamma, \Theta$, and $(B)^*$-rules are inference rules of the form

$$(B^*) \quad \frac{Q_1(\bar{s}_1(\bar{t}_0, \ldots, \bar{t}_p, \bar{a})),}{R_1(\bar{t}_1), \ldots, R_p(\bar{t}_p), A_1(\bar{t}_0, \ldots, \bar{t}_p), \ldots, A_q(\bar{t}_0, \ldots, \bar{t}_p), \Gamma \to \Theta; \ \ldots;}$$

$$\frac{Q_r(\bar{s}_r(\bar{t}_0, \ldots, \bar{t}_p, \bar{a})), R_1(\bar{t}_1), \ldots}{\dfrac{\ldots, R_p(\bar{t}_p), A_1(\bar{t}_0, \ldots, \bar{t}_p), \ldots, A_q(\bar{t}_0, \ldots, \bar{t}_p), \Gamma \to \Theta}{R_1(\bar{t}_1), \ldots, R_p(\bar{t}_p), A_1(\bar{t}_0, \ldots, \bar{t}_p), \ldots, A_q(\bar{t}_0, \ldots, \bar{t}_p), \Gamma \to \Theta}}$$

where $\bar{a}$ is sequence of distinct free variables which do not occur in $\bar{t}_0, \ldots, \bar{t}_p, \bar{s}_1, \ldots, \bar{s}_r, A_1, \ldots, A_q, \Gamma, \Theta$.

If A is a uniqueness condition of $\boldsymbol{R}$ or an existence condition of $\boldsymbol{R}$, then $S(A)$ is the set of subformulas of A which are obtained from A by deleting some outermost occurrences of quantifiers. If K is a set of uniqueness conditions and existence conditions of $\boldsymbol{R}$, then $S(K)$ is the set $\bigcup_{A \in K} S(A)$.

A sequent is said to be $\boldsymbol{R}$-*positive* if every formula in Γ is $\boldsymbol{R}$-negative and every formula in Θ is $\boldsymbol{R}$-positive. $\boldsymbol{R}$-sequents are sequents of the form: $\Gamma_0, \Gamma \to \Theta$, where Γ_0 is a finite set of $\boldsymbol{R}$-atomic formulas and $\Gamma \to \Theta$ is $\boldsymbol{R}$-positive. Suppose that $\boldsymbol{U}$ is a finite set of uniqueness condition of $\boldsymbol{R}$ and $\boldsymbol{E}$ is a finite set of existence conditions of $\boldsymbol{R}$. Let $L_3(\boldsymbol{U}, \boldsymbol{E})$ be the system obtained from

L_2 by adding every (B)-rule for each $B \in \boldsymbol{U} \cup \boldsymbol{E}$, and $L_4(\boldsymbol{U}, \boldsymbol{E})$ the system obtained from L_2 by adding every $(B)^*$-rule for each $B \in \boldsymbol{U} \cup \boldsymbol{E}$. Then, the following three lemmas are easily proved by induction on derivations. So, we omit them.

LEMMA M. *Every sequent in a derivation of an R-sequent in* $L_i(\boldsymbol{U}, \boldsymbol{E})$ *is also an* $\boldsymbol{R}$*-sequent, where* $i = 3, 4$.

LEMMA N. *Suppose* $\Gamma_0 \subseteq S(\boldsymbol{U} \cup \boldsymbol{E})$, $\Gamma \to \Theta$ *is an* $\boldsymbol{R}$*-sequent,* $K_1(\bar{a}_1), \ldots, K_n(\bar{a}_n)$, $M_1, \ldots, M_m$ *are non-empty finite sets of* $\boldsymbol{R}$*-atomic formulas, and* $\bar{a}_1, \ldots, \bar{a}_n$ *are sequences of free variables.*

If the sequent

$$\exists \bar{v}_1 \bigvee K_1(\bar{v}_1), \ldots, \exists \bar{v}_n \bigvee K_n(v_n), \bigvee M_1, \ldots, \bigvee M_m, \Gamma_0, \Gamma \to \Theta$$

is provable in L_2, *then the sequent*

$$A_1(\bar{a}_1), \ldots, A_n(\bar{a}_n), B_1, \ldots, B_m, \Gamma \to \Theta$$

is provable in $L_3(\boldsymbol{U}, \boldsymbol{E})$ *for each* $A_1(\bar{a}_1) \in K_1(\bar{a}_1), \ldots, A_n(\bar{a}_n) \in K_n(\bar{a}_n)$, $B_1 \in M_1, \ldots, B_m \in M_m$.

LEMMA O. *If an* $\boldsymbol{R}$*-sequent* $\Gamma \to \Theta$, $R(\bar{t})$ *is provable in* $L_4(\boldsymbol{U}, \boldsymbol{E})$ *and* $R \in \boldsymbol{R}$, *then either* $\Gamma \to \Theta$ *is provable in* $L_4(\boldsymbol{U}, \boldsymbol{E})$ *or* $R(\bar{t}) \in \Gamma$.

Note that Lemma N above is an immediate consequence of the normal derivation theorem in [5] if $L = LK$.

By Lemma N, we have

LEMMA P. *If* $\Gamma \to \Theta$ *is an* $\boldsymbol{R}$*-sequent and the sequent* $\boldsymbol{U}, \boldsymbol{E}, \Gamma \to \Theta$ *is provable in* L_2, *then the sequent* $\Gamma \to \Theta$ *is provable in* $L_3(\boldsymbol{U}, \boldsymbol{E})$.

Also, by using Lemma O, we can replace every (B)-rule by $(B)^*$-rule for every $B \in \boldsymbol{U} \cup \boldsymbol{E}$ in a derivation of an $\boldsymbol{R}$-sequent in $L_3(\boldsymbol{U}, \boldsymbol{E})$. Therefore, we have

LEMMA Q. *If an* $\boldsymbol{R}$*-sequent* $\Gamma \to \Theta$ *is provable in* $L_3(\boldsymbol{U}, \boldsymbol{E})$, *then this sequent is also provable in* $L_4(\boldsymbol{U}, \boldsymbol{E})$.

LEMMA R. *Suppose that* X *is a finite set of* $\boldsymbol{R}$*-atomic formulas and* $\Gamma \to \Theta$ *is an* $\boldsymbol{R}$*-positive sequent. If the sequent* $X, \Gamma \to \Theta$ *is provable in* $L_4(\boldsymbol{U}, \boldsymbol{E})$, *then the sequent* $C, \Gamma \to \Theta$ *is provable in* L *for some approximation* C *of* $\boldsymbol{U}$ *by* $\boldsymbol{E}$ *over* X.

Proof. By induction on a derivation $\mathcal{D}$ of the sequent $X, \Gamma \to \Theta$ in $L_4(\boldsymbol{U}, \boldsymbol{E})$. Note that every sequent in $\mathcal{D}$ is an $\boldsymbol{R}$-sequent by Lemma M.

If $\mathcal{D}$ itself is an axiom sequent, then let C be the formula $\bigwedge X \wedge \bigwedge_{A \in \boldsymbol{U}} A[X]$, which has the desired properties.

If $\mathcal{D}$ is not an axiom sequent and the last rule of $\mathcal{D}$ is not any $(B)^*$-rule $(B \in \boldsymbol{U} \cup \boldsymbol{E})$, then we can easily obtained an approximation C of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X which satisfies the required property by hypothesis of induction. So, we only treat the case the last rule of $\mathcal{D}$ is a $(B)^*$-rule for some $B \in \boldsymbol{U} \cup \boldsymbol{E}$. If B is a uniqueness condition of $\boldsymbol{R}$ of the form

$$\forall \bar{x}_1 \ldots \forall \bar{x}_p \Big(\bigwedge_{i=1}^{p} R_i(\bar{x}_i) . \supset A(\bar{x}_1, \ldots, \bar{x}_p)\Big),$$

then $\mathcal{D}$ has the form

$$(B)^* \frac{A(\bar{t}_1, \ldots, \bar{t}_p), X, \Gamma \xrightarrow{D_1} \Theta}{X, \Gamma \to \Theta},$$

where $R_1(\bar{t}_1) \in X, \ldots, R_p(\bar{t}_p) \in X$.

By hypothesis of induction, there is an approximation C_1 of $\boldsymbol{U}$ by $\boldsymbol{E}$ over X such that the sequent $A(\bar{t}_1, \ldots, \bar{t}_p), C_1, \Gamma \to \Theta$ is provable in L. Since $R_1(\bar{t}_1) \in X, \ldots, R_p(\bar{t}_p) \in X$, the sequent $\bigwedge X \wedge B[X] \to A(\bar{t}_1, \ldots, \bar{t}_p)$ is provable in L, hence the sequent $\bigwedge X \wedge \bigwedge_{F \in \boldsymbol{U}} F[X] \to A(\bar{t}_1, \ldots, \bar{t}_p)$ is provable in L. Therefore, the sequent $\bigwedge X \wedge \bigwedge_{F \in \boldsymbol{U}} F[X], C_1, \Gamma \to \Theta$ is provable in L. Let $C = C_1 \wedge (\bigwedge X \wedge \bigwedge_{F \in \boldsymbol{U}} F[X])$, then this C satisfies the required properties. If B is an existence condition of $\boldsymbol{R}$ of the form

$$\forall \bar{x}_0 \ldots \forall \bar{x}_p \Big(\bigwedge_{i=1}^{p} R_i(\bar{x}_i) \wedge \bigwedge_{j=1}^{q} A_j(\bar{x}_1, \ldots, \bar{x}_p) . \supset \exists \bar{y} \bigvee_{k=1}^{r} Q_k(\bar{s}_k(\bar{x}_0, \ldots, \bar{x}_p, \bar{y}))\Big),$$

then $\mathcal{D}$ has the form

$$\frac{\mathcal{D}_1, \ldots, \mathcal{D}_r}{X, \Gamma \to \Theta}$$

where $\mathcal{D}_i$ is a derivation of the sequent $Q_i(\bar{s}_i(\bar{t}_0, \ldots, \bar{t}_p, \bar{a})), X, \Gamma \to \Theta$ in $L_4(\boldsymbol{U}, \boldsymbol{E})$ for each $i = 1, \ldots, r$, where $R_1(\bar{t}_1) \in X, \ldots, R_p(\bar{t}_p) \in X$, $A_1(\bar{t}_0, \ldots, \bar{t}_p) \in \Gamma, \ldots, A_q(\bar{t}_0, \ldots, \bar{t}_p) \in \Gamma$, and each member of $\bar{a}$ does not occur in $\bar{t}_0, \ldots, \bar{t}_p, \bar{s}_1, \ldots, \bar{s}_r, A_1, \ldots, A_q, X, \Gamma, \Theta$. By hypotheses of induction there is an approximation $C_i(\bar{a}, \bar{a}_0, \ldots, \bar{a}_p)$ of $\boldsymbol{U}$ by $\boldsymbol{E}$ over $X \cup \{Q_i(\bar{s}_i(\bar{a}_0, \ldots, \bar{a}_p, \bar{a}))\}$ such that the sequent $C_i(\bar{a}, \bar{t}_0, \ldots, \bar{t}_p), \Gamma \to \Theta$ is provable in L for each $i = 1, \ldots, r$. Hence, the sequent $\exists \bar{y} \bigvee_{i=1}^{r} C_i(\bar{y}, \bar{t}_0, \ldots, \bar{t}_p)$, $\Gamma \to \Theta$ is provable in L. Since $A_1(\bar{t}_0, \ldots, \bar{t}_p) \in \Gamma, \ldots, A_q(\bar{t}_0, \ldots, \bar{t}_p) \in \Gamma$, the sequent

$$A_1(\bar{t}_0, \ldots, \bar{t}_p) \wedge \ldots \wedge A_q(\bar{t}_0, \ldots, \bar{t}_p) . \supset \exists \bar{y} \bigvee_{i=1}^{r} C_i(\bar{y}, \bar{t}_0, \ldots, \bar{t}_p), \Gamma \to \Theta$$

is provable in L.

Let C be the formula

$$\bigwedge_{\bar{u}_1 \in X(R_1)} \ldots \bigwedge_{\bar{u}_p \in X(R_p)} \forall \bar{x}_0 \big(A_1(\bar{x}_0, \bar{u}_1, \ldots, \bar{u}_p) \wedge \ldots \wedge A_q(\bar{x}_0, \bar{u}_1, \ldots, \bar{u}_p) . \supset \exists \bar{y} \bigvee_{i=1}^{r} C_i(\bar{y}, \bar{x}_0, \bar{u}_1, \ldots, \bar{u}_p)\big).$$

Since $R_1(\bar{t}_1) \in X, \ldots, R_p(\bar{t}_p) \in X$, the sequent $C, \Gamma \to \Theta$ is provable in L. Also,

C is an approximation of $\dot{U}$ by E over X. This completes our proof of Lemma R. ■

Now, we give a proof of Theorem A. Suppose that A is an $\boldsymbol{R}$-positive formula in L, X is a finite set of $\boldsymbol{R}$-atomic formulas, U is a finite set of uniqueness conditions of $\boldsymbol{R}$, and E is a finite set of existence conditions of $\boldsymbol{R}$ The "if-part" of Theorem A is obvious because that the formula $\bigwedge X \wedge \bigwedge U \wedge \bigwedge E. \supset B$ is provable in L, for every approximation B of U by E over X. Also, if E consists of simple existence conditions only, for any approximation B of U by E over X, we can find a simple approximation B' of U by E over X such that $B' \supset B$ is provable in L. So, it is sufficient to prove the "only-if-part" of Theorem A.

Assume that the formula $\bigwedge X \wedge \bigwedge U \wedge \bigwedge E. \supset A$ is provable in L. Then, the sequent $X, U, E \to A$ is provable in L. By Lemma K, the sequent $X, U, E \to A^{=}$ is provable in L_1. Then, this sequent is also provable in L_2 by Lemma L. Since $A^{=}$ is $\boldsymbol{R}$-positive, the sequent $X \to A^{=}$ is an $\boldsymbol{R}$-sequent. By Lemma P, this sequent is provable in $L_3(U, E)$. By Lemma Q, the sequent $X \to A^{=}$ is provable in $L_4(U, E)$. By Lemma R, there is an approximation B of U by E over X such that the sequent $B \to A^{=}$ is provable in L. Since the sequent $A^{=} \to A$ is provable in L, the sequent $B \to A$ is provable in L. Hence, the formula $B \supset A$ is provable in L. This completes our proof of Theorem A.

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Accepté par la Rédaction le 30. 11. 1981

On the homotopical classification of DJ-mappings of infinitely dimensional spheres

by

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Abstract. This paper contains some results which concern the DJ-homotopical classification of DJ-mappings of the sphere in an infinitely dimensional Hilbert space into itself.

In connection with the appearance of the definition of the category in which a sphere in the infinitely dimensional Banach space is not contractible, we have to consider the homotopical classification of transformations of such a sphere into itself within this category. This problem has been presented by B. Nowak [5]. It will be a certain simplification to notice that the set of homotopy classes is a group, as it is in the finite-dimensional case. This paper is an attempt to present certain numerical invariants of the homotopy classes. However, we will not be able to prove that there is a one-to-one correspondence between them.

We will first define objects and morphisms of DJ-category.

DEFINITION. A pair $(X, (X_n)_{n\in N})$ where X is a metric space and $(X_n)_{n\in N}$ is an increasing sequence of its subspaces such that

(1) $$X = \overline{\bigcup_{n\in N} X_n}$$

is called a *metric space with filtration.*

DEFINITION. Let $(X, (X_n)_{n\in N})$ and $(Y, (Y_n)_{n\in N})$ be two metric spaces with filtration and d_Y- the distance in Y. A uniformly continuous transformation $f\colon X \to Y$ such that

(2) $$\lim_{n\to\infty} \sup_{x\in X_n} d_Y(f(x), Y_n) = 0$$

is called a DJ-*mapping*. If condition (2) is replaced by:

(3) there is $n_0 \in N$ such that for $n \geqslant n_0$: $f(X_n) \subset Y_n$,

then f is called an FJ-*mapping.*

FJ-mappings are a particular case of DJ-mappings.

We will consider an infinitely dimensional Hilbert space H with a filtration containing finite-dimensional linear subspaces $(H_n)_{n\in N}$. A filtration induced on $S_H = \{x \in H\colon \|x\| = 1\}$ makes the unit sphere a metric space with